Lecture 1

Cutting Planes in Integer Programming

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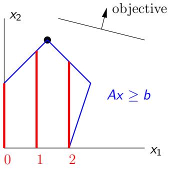
Brief history

First Algorithms	Polynomial Algorithms
Solving systems of linear equations	
• Babylonians 1700BC • Gauss 1801	• Edmonds 1967
Solving systems of linear inequalities	
• Fourier 1822 • Dantzig 1951	• Khachyan 1979 • Karmarkar 1984
Solving systems of linear inequalities in integers	
• Gomory 1958	• Lenstra 1983

Mixed Integer Linear Programming

$$\min cx \quad x \in S$$

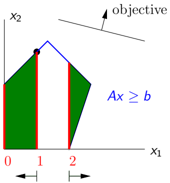
where $S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \; Ax \geq b\}$



Linear Relaxation

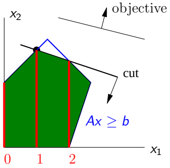
$$\min cx \quad x \in P$$

where $P := \{x \in \mathbb{R}_+^n : \; Ax \geq b\}$



Branch-and-bound

Land and Doig 1960



Cutting Planes

Dantzig, Fulkerson and Johnson 1954

Gomory 1958

Fractional Cuts Gomory 1958

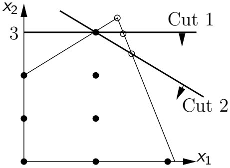
Consider a single constraint : $S := \{x \in \mathbb{Z}^n_+ : \sum_{j=1}^n a_j x_j = b\}$.

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$,
and $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$.

THEOREM $\sum_j f_j x_j \geq f_0$ is a valid inequality for S.

EQUIVALENT FORM $\sum_j \lfloor a_j \rfloor x_j \leq \lfloor b \rfloor$.

APPLICATION



$$\begin{array}{lrl} \max & 5.5x_1 + 2.1x_2 & \\ & -x_1 + x_2 & \leq 2 \\ & 8x_1 + 2x_2 & \leq 17 \\ & x_1,\ x_2 \geq 0 & \\ & x_1,\ x_2 \text{ integer.} & \end{array}$$

$$\begin{array}{lrrcl} z & +0.58x_3 & +0.76x_4 & = & 14.08 \\ x_2 & +0.8x_3 & +0.1x_4 & = & 3.3 \\ x_1 & -0.2x_3 & +0.1x_4 & = & 1.3 \\ & & x_1, x_2, x_3, x_4 \geq 0. & & \end{array}$$

Cut $0.8x_3 + 0.1x_4 \geq 0.3$

or $x_2 \leq 3$.

EXERCISE Finish solving this integer program using fractional cuts.

Polyhedral Theory

$P := \{x \in \mathbb{R}^n_+ : \ Ax \geq b\}$ Polyhedron

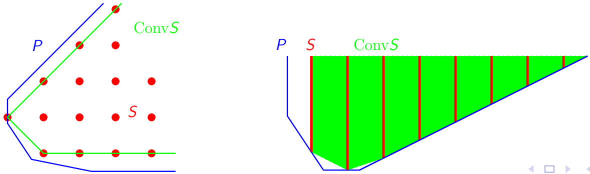
$S := P \cap (\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+)$ Mixed Integer Linear Set

$$\text{Conv } S := \{x \in \mathbb{R}^n : \ \exists x^1, \ldots, x^k \in S, \ \lambda \geq 0, \ \sum \lambda_i = 1 \text{ such that } x = \lambda_1 x^1 + \ldots + \lambda_k x^k\}$$

THEOREM Meyer 1974

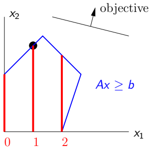
If A, b have rational entries, then Conv S is a polyhedron.

Idea of Proof Using a theorem of Minkowski 1896 and Weyl 1935 :
P is a polyhedron if and only if $P = Q + C$ where Q is a polytope and C is a polyhedral cone.



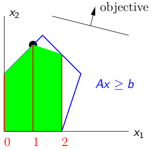
Thus

$$\min \; cx$$
$$x \in S$$



can be rewritten as the LP

$$\min \; cx$$
$$x \in \text{Conv } S$$



We are interested in the constructive aspects of Conv S.

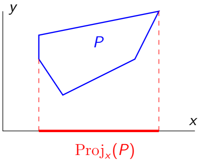
REMARK The number of constraints of Conv S can be exponential in the size of $Ax \geq b$, BUT
1) often a partial representation of Conv S suffices (Examples : Dantzig, Fulkerson, Johnson 1954, Gomory 1958) ;
2) Conv S can sometimes be obtained as the projection of a polyhedron with a polynomial number of variables and constraints.

Projections

Let $P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : \; Ax + Gy \geq b\}$

DEFINITION

$\text{Proj}_x(P) := \{x \in \mathbb{R}^n : \; \exists y \in \mathbb{R}^k \text{ such that } Ax + Gy \geq b\}$



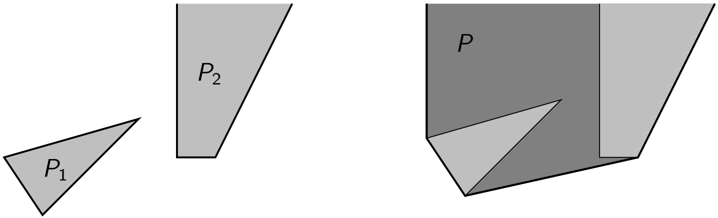
THEOREM

$\text{Proj}_x(P) = \{x \in \mathbb{R}^n : \; vAx \geq vb \text{ for all } v \in Q\}$
where $Q := \{v \in \mathbb{R}^m : \; vG = 0, v \geq 0\}$.

PROOF

Let $x \in \mathbb{R}^n$. Farkas's lemma (Farkas 1894) implies that
$Gy \geq b - Ax$ has a solution y if and only if
$v(b - Ax) \leq 0$ for all $v \geq 0$ such that $vG = 0$. ■

Union of Polyhedra Balas 1974



(a) (b)

We first model that a point belongs to the union of k polytopes in $\mathbb{R}^n$, namely bounded sets of the form

$$\begin{aligned} A_i y &\leq b_i \\ 0 \leq y &\leq u_i, \end{aligned}$$

for $i = 1, \ldots, k$.

The same modeling question is more complicated for unbounded polyhedra and will be discussed later.

Union of Polytopes

A way to model that a point $y \in \mathbb{R}^n$ belongs to the union of k polytopes in $\mathbb{R}^n$ is to introduce k variables $x_i \in \{0,1\}$, indicating whether y is in the ith polytope, and k vectors of variables $y_i \in \mathbb{R}^n$. The vector $y \in \mathbb{R}^n$ belongs to the union of the k polytopes

$$\begin{aligned} A_i y &\leq b_i \\ 0 \leq y &\leq u_i, \end{aligned}$$

if and only if

$$\begin{array}{rcll} \sum_{i=1}^{k} y_i & = & y & \\ A_i y_i & \leq & b_i x_i & i = 1, \ldots, k \\ 0 \leq \; y_i & \leq & u_i x_i & i = 1, \ldots, k \\ \sum_{i=1}^{k} x_i & = & 1 & \\ x \in \{0,1\}^k. & & & \end{array}$$

Union of Polytopes

PROPOSITION The convex hull of solutions to

$$\begin{array}{rcll} \sum_{i=1}^{k} y_i & = & y & \\ A_i y_i & \leq & b_i x_i & i = 1, \ldots, k \\ 0 \leq \; y_i & \leq & u_i x_i & i = 1, \ldots, k \\ \sum_{i=1}^{k} x_i & = & 1 & \\ x \in \{0,1\}^k. & & & \end{array}$$

is

$$\begin{array}{rcll} \sum_{i=1}^{k} y_i & = & y & \\ A_i y_i & \leq & b_i x_i & i = 1, \ldots, k \\ 0 \leq \; y_i & \leq & u_i x_i & i = 1, \ldots, k \\ \sum_{i=1}^{k} x_i & = & 1 & \\ x \in [0,1]^k. & & & \end{array}$$

Union of Polytopes

PROOF

Let Q and P be the 0,1 formulation and the polytope, respectively, given in the statement of the proposition. It suffices to show that any point $\bar{z} := (\bar{y}, \bar{y}_1, \ldots, \bar{y}_k, \bar{x}_1, \ldots, \bar{x}_k)$ in P is a convex combination of solutions to Q. For t such that $\bar{x}_t \neq 0$, define the point $z^t = (y^t, y_1^t, \ldots, y_k^t, x_1^t, \ldots, x_k^t)$ where

$$y^t := \frac{\bar{y}_t}{\bar{x}_t}, \qquad y_i^t := \begin{cases} \frac{\bar{y}_t}{\bar{x}_t} & \text{for } i = t, \\ 0 & \text{otherwise}, \end{cases} \qquad x_i^t := \begin{cases} 1 & \text{for } i = t, \\ 0 & \text{otherwise}. \end{cases}$$

The z^ts are solutions of Q. We claim that $\bar{z}$ is a convex combination of these points, namely $\bar{z} = \sum_{t:\bar{x}_t \neq 0} \bar{x}_t z^t$. To see this, observe first that $\bar{y} = \sum \bar{y}_i = \sum_t \bar{y}_t = \sum_{t:\bar{x}_t \neq 0} \bar{x}_t y^t$. We leave it as an exercise to show that $\bar{y}_i = \sum_{t:\bar{x}_t \neq 0} \bar{x}_t y_i^t$ and $\bar{x}_i = \sum_{t:\bar{x}_t \neq 0} \bar{x}_t x_i^t$ for $i = 1, \ldots, k$. □

Union of Polyhedra

THEOREM Balas 1974

Given k polyhedra $P_i := \{x \in \mathbb{R}^n \ : \ A_i x \leq b^i\},\ i = 1, \dots k$, let $C_i := \{x \ : \ A_i x \leq 0\}$, and let $R^i \subset \mathbb{R}^n$ be a finite set such that $C_i = \text{cone}(R^i)$. For every $i \in \{1, \dots, k\}$ such that $P_i \neq \emptyset$, let $V^i \subset \mathbb{R}^n$ be a finite set such that $P_i = \text{conv}(V^i) + \text{cone}(R^i)$. Consider the polyhedron $P := \text{conv}(\bigcup_{i: P_i \neq \emptyset} V^i) + \text{cone}(\bigcup_{i=1}^k R^i)$ and let $Y \subseteq \mathbb{R}^n \times (\mathbb{R}^n)^k \times \mathbb{R}^k$ be the polyhedron described by the following system

$$\begin{array}{rcll} A_i x^i & \leq & \delta_i b^i & i = 1, \dots, k \\ \sum_{i=1}^k x^i & = & x & \\ \sum_{i=1}^k \delta_i & = & 1 & \\ \delta_i & \geq & 0 & i = 1, \dots, k. \end{array}$$

Then $P = \text{proj}_x(Y) := \{x \in \mathbb{R}^n \ : \ \exists (x^1, \dots, x^k, \delta) \in (\mathbb{R}^n)^k \times \mathbb{R}^k \text{ s.t. } (x, x^1, \dots, x^k, \delta) \in Y\}$.

Union of Polyhedra

Balas' theorem gives an extended formulation of a polyhedron P whose size is approximately the sum of the sizes of the formulations that describe the polyhedra P_i.

This polyhedron P contains the convex hull of $\bigcup_{i=1}^k P_i$ but in general this inclusion is strict. Indeed, the recession cone of P contains cone $C_i, i = 1, \ldots, k$, even if P_i is empty. Furthermore, even if the polyhedra P_i are all nonempty but have different recession cones, the set $\mathrm{conv}(\bigcup_{i=1}^k P_i)$ may not be closed, and therefore it may not be a polyhedron. For example, in $\mathbb{R}^2$, the convex hull of a line L and a point not in L is not a closed set.

LEMMA ABOUT NONEMPTY POLYHEDRA

Let $P_1, \ldots, P_k \subseteq \mathbb{R}^n$ be nonempty polyhedra. Then

$$\overline{\mathrm{conv}}(\cup_{i=1}^k P_i) = P.$$

Cone Condition for the Union of Polyhedra

THEOREM Balas 1974

Let $P_i := \{x \in \mathbb{R}^n : A_i x \leq b^i\}$ be k polyhedra such that $\cup_{i=1}^k P_i \neq \emptyset$, and let Y be the polyhedron defined earlier. Let $C_i := \{x : A_i x \leq 0\}$ and let $R^i \subset \mathbb{R}^n$ be a finite set such that $C_i = \mathrm{cone}(R^i)$, $i = 1, \ldots, k$. Then $\overline{\mathrm{conv}}(\cup_{i=1}^k P_i)$ is the projection of Y onto the x-space if and only if $C_j \subseteq \mathrm{cone}(\bigcup_{i : P_i \neq \emptyset} R^i)$ for every $j = 1, \ldots, k$.

CORROLARY If $P_1, \ldots, P_k$ are nonempty polyhedra with identical recession cones, then $\mathrm{conv}(\cup_{i=1}^k P_i)$ is a polyhedron.

Modeling a split disjunction

We are given a linear system $Ax \leq b$ in $\mathbb{R}^n$, and we want to further impose the disjunctive constraint $\quad cx \leq d_1 \quad$ or $\quad cx \geq d_2$, where $c \in \mathbb{R}^n$ and $d_1 < d_2$.

If we define $P := \{x \in \mathbb{R}^n \, : \, Ax \leq b\}$, $P_1 := \{x \in P \, : \, cx \leq d_1\}$, $P_2 := \{x \in P \, : \, cx \geq d_2\}$, the set of feasible solutions is $P_1 \cup P_2$.

The next lemma shows that $\mathrm{conv}(P_1 \cup P_2)$ is a polyhedron.

LEMMA $\mathrm{conv}(P_1 \cup P_2)$ is the projection onto the space of x variables of the polyhedron Q defined by

$$\begin{array}{rcl} Ax^1 & \leq & \lambda b \\ cx^1 & \leq & \lambda d_1 \\ Ax^2 & \leq & (1-\lambda) b \\ cx^2 & \geq & (1-\lambda) d_2 \\ x^1 + x^2 & = & x \\ 0 \leq \lambda & \leq & 1. \end{array}$$

Note that, to prove this, we cannot apply the above Corollary because P_1 and P_2 may have different recession cones.

Modeling a split disjunction

PROOF

The lemma holds when $P_1 = P_2 = \emptyset$ by Balas' theorem.
By symmetry, we assume in the remainder that $P_1 \neq \emptyset$.
Let $C_1 = \{r : Ar \leq 0, cr \leq 0\}$ and $C_2 = \{r : Ar \leq 0, cr \geq 0\}$.
Note that $\mathrm{rec}(P) = C_1 \cup C_2$.
Let $\bar{x} \in P_1$. We observe that, if there exists a vector $r \in C_2 \setminus C_1$, we have $P_2 \neq \emptyset$ and $\bar{x} + r \in \mathrm{conv}(P_1 \cup P_2)$. Indeed, $cr > 0$ and, if we let $\lambda := \max(1, \frac{d_2 - c\bar{x}}{cr})$, the point $\bar{x} + \lambda r$ is in P_2 and $\bar{x} + r$ is in the line segment joining $\bar{x}$ and $\bar{x} + \lambda r$.
The above observation shows that, if $P_2 = \emptyset$, then $C_2 \subseteq C_1$, therefore the cone condition holds in this case. It also trivially holds when both $P_1, P_2 \neq \emptyset$. Thus, in all cases, the cone condition theorem implies that $\overline{\mathrm{conv}}(P_1 \cup P_2)$ is the projection onto the space of x variables of the polyhedron Q defined in the statement of the lemma.

Modeling a split disjunction

Therefore, to prove the lemma, we only need to show $\overline{\mathrm{conv}}(P_1 \cup P_2) = \mathrm{conv}(P_1 \cup P_2)$. We assume $P_1, P_2 \neq \emptyset$ otherwise the statement is obvious. Let $Q_1, Q_2 \subset \mathbb{R}^n$ be two polytopes such that $P_1 = Q_1 + C_1$ and $P_2 = Q_2 + C_2$. By the lemma on unions of nonempty polyhedra, and because $\mathrm{rec}(P) = C_1 \cup C_2$, $\overline{\mathrm{conv}}(P_1 \cup P_2) = \mathrm{conv}(Q_1 \cup Q_2) + \mathrm{rec}(P)$, thus we only need to show that $\mathrm{conv}(Q_1 \cup Q_2) + \mathrm{rec}(P) \subseteq \mathrm{conv}(P_1 \cup P_2)$. Let $\bar{x} \in \mathrm{conv}(Q_1 \cup Q_2) + \mathrm{rec}(P)$. Then there exist $x^1 \in Q_1$, $x^2 \in Q_2$, $0 \leq \lambda \leq 1$, $r \in \mathrm{rec}(P)$, such that $\bar{x} = \lambda x^1 + (1 - \lambda)x^2 + r$. By symmetry we may assume $\lambda > 0$.

By the initial observation, $x^1 + \frac{r}{\lambda} \in \mathrm{conv}(P_1 \cup P_2)$, thus $\bar{x} = \lambda(x^1 + \frac{r}{\lambda}) + (1 - \lambda)x^2 \in \mathrm{conv}(P_1 \cup P_2)$. $\square$

Exercises

In "Courses Material" on the webpage

http ://eventos.cmm.uchile.cl/discretas2016/

do the following exercises in Course Notes "Cutting planes in integer programming"

Exercise 1.5

Exercise 1.7

Optional : Exercise 1.8