CUT-GENERATING FUNCTIONS

or The Infinite Relaxation

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Mixed Integer Linear Programming

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax = b & \\ & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\ & x_j \geq 0 & \text{for } j = 1, \ldots, n. \end{array}$$

Cutting plane approach to solving MILP:

- First solve the LP relaxation. Basic optimal solution:

$$x_i \quad = \quad f_i + \sum_{j \in N} r^j x_j \qquad \text{for } i \in B.$$

- If $f_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add cutting planes.

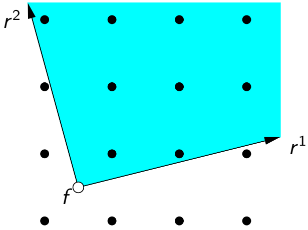
Setting the Stage for Cutting Plane Formulas

Gomory 1969: Relax nonnegativity on the basic variables.

In addition, Andersen, Louveaux, Weismantel and Wolsey 2007 suggested to relax integrality on the nonbasic variables x_j.

$$\begin{array}{rcl} y & = & f + \sum_{j=1}^{k} r^j x_j \\ y & \in & \mathbb{Z}^q \\ x & \geq & 0 \end{array}$$

Example



Feasible set $\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{Z}^2 :$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

where $x_1 \geq 0, x_2 \geq 0\}$

Formulas for Cutting Planes (for the Andersen et al model)

$$\begin{array}{rcl} y &=& f + \sum_{j=1}^{k} r^j x_j \\ y &\in& \mathbb{Z}^q \\ x &\geq& 0 \end{array}$$

Every inequality cutting off the point $(\bar{x}, \bar{y}) = (0, f)$ is of the form

$$\sum_{j=1}^{k} \alpha_j x_j \geq 1.$$

We are interested in "formulas" for deriving such inequalities.

More formally, we are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

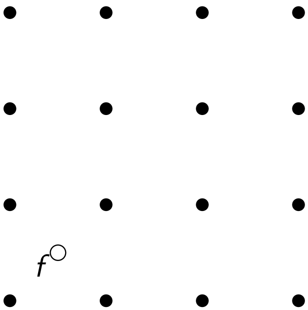
$$\sum_{j=1}^{k} \psi(r^j) x_j \geq 1$$

is valid for every choice of k and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$.

We refer to such functions ψ as cut-generating functions.

We are interested in minimal cut-generating functions.

Cut-Generating Functions



We are given $f \notin \mathbb{Z}^q$. Can we generate cut-generating functions from this information?

These functions should generate valid cuts for any integer program with q basic integer variables: We know that any feasible solution to the integer program must satisfy y integral, and we want to cut off the point $y = f$ since $f \notin \mathbb{Z}^q$.

Theorem

Borozan and Cornuéjols (MOR 2009)

Let $f \in \mathbb{R}^q \setminus \mathbb{Z}^q$.

If $\psi : \mathbb{R}^q \to \mathbb{R}$ is a minimal cut-generating function, then ψ is

- nonnegative
- piecewise linear
- positively homogeneous
- and convex.

Furthermore $K_\psi := \{y \in \mathbb{R}^q : \ \psi(y - f) \leq 1\}$ is a maximal $\mathbb{Z}^q$-free convex set containing f in its interior.

Conversely, for any maximal $\mathbb{Z}^q$-free convex set K containing f in its interior, the gauge of $K - f$ is a minimal cut-generating function.

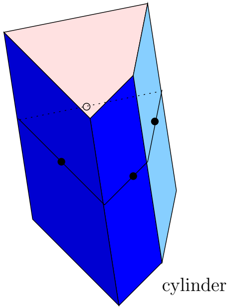
DEFINITION Gauge of a convex set S containing the origin:
$\gamma_S(r) := \inf\{t > 0 \ : \ \frac{r}{t} \in S\},$ for all $r \in \mathbb{R}^n$.

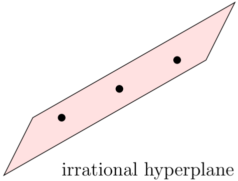
Lovász' Theorem

Lovász MPS Tokyo 1988

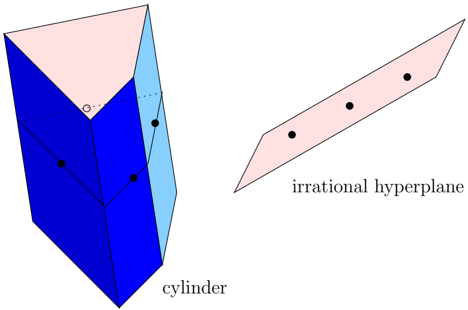
THEOREM A set $K \subset \mathbb{R}^q$ is a maximal $\mathbb{Z}^q$-free convex set if and only if

- either K is a polyhedron of the form $K = P + L$ where P is a polytope, L is a rational linear space, $\dim(P) + \dim(L) = p$, K does not contain any point of $\mathbb{Z}^q$ in its interior and there is a point of $\mathbb{Z}^q$ in the relative interior of each facet of K.
- or K is an irrational hyperplane.





Consequence of the Lovász and Borozan-Cornuéjols Theorems



If $K = \{y \in \mathbb{R}^q : a_i(y - f) \leq 1, i = 1, \ldots, t\}$,

then the gauge of $K - f$ is $\psi(r) = max_{i=1,\ldots,t} a_i r$.

Every minimal cut-generating function is of this form.

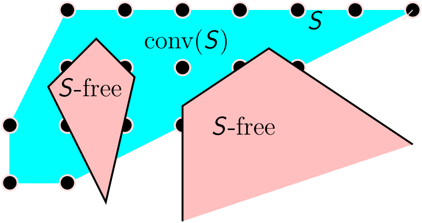
Generalizing the Andersen et al model

$$\begin{aligned} y &= f + \sum_{j=1}^{k} r^j x_j \\ y &\in S \\ x &\geq 0 \end{aligned}$$

where $S = P \cap \mathbb{Z}^q$ and P is a rational polyhedron.

QUESTIONS: Can one define cut-generating functions?

What about maximal S-free sets?

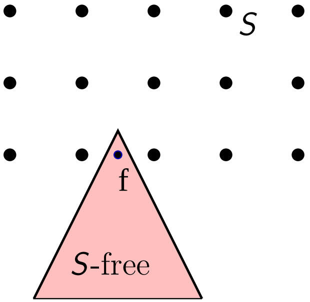


Generalizing the Andersen et al model

If $K = \{y \in \mathbb{R}^q : a_i(y - f) \leq 1, i = 1, \dots, t\}$,
let $\psi_K(r) = max_{i=1,\dots,t} a_i r$.

THEOREM Basu, Conforti, Cornuéjols, Zambelli SIDMA 2010

For every cut-generating function ψ, there exists a maximal S-free convex set K with f in its interior such that $\psi_K \leq \psi$.
Conversely, if K is a maximal S-free convex set K with f in its interior, then ψ_K is a minimal cut-generating function.



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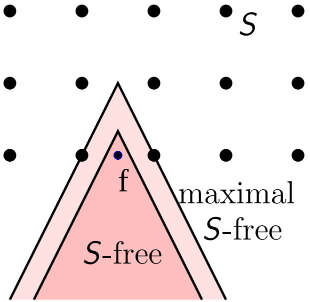

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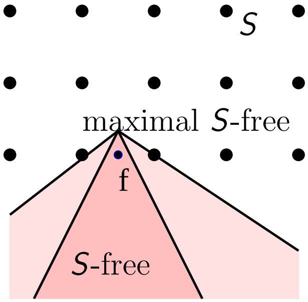


Integer Lifting

Integer Lifting

We now consider a system of the form

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j + \sum_{i=1}^{\ell} \rho^i y_i \\ x & \in & S := P \cap \mathbb{Z}^q \\ s & \geq & 0 \\ y & \geq & 0, \quad y \in \mathbb{Z}^\ell. \end{array}$$

We are interested in pairs of functions $\psi : \mathbb{R}^q \to \mathbb{R}$ and $\pi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) s_j + \sum_{i=1}^{\ell} \pi(\rho^i) y_i \geq 1$$

is valid for every choice of integers k, ℓ and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$ and $\rho^1, \ldots, \rho^\ell \in \mathbb{R}^q$.

Gomory and Johnson since the 1970's: Construct π first, then ψ. We turn things around! We start from ψ.

DEFINITION The function π is called a lifting of ψ.

REMARK If ψ is a cut-generating function and π is a minimal lifting of ψ, then $\pi \leq \psi$.

An Equivalent Formulation

The following formulation is equivalent for all $h : \mathbb{R}^q \longrightarrow \mathbb{Z}$.

$$
\begin{aligned}
x &= f + \sum_{j=1}^{k} r^j s_j + \sum_{i=1}^{\ell} \rho^i y_i \\
z &= 0 + \sum_{j=1}^{k} 0 s_j + \sum_{i=1}^{\ell} h(\rho^i) y_i \\
z &\in \mathbb{Z} \\
x &\in S \\
s &\geq 0 \\
y &\geq 0, \quad y \in \mathbb{Z}^{\ell}.
\end{aligned}
$$

A Relaxation: Now we relax the integrality of the y variables.

This is a problem of the form that we understand: minimal inequalities correspond to maximal lattice-free convex sets. We have increased the dimension by 1.

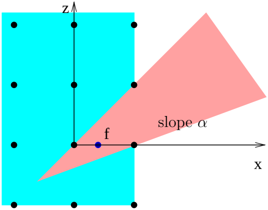
Let $\psi(r^j) := \tilde{\psi}(\begin{pmatrix} r^j \\ 0 \end{pmatrix})$ and $\pi^h(\rho^i) := \tilde{\psi}(\begin{pmatrix} \rho^i \\ h(\rho^i) \end{pmatrix})$

$$
\sum_{j=1}^{k} \psi(r^j) s_j + \sum_{i=1}^{\ell} \pi^h(\rho^i) y_i \geq 1
$$

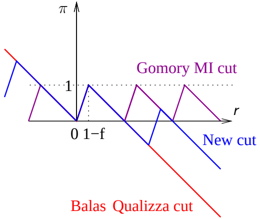
Example Cornuéjols, Kis and Molinaro ORL 2013

Consider a single basic row,
with integer basic variable $x \leq 1$,
and both continuous and integer
nonbasic variables.

Introduce a new basic variable $z \in \mathbb{Z}$.



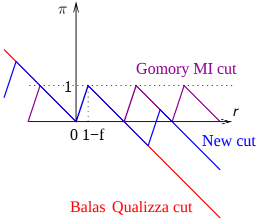
This yields a new cut that is identical to the Gomory mixed integer cut on the continuous variables but different on the integer variables: $\pi_\alpha(r) = \min\{\frac{-r+\lceil \alpha r \rceil}{f}, \frac{r}{1-f} - \frac{\lfloor \alpha r \rfloor (1-\alpha(1-f))}{\alpha f(1-f)}\}$.



QUESTION Starting from a minimal cut-generating function $\psi : \mathbb{R}^q \to \mathbb{R}$, what can we say about a minimal lifting function π?

We already observed that $\pi \leq \psi$.
Can we guarantee that $\pi(r) = \psi(r)$ for some vectors r?

THEOREM Let ψ be a minimal cut-generating function and π a minimal lifting of ψ. Then there exists $\epsilon > 0$ such that ψ and π coincide on a ball of radius ϵ centered at the origin.



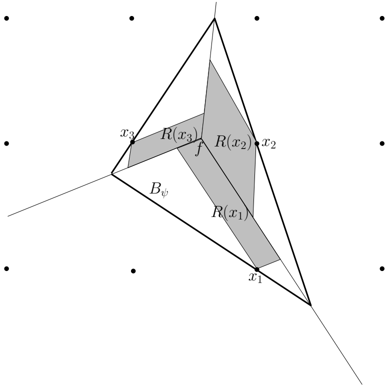
Let R be the region where π and ψ coincide.

What can be said about this region R?

Integer Lifting

Basu, Campelo, Conforti, Cornuéjols, Zambelli MP 2013

THEOREM Let ψ be a minimal cgf and let π be a minimal lifting of ψ. Then $\pi(r) = \psi(r)$ for $r \in R := \bigcup_t R(x_t)$ where the union is taken over all points $x_t \in S$ on the boundary of the maximal S-free convex set K_ψ defining ψ and the $R(x_t)$s are parallelepipeds as shown in grey in the figure.
Conversely, if $r \notin R$, there exists a minimal lifting π where $\pi(r) < \psi(r)$.

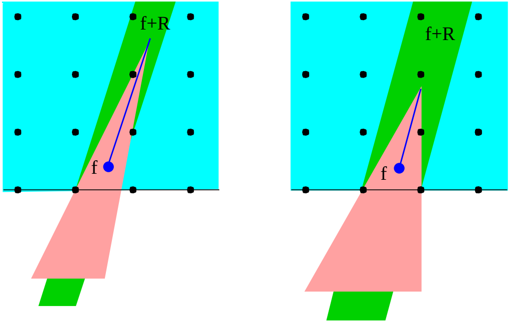


THEOREM Let ψ be a minimal cgf. Assume $S = \mathbb{Z}^n$.
Then ψ has a unique minimal lifting π if and only if $R + \mathbb{Z}^q$ covers $\mathbb{R}^q$.

OPEN PROBLEM Does this result hold for general $S := P \cap \mathbb{Z}^n$?

Sufficiency:

THEOREM Consider a minimal cut-generating function ψ.
Let L be the lineality space of conv(S).
If $R + (\mathbb{Z}^q \cap L)$ covers $\mathbb{R}^q$, then ψ has a unique minimal lifting π.



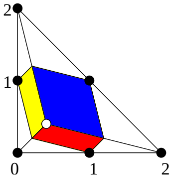
Necessity?

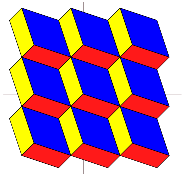
Bodies with a Unique Lifting

Dey, Wolsey IPCO 2008

THEOREM In the plane, the splits, Type 1 and Type 2 triangles have a unique lifting. The Type 3 triangles and most quadrilaterals do not.

Example: The region $f + R$ and its integer translates.

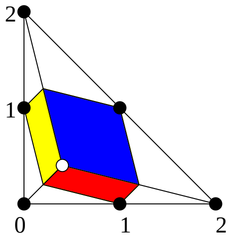


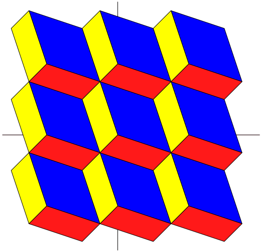


THEOREM Averkov and Basu IPCO 2014

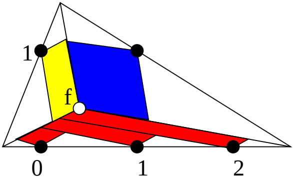
Let K be a maximal $\mathbb{Z}^q$-free polytope ($q \geq 2$). Then K is either a body with a unique lifting for all $f \in \text{int}(K)$, or a body with multiple liftings for all $f \in \text{int}(K)$.

THEOREM Let K be a maximal $\mathbb{Z}^q$-free simplex such that each facet of K has exactly one integer point in its relative interior. Then K is a body with a unique lifting if and only if all the vertices of K are integral, i.e., K is a unimodular transformation of $\mathrm{conv}\{0, qe^1, \ldots, qe^q\}$.





THEOREM
Let K be a maximal $\mathbb{Z}^q$-free 2-partitionable simplex with hyperplanes H_1, H_2 such that H_1 defines a facet of K and this is the only facet of K with more than one lattice point in its relative interior.
Then K is a body with a unique lifting if and only if $K \cap H_2$ is an affine unimodular transformation of $\text{conv}\{0, (q-1)e^1, \ldots, (q-1)e^{q-1}\}$.



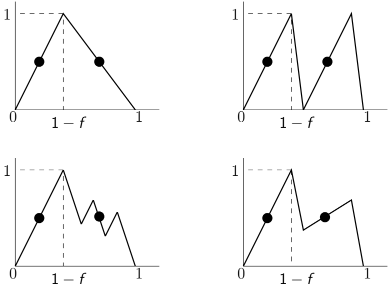
Minimal cut-generating functions for the pure integer case

$$f + \sum_{j=1}^{k} r_j s_j \in \mathbb{Z}^m, \quad s_j \in \mathbb{Z}_+ \text{ for } j = 1, \ldots, k. \qquad \text{(PureIP)}$$

A function $\pi : \mathbb{R}^m \to \mathbb{R}$ is periodic if $\pi(r) = \pi(r + w)$ for all $r \in [0, 1]^m$ and $w \in \mathbb{Z}^m$.

Also, π is said to satisfy the symmetry condition if $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^m$.

Finally, π is subadditive if $\pi(a + b) \leq \pi(a) + \pi(b)$.



THEOREM (Gomory and Johnson 1972) Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a non-negative function. Then π is a minimal cut-generating function for (PureIP) if and only if $\pi(0) = 0$, π is periodic, subadditive and satisfies the symmetry condition.

Extreme cut-generating functions

A cut-generating function π is extreme if it cannot be written as a convex combination of two other cut-generating functions.

Basu, Hildebrand and Köppe address the issue of checking the extremality of a cut generating function.

A deep result on the infinite relaxation is a sufficient condition for extremality in the restricted setting $m = 1$, the so-called 2-slope theorem of Gomory and Johnson 1972.

THEOREM (2-slope theorem)
Let $\pi : \mathbb{R} \to \mathbb{R}$ be a minimal cut-generating function. If π is a continuous piecewise linear function with only two slopes, then π is extreme.

Gomory and Johnson 2003 conjectured that continuous extreme cut-generating functions are always piecewise linear. Basu, Conforti, Cornuéjols and Zambelli MP 2012 disproved this conjecture.

Exercises

In "Courses Material" on the webpage

http://eventos.cmm.uchile.cl/discretas2016/

do the following exercises in Course Notes "Cutting planes in integer programming"

Exercise 4.3

Exercise 4.7

Optional: Exercise 4.8