Problem Set 1 - Introduction to pseudorandom graphs

1. Using Chernoff's inequality and the union bound, prove that if $p = p(n) \leq 0.99$, then, asymptotically almost surely, the random graph $G(n, p)$ has the following properties. For any subset $X \subseteq V(G)$,
$$\left| e(X) - p\binom{|X|}{2} \right| = O(\sqrt{p|X|^3 \log(2n/|X|)})$$
and, for any subsets $X, Y \subseteq V(G)$ with $|X| \leq |Y|$,
$$|e(X,Y) - p|X||Y|| = O(\sqrt{p|X||Y|^2 \log(2n/|Y|)})).$$
Deduce that $G(n, p)$ is $(p, O(\sqrt{pn}))$-jumbled.

2. Suppose that G is a bipartite graph of density $1/2$ between two vertex sets A and B, each of order $n/2$. Show that there exist subsets $A' \subset A$ and $B' \subset B$ such that
$$|e(A', B') - \frac{1}{2}|A'||B'|| \geq cn^{3/2},$$
for some positive constant c. What does this say about the jumbledness of graphs?

   [*Hint:* Consider a random subset of A of order $|A|/2$ and think about how the neighbourhoods of vertices in B intersect this set.]

3. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a d-regular graph G. Prove that $\lambda_1 = d$ and $|\lambda_i| \leq d$ for all $2 \leq i \leq n$. Show also that if G is connected then the λ_1-eigenvector v_1 is proportional to the vector $(1, 1, \ldots, 1)^t \in \mathbb{R}^n$ and $\lambda_i < d$ for all $i \geq 2$. What happens if G is not connected?

4. Following the notation of the previous question, show that $\lambda_n = -d$ if and only if G is bipartite.