Regularisation by noise in PDEs
Addition of noise has positive effects on the theory of the equation (in a pathwise sense)

→ ODEs:

\[
X_t = x + \int_0^t b(X_s)ds + W_t
\]

Many results: Veretennikov, Davie, Krylov-Röckner, Flandoli, Attanasio, Fedrizzi, Proske, Aryasova–Pilipenko,… Essentially bounded \(b\). More precisely, Ladyzhenskaya-Prodi-Serrin (LPS) condition :

\[
b \in L_t^q L_x^p \quad \frac{d}{p} + \frac{2}{q} < 1.
\]

→ Transport (or continuity) equation (Stratonovich integral):

\[
d_t u(t, x) + b(x) \cdot \nabla u(t, x) dt = \nabla u(t, x) \circ dW_t
\]


Stochastic vector advection equation (Flandoli–Maurelli–Neklyudov):

\[ \frac{d}{dt}B + \text{curl}(\mathbf{v} \times B)dt + \sigma \sum_{k=1}^{d} \text{curl}(\mathbf{e}_k \times B) \circ dW^k_t = 0. \]

Noise avoid blow-up of \( \|B(t, \cdot)\|_{L^\infty_x} \) for \( \mathbf{v} \in C^\alpha \) with \( \alpha \in (0, 1) \).

Non-linear PDEs with transport structure. Point vortices in 2d (Flandoli–G.–Priola), Vlasov–Poisson (Delarue–Flandoli–Vincenzi).

\[ du(t, x) + u(t, x) \cdot \nabla u(t, x) = \sum_{k=1}^{N} \sigma_k(x) \cdot \nabla u(t, x) \circ dW^k_t. \]

(Hypoelliptic) Noise helps to avoid collapse due to peculiar configurations.

Modulated non-linear Schrödinger equation in \( d = 1 \). De Bouard–Debussche, Debussche–Tsutsumi.

\[ d_t \varphi(t, x) = i \Delta \varphi(t, x) \circ dW_t + i|\varphi(t, x)|^{p-2}\varphi(t, x)dt \]

Motivated by homogeneisation in optical wave–guides with dispersion management.

Averaging lemmas for kinetic equations. (Fedrizzi–Flandoli–Priola–Vovelle, Lions–Perthame–Souganidis, Gess–Souganidis)
Goal: provide a deterministic framework to discuss regularization by “perturbations/modulation” for the following model PDEs:

- **Transport equation:** $x \in \mathbb{R}^d$, $t \geq 0$, $w: \mathbb{R} \to \mathbb{R}^d$, $b: \mathbb{R}^d \to \mathbb{R}^d$

  $$\partial_t u(t, x) + \dot{w}_t \cdot \nabla u(t, x) + b(x) \cdot \nabla u(t, x) = 0, \quad u(0, \cdot) = u_0.$$ 

- **Non-linear Schrödinger equation:** $x \in \mathbb{T}$, $\mathbb{R}$, $t \geq 0$, $w: \mathbb{R} \to \mathbb{R}$

  $$\partial_t \varphi(t, x) = i \Delta \varphi(t, x) \dot{w}_t + i |\varphi(t, x)|^{p-2} \varphi(t, x).$$

- **Korteweg–de Vries equation:** $x \in \mathbb{T}$, $\mathbb{R}$, $t \geq 0$, $w: \mathbb{R} \to \mathbb{R}$

  $$\partial_t u(t, x) = \partial_x^3 u(t, x) \dot{w}_t + \partial_x u(t, x)^2.$$ 

- By defining a suitable notion of "irregular" $w$ we are able to show, in a quantitative way, that the more $w$ is irregular the more some properties of these equations improves.

- The sample paths of Brownian motion or fractional Brownian motion and similar processes have almost surely this kind of irregularity.

[Joint work with Remi Catellier and Khalil Chouk]
Consider the linear transport PDE

$$\partial_t u(t, x) + \dot{w}_t \cdot \nabla u(t, x) = f(x), \quad u(0, \cdot) = 0.$$ 

Solutions are given explicitly by

$$u(t, x) = \int_0^t f(x + w_s - w_t) ds = T_t^w f(x - w_t)$$

where for any continuous function $w: [0, 1] \to \mathbb{R}^d$ we define the averaging operator

$$T_t^w f(x) = \int_0^t f(x + w_s) ds, \quad T_{t,s}^w f = T_t^w f - T_s^w f$$

acting on functions (or distributions) $f: \mathbb{R}^d \to \mathbb{R}$.

**Question:** What is the relation between $w$, the regularity of $f$ and that of $u(t, \cdot)$?

If $w$ is smooth we do not expect anything special to happen and $u$ to have the same regularity of $f$. 

\( d = 1, \ w_t = t. \) Then if \( F'(x) = f(x) \) we have \( T_t^w f(x) = \int_0^t F'(x + s) ds = F(x + t) - F(x) \) and \( T^w : L^\infty \to \text{Lip}: \)

\[
|T_t^w f(x) - T_t^w f(y)| \leq \| f \|_\infty |x - y|, \quad |T_{t,s}^w f(x)| \leq \| f \|_\infty |t - s|
\]

\( \triangleright \) Tao–Wright: if \( w \) “wiggles enough” then \( T_t^w \) maps \( L^q \) into \( L^{q'} \) with \( q' > q \).

\( \triangleright \) Davie: if \( w \) is a sample of BM then a.s. (the exceptional set depends on \( f \))

\[
|T_{t,s}^w f(x) - T_{t,s}^w f(y)| \leq C_w \| f \|_\infty |x - y|^{1-} |t - s|^{1/2-}
\]

**Problem:** study the mapping properties of \( T^w \) with \( w \) sample path of a stochastic process.
Consider

\[ Y_t^w(\xi) = \int_0^t e^{i(\xi, w_s)} ds \]

then \( T_t^w f = \mathcal{F}^{-1}(Y_t^w \mathcal{F}(f)) \). Mapping properties of \( T^w \) in \((H^s)_{s \in \mathbb{R}}\) spaces can be discussed in terms of \( Y^w \):

\[ \|T_{t,s}^w f\|_{H^s} = \left\| (1 + \xi^2)^{s/2} Y_{t,s}^w(\xi) \mathcal{F} f(\xi) \right\|_{H^s}. \]

In our setting more convenient to look at the scale \((\mathcal{F}L^\alpha)_{\alpha}\):

\[ \|f\|_{\mathcal{F}L^\alpha} = \int |f(\xi)|(1 + \xi^2)^{\alpha/2} d\xi \]

since \( \mathcal{F}L^\alpha \subseteq C^\alpha \).

**Definition 1** (Catellier–G.) *We say that \( w \) is \((\rho, \gamma)\)--irregular if there exists a constant \( K \) such that for all \( \xi \in \mathbb{R}^d \) and \( 0 \leq s \leq t \leq 1 \):

\[ |Y_{t,s}^w(\xi)| \leq K(1 + |\xi|)^{-\rho}|t - s|^\gamma. \]
Where we find irregularity?

- In $d = 1$ smooth functions are $(\rho, \gamma)$ irregular for $\rho + \gamma = 1$. In particular if we insist on $\gamma > 1/2$ we have $\rho < 1/2$.

- Not easy to say if a function is irregular.

**Theorem**  The fBM of Hurst index $H$ is $\rho$–irregular for any $\rho < 1/2H$.

⇒ there exists functions of arbitrarily high irregularity and arbitrarily $L^\infty$-near any given continuous function.

**Lemma**  An irregular function cannot be too regular.

**Proof.** If $w \in C^\theta$ with $\alpha \theta + \gamma > 1$ and $\alpha \in [0, 1]$, using the Young integral, we find

$$|t - s| = |e^{ia}(t - s)| = \left| \int_s^t \underbrace{e^{ia - iawr}}_{C^\alpha \theta} \underbrace{drY_r(a)}_{C^\gamma} \right|$$

$$\leq C K_w (|t - s|^\gamma + |t - s|^{\alpha \theta + \gamma} |a|^{\alpha}) \|w\| \theta (1 + |a|)^{-\rho} \to 0$$

if $t > s$ and $\alpha < \rho$. This implies that is not possible that $\theta > (1 - \gamma)/\rho$. 
Facts about irregularity

- For $d > 1$ smooth functions are not irregular: if $|t - s| \ll 1$
  \[
  \int_s^t e^{i\langle a, w_r \rangle} dr \simeq \int_s^t e^{i\langle a, w_s' \rangle(t-s)} dr \simeq (1 + |\langle a, w_s' \rangle|)^{-1} \mathcal{J}(1 + |a|)^{-\rho}.
  \]

- If $w$ is $\rho$–irregular and $\varphi$ is a $C^1$ perturbation then $w + \varphi$ is at least $\rho - (1 - \gamma)$ irregular since:
  \[
  Y_{t,s}^{w+\varphi}(\xi) = \int_s^t e^{i\langle \xi, w_r + \varphi_r \rangle} dr = \int_s^t e^{i\langle \xi, \varphi_r \rangle} dr Y_{s,\varphi_r}(\xi)
  \]
  and we can use Young integral estimates.

- If $W$ is a fBM and $\Phi$ an adapted smooth perturbation then $W + \Phi$ is as irregular as $W$ (via Girsanov theorem).

- Other results (see Catellier thesis): relation with intersection local times, irregularity for $\alpha$-stable Levy processes, relation with local non-determinism.
**Theorem** If \( w \) is \( \rho \)-irregular then

\[
T^w : H^s \to H^{s+\rho}
\]

and

\[
T^w : FL^\alpha \to FL^{\alpha+\rho}.\]

**Proof.** Indeed

\[
\|T^w_{t,s} f\|_{FL^{\alpha+\rho}} = \int d\xi (1 + |\xi|)^{\alpha+\rho} |Y^w_{t,s}(\xi)(\mathcal{F}f)(\xi)|
\]

\[
\leq K_w |t-s|^{\gamma} \int d\xi (1 + |\xi|)^{\alpha}|(\mathcal{F}f)(\xi)| = K_w |t-s|^{\gamma} \|f\|_{FL^\alpha}.
\]

**Remark** More difficult to understand the mapping properties in other spaces, for example Hölder spaces \( C^\alpha \). Only partial results available. Wide open problem.
Consider the transport equation with a perturbation:

$$\partial_t u(t, x) + \dot{w}_t \cdot \nabla u(t, x) + b(x) \cdot \nabla u(t, x) = 0, \quad u(0, \cdot) = u_0.$$ 

In the Lipshitz case there is only one solution $u$ given by the method of characteristics:

$$u(t, x) = u_0(\phi_t^{-1}(x))$$

where $\phi_t(x) = x_t$ is the flow of the ODE

$$\left\{ \begin{array}{l}
\dot{x}_t = b(x_t) + \dot{w}_t \\
x_0 = x
\end{array} \right.$$ 

Uniqueness of solutions is related to the uniqueness (and smoothness) theory of the flow.
In order to exploit the averaging properties of $w$ in the study of the ODE

$$x_t = x_0 + \int_0^t b(x_s)ds + w_t$$

we rewrite it in order to make the action of the averaging operator explicit: let $\theta_t = x_t - w_t$:

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s)ds = \theta_0 + \int_0^t (d_s G_s)(\theta_s)$$

where $G_s(x) = T^w_s b(x)$ so that $d_s G_s(x) = f(w_s + x)$.

If we assume that $G$ is $C^\gamma$ in time ($\gamma > 1/2$) with values in a space of regular enough functions we can study this equation as a Young type equation for $\theta \in C^\gamma$.

**Non-linear Young integral:**

$$\int_0^t (d_s G_s)(\theta_s) = \lim_{\Pi} \sum_i G_{t_{i+1},t_i}(\theta_{t_i})$$

This limit exists if $\theta \in C^\gamma_t$ and $G \in C^\gamma_t C^\nu_x$ with $\gamma(1 + \nu) > 1$. The integral is in $C^\gamma_t$. 
Theorem: The integral equation

\[ \theta_t = \theta_0 + \int_0^t (d_s G_s)(\theta_s) \]

is well defined for \( \theta \in C^\gamma \) and \( G \in C^\gamma_t C^\nu_x,_{loc} \) with \( (1 + \nu) \gamma > 1 \).

- Existence of global solutions if \( G \) of linear growth.
- Uniqueness if \( G \in C^\gamma_t C^\nu_{x,loc} + 1 \) and differentiable flow.
- Smooth flow if \( G \in C^\gamma_t C^\nu_x + k \).

Theorem: The equation

\[ x_t = x_0 + \int_0^t b(x_s)ds + w_t \]

has a unique solution for \( w \) \( \rho \)-irregular and \( b \in \mathcal{FL}^\alpha \) for \( \alpha > 1 - \rho \). In this case we can take \( \theta \in C^1 \) above and the condition for uniqueness (and Lipshitz flow) is \( G \in C^\gamma_t C^3_x \).
Say that \( x \) is controlled by \( w \) if \( \theta = x - w \in C^\gamma \). In this case we have

\[
I_x(b) = \int_0^t b(x_s)ds = \int_0^t (ds T^w b)(\theta_s)
\]

and the r.h.s. is well defined as soon as \( T^w b \in C_t^\gamma C_x^\nu \).

If \( w \) is \( \rho \) irregular and \( b \in FL^\alpha \) then \( T^w b \in C_t^\gamma FL_x^{\alpha+\rho} \) so if \( \alpha + \rho \geq \nu \) we have \( T^w b \in C_t^\gamma C_x^\nu \).

In this case \( I_x(b) \) can be extended by continuity to all \( b \in FL^\alpha \) and in particular we have given a meaning to

\[
\int_0^t b(x_s)ds
\]

when \( b \) is a distribution provided \( x \) is controlled by a \( \rho \)-irregular path.

For controlled paths the ODE

\[
x_t = x_0 + \int_0^t b(x_s)ds + w_t
\]

make sense even for certain distributions \( b \) as a Young equation for \( \theta = x - w \).
We want to give a meaning and study the uniqueness problem for the transport equation

\[(\partial_t + b(x) \cdot \nabla + \dot{w}_t \cdot \nabla)u(t, x) = 0\]

for \(u \in L^\infty\) and \(w \in C^\sigma\) with \(\sigma > 1/3\) such that \((w, \mathbb{W})\) is a geometric \(\sigma\)-Hölder rough path such that \(w\) is \(\rho\)-irregular. For the moment only in the case \(\text{div} \ b = 0\).

**Weak formulation:** We consider \(u\) as a distribution: \(u_t(\varphi) = \int \varphi(x) u(t, x) \, dx\) for all \(\varphi \in L^1(\mathbb{R}^d)\). The integral formulation of the equation is

\[u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b \varphi)) \, dr + \int_s^t u_r(\nabla \varphi) \, d_r w_r\]

for all \(\varphi \in \mathcal{S}(\mathbb{R}^d)\) and \(0 \leq s \leq t\).

We need to give a meaning to such an integral equation in order to discuss the regularization by noise phenomenon. (No way out!)

**Possible via the theory of controlled rough paths (G. 2004).**
Let \((X, \dot{X})\) be a \(\sigma\)-Hölder rough path with \(\sigma > 1/3\):

\[
X_{t,s} = X_{t,u} + X_{u,s} + (X_t - X_u) \otimes (X_u - X_s), \quad |X_t - X_s| + |X_{s,t}|^{1/2} = O(|t - s|^{\sigma})
\]

We say that \(y \in C^\sigma_t\) is **controlled by** \(X\) if there exists \(y^X \in C^\sigma_t\) such that

\[
y_t - y_s - y^X_s (X_t - X_s) =: y^y_{s,t} = O(|t - s|^{2\sigma}).
\]

For a controlled path \(y\) we can define the integral against \(X\) by compensated Riemann sums:

\[
I_t = \int_0^t y_s dX_s := \lim_{\Pi} \sum_i y_{t_i}(X_{t_{i+1}} - X_{t_i}) + y^X_{t_i}X_{t_{i+1}, t_i}
\]

This integral is the only function (up to constants) which has the following property

\[
I_t - I_s = y_s(X_t - X_s) + y^X_s X_{t,s} + O(|t - s|^{3\sigma}).
\]

In particular, the integral is itself controlled by \(X\) and \(I^X = y\).
**Definition** We say that \( u \) is a function controlled by \( w \) if for all \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) we have

\[
 u_t(\varphi) - u_s(\varphi) = u_s^w(\varphi)(w_t - w_s) + u_{t,s}^\#(\varphi)
\]

where \( u_s^w(\varphi) \in C^\sigma \) and \( |u_{t,s}^\#(\varphi)| \lesssim |t - s|^{2\sigma} \).

**Definition** If \( u \) is controlled we say that it is a \( L^\infty \) solution of the rough transport equation (RTE) if

\[
 u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b \varphi))dr + \int_s^t u_r(\nabla \varphi)dr \, w_r
\]

holds for all \( \varphi \in \mathcal{S}(\mathbb{R}^d), \ 0 \leq s \leq t \).

**Remark:** If \( \sigma > 1/2 \) we can just assume that \( u_t(\nabla \varphi) \in C^\sigma_t \) so that the rough integral becomes a Young integral.

Equivalently, \( u \) is a solution to the RTE iff

\[
 u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b \varphi))dr + u_s(\nabla \varphi)(w_t - w_s) + u_s(\nabla^2 \varphi)\mathbb{W}_{t,s} + O(|t - s|^{3\sigma})
\]
Lemma If \( b \) is Lipshitz there exists a solution to the RTE given by \( u(t, x) = u_0(\phi_t^{-1}(x)) \).

Theorem Let \( b \in F^\alpha \) for \( \alpha > 0 \) and \( \alpha + \rho > 3/2 \) and let \( w \) be \( \rho \)-irregular. Then there exists a unique solution to the RTE given by the method of characteristics.

Proof. Approximate \( b \) by \( b_\varepsilon \), then by the previous Lemma there exists a unique solution \( u_\varepsilon \) to the RTE. Analysis of the approximate flow \( \phi_\varepsilon \) shows that this solution converges to a controlled solution \( u \) of the RTE with vectorfield \( b \). Since \( \phi \) is Lipschitz we can prove again uniqueness. \( \square \)

Remark The above result is path-wise. In particular \( b \) can depend on \( w \).

Remark If \( b \in C^\alpha \), \( b \) deterministic and \( w \) is a fBm of Hurst index \( H \) then the uniqueness holds almost surely when \( \alpha > 1 - 1/(2H) \) and \( \alpha > 0 \). This recovers the results of Flandoli–Gubinelli–Priola for the Brownian case but extend them well beyond the Brownian context.
Two simple dispersive models with $\rho$-irregular modulation $w$:

- **Non-linear Schrödinger equation:** $x \in \mathbb{T}, \mathbb{R}, \mathbb{R}^2$, $t \geq 0$

  \[
  \partial_t \phi(t, x) = i \Delta \phi(t, x) \partial_t w_t + i |\phi(t, x)|^{p-2} \phi(t, x).
  \]

- **Korteweg–de Vries equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geq 0$

  \[
  \partial_t u(t, x) = \partial_x^3 u(t, x) \partial_t w_t + \partial_x (u(t, x))^2.
  \]

To be compared to the non-modulated setting where $\partial_t w_t = 1$ and studied in the scale of $(H^s)_s$ spaces.

The equations are understood in the mild formulation

\[
  u(t) = U_t^w u(0) + \int_0^t U_t^w (U_s^w)^{-1} \partial_x (u(s))^2 ds.
\]

with $U_t^w = e^{iw_t \partial_x^3}$. (similarly for NLS). Here $w$ can be an arbitrary continuous function.
Rewrite the mild formulation as \((U_t^w = e^{\partial_x^2 w_t})\)

\[ v(t) = (U_t^w)^{-1} u(t) = u(0) + \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 ds. \]

**Theorem** Let

\[ X_t(\varphi) = X_t(\varphi, \varphi) = \int_0^t (U_s^w)^{-1} \partial_x (U_s^w \varphi)^2 ds \]

If \( w \) is \( \rho \) irregular then \( X \in C^\gamma \text{Lip}_{\text{loc}}(H^\alpha) \) for \( \alpha > -\rho \) and \( \rho > 3/4 \).

For \( v \in C^\gamma H^\alpha \) we can give a meaning to the non–linearity as a Young integral

\[ \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 ds := \int_0^t (d_s X_s)(v(s)) := \lim_{\Pi} \sum_i X_{t_i+1}(v(t_i)) - X_{t_i}(v(t_i)) \]

The continuity of the Young integral implies that if \( v_n \to v \) in \( C^\gamma H^\alpha \) then

\[ \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 ds = \lim_n \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v_n(s))^2 ds \]
**Theorem** The Young equation for \( v \in C^\gamma H^\alpha \):

\[
v(t) = u(0) + \int_0^t (d_s X_s)(v(s))
\]

has local solutions for initial conditions in \( H^\alpha \) with locally Lipshitz flow. Uniqueness in \( C^\gamma H^\alpha \).

- Equivalent “differential” formulation:

\[
v(t) - v(s) = X_{t,s}(v(s)) + O(|t - s|^{2\gamma}), \quad v(0) = u_0
\]

**Regularization by modulation.** In the non-modulated case it is known that there cannot be a continuous flow for \( \alpha \leq -1/2 \) on \( \mathbb{T} \) and \( \alpha \leq -3/4 \) on \( \mathbb{R} \).

- Global solutions thanks to the \( L^2 \) conservation and smoothing for \( \alpha > 0 \) or an adaptation of the I-method for \( -3/2 \leq \alpha < 0 \) and \( \alpha > -\rho/(3 - 2\gamma) \).
- **NLS:** 1d, global solutions for \( \alpha \geq 0 \) and \( \rho > 1/2 \). 2d, local solutions for \( \alpha \geq 1/2 \).
- Global solutions for 1d NLS with \( \alpha > 0 \) come from a smoothing effect of the non-linearity which is due to the irregularity of the driving function.
A different line of attack to the modulated Schrödinger equation comes from the application of the following Strichartz type estimate which can be proved under the same $\rho$-irregularity assumption.

**Theorem** Let $T > 0$, $p \in (2, 5]$, $\rho > \min \left( \frac{3}{2} - \frac{2}{p}, 1 \right)$ then there exists a finite constant $C_{w,T} > 0$ and $\gamma^*(p) > 0$ such that the following inequality holds:

$$\left\| \int_0^T U^w(U_s^w)^{-1} \psi_s \, ds \right\|_{L^p([0,T],L^2\mathbb{R})} \leq C_{w,T} \gamma^*(p) \| \psi \|_{L^1([0,T],L^2\mathbb{R})}$$

for all $\psi \in L^1([0,T],L^2\mathbb{R})$.

In the deterministic case the Strichartz estimate does not have the factor of $T$ in the critical case $p = 5$. This is a sign of a *mild* regularization effect of the noise.

**Remark** Similar path–wise statements (in $w$) holds true for averaging lemmas in kinetic equations with irregular perturbations (similar to the results of Lions–Perthame–Souganidis in the Brownian case).
As an application we obtain global well-posedness for the modulated NLS equation with generic power nonlinearity \( i e: \mathcal{N}(\phi) = |\phi|^\mu \phi \) (Debussche–de Bouard, Debussche–Tsutsumi).

**Theorem** Let \( \mu \in (1, 4], \ p = \mu + 1, \ \rho > \min (1, 3/2 - \frac{2}{p}) \) and \( u^0 \in L^2(\mathbb{R}) \) then there exists \( T^* > 0 \) and a unique \( u \in L^p([0, T], L^{2p}(\mathbb{R})) \) such that the following equality holds:

\[
    u_t = U_t^w u^0 + i \int_0^t U_t^w (U_s^w)^{-1} (|u_s|^\mu u_s) \, ds
\]

for all \( t \in [0, T^*] \). Moreover we have that \( ||u_t||_{L^2(\mathbb{R})} = ||u_0||_{L^2(\mathbb{R})} \) and then we have a global unique solution \( u \in L^p_{lo}(\mathbb{R}) \) and \( u \in C([0, +\infty), L^2(\mathbb{R})) \). If \( u^0 \in H^1(\mathbb{R}) \) then \( u \in C([0, \infty), H^1(\mathbb{R})) \).
Thanks.