Branching processes in random environment.

Juan Carlos Pardo

CIMAT, Mexico
Programme of the short course:

- Galton-Watson (GW) processes
- GW processes in random environment (GWRE).
- Scaling limits of GWRE
- CB processes in random environment.
Galton-Watson processes

We are interested in studying the dynamics of a population that generates individuals of the same kind and independently.
Galton-Watson processes

We are interested in studying the dynamics of a population that generates individuals of the same kind and independently.

Let $Z_0, Z_1, Z_2, \ldots$, be a sequence of r.v.’s such that $Z_n$ represents the number of individuals of a given population presented at generation $n$. For simplicity, we assume that $Z_0 = 1$. 
Galton-Watson processes

We are interested in studying the dynamics of a population that generates individuals of the same kind and independently.

Let $Z_0, Z_1, Z_2, \ldots$, be a sequence of r.v.'s such that $Z_n$ represents the number of individuals of a given population presented at generation $n$. For simplicity, we assume that $Z_0 = 1$.

Moreover, we introduce $\{X_{i,n}; n \geq 0, i \geq 1\}$ an i.i.d. sequence of r.v.'s where each $X_{i,n}$ represents the number of offspring of the $i$-th individual at generation $n$. 
Galton-Watson processes

We are interested in studying the dynamics of a population that generates individuals of the same kind and independently.

Let \( Z_0, Z_1, Z_2, \ldots \) be a sequence of r.v.'s such that \( Z_n \) represents the number of individuals of a given population presented at generation \( n \). For simplicity, we assume that \( Z_0 = 1 \).

Moreover, we introduce \( \{X_{i,n}; n \geq 0, i \geq 1\} \) an i.i.d. sequence of r.v.'s where each \( X_{i,n} \) represents the number of offspring of the \( i \)-th individual at generation \( n \).

In other words, \( Z_{n+1} \) can be written as follows:

\[
Z_{n+1} = \sum_{i=1}^{Z_n} X_{i,n}.
\]
From the later identity, we observe that if $Z_n = 0$, then $Z_{n+m} = 0$ implying that 0 is an absorbing state.
From the later identity, we observe that if $Z_n = 0$, then $Z_{n+m} = 0$ implying that 0 is an absorbing state.

Let us denote by $\mathbb{P}$ for the law of the process. Thus,

$$\mathbb{P}(X_{i,n} = k) = \rho_k \quad \text{and} \quad \sum_{k=0}^{\infty} \rho_k = 1.$$
From the later identity, we observe that if $Z_n = 0$, then $Z_{n+m} = 0$ implying that 0 is an absorbing state.

Let us denote by $\mathbb{P}$ for the law of the process. Thus,

$$\mathbb{P}
(X_{i,n} = k)
= \rho_k
\quad \text{and} \quad
\sum_{k=0}^{\infty} \rho_k = 1.
$$

Moreover, the process $(Z_n, \geq 0)$ is a Markov chain whose transition probabilities are given

$$P_{ij} = \mathbb{P}
(Z_{n+1} = j \mid Z_n = i)
= \mathbb{P}
\left(
\sum_{k=1}^{i} X_{k,n} = j
\right).$$
Now, we introduce the moment-generating function of $Z_n$

$$f_n(s) = \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)s^k \quad \text{for } n \geq 0.$$
Now, we introduce the moment-generating function of $Z_n$

$$f_n(s) = \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k) s^k \quad \text{for } n \geq 0.$$ 

Observe that

$$f_0(s) = s \quad \text{and} \quad f_1(s) = \sum_{k=0}^{\infty} \mathbb{P}(X_{1,0} = k) s^k.$$
Now, we introduce the moment-generating function of $Z_n$

$$f_n(s) = \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)s^k \quad \text{for } n \geq 0.$$ 

Observe that

$$f_0(s) = s \quad \text{and} \quad f_1(s) = \sum_{k=0}^{\infty} \mathbb{P}(X_{1,0} = k)s^k.$$ 

Moreover, a straightforward computation allow us to deduce

$$f_{n+1}(s) = \sum_{k=0}^{\infty} \mathbb{P}(Z_{n+1} = k)s^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}(Z_{n+1} = k|Z_n = j)\mathbb{P}(Z_n = j)s^k$$

$$= \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\mathbb{E}\left[\sum_{i=1}^{j} X_{i,n}\right] = \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\left(\mathbb{E}[s^{X_{1,0}}]\right)^j$$

$$= \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)[f(s)]^j = f_n(f(s)).$$
The latter implies

\[ f_{n+1}(s) = f_{n-k}(f_{k+1}(s)) \quad \text{for} \quad k \in \{0, 1, 2, \ldots, n - 1\}. \]
The latter implies

\[ f_{n+1}(s) = f_{n-k}(f_{k+1}(s)) \quad \text{for} \quad k \in \{0, 1, 2, \ldots, n - 1\}. \]

As a consequence, we can compute the mean and variance of the process at time \( n \),
The latter implies

\[ f_{n+1}(s) = f_{n-k}(f_{k+1}(s)) \quad \text{for} \quad k \in \{0, 1, 2, \ldots, n - 1\}. \]

As a consequence, we can compute the mean and variance of the process at time \( n \), i.e. let \( \mu = \mathbb{E}[X_1] < \infty \) and \( \sigma^2 = \text{Var}[X_1] < \infty \), then

\[ \mathbb{E}[Z_n] = \mu^n \quad \text{and} \quad \text{Var}[Z_n] = \begin{cases} n\sigma^2 & \text{if } \mu = 1, \\ \sigma^2 \mu^{(n-1)} \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1. \end{cases} \]

Observe that accordingly as \( \mu < 1 \), \( \mu = 1 \) or \( \mu > 1 \), the expectation \( \mathbb{E}[Z_n] \) decrease, is a constant or increases. Respectively, we say that the process \( Z \) is subcritical, critical or supercritical.
The previous iteration also help us to compute the extinction probability of the process $Z$. Let us denote

$$\{\text{Ext}\} = \{\text{there is } n : Z_n = 0\},$$

and $\eta = \mathbb{P}(\text{Ext})$.

**Theorem**

*If extinction does not occur, then* $\lim_{n \to \infty} Z_n = +\infty$. *Moreover, we have*

$$\lim_{n \to \infty} \mathbb{P}(Z_n = 0) = \mathbb{P}(\text{Ext}) = \eta,$$

*and* $\eta$ *is the smallest non-negative root of* $s = f_1(s)$. *In particular,* $\eta = 1$ *if* $\mu \leq 1$ *and* $\eta < 1$ *if* $\mu > 1$ *whenever* $\mathbb{P}(Z_1 = 1) < 1$. 
Idea of the proof: Observe $f_n(0) = P(Z_n = 0)$. Since 
$\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$ and therefore

$$
\eta = P \left( \bigcup_{n \geq 1} \{Z_n = 0\} \right) = \lim_{n \to \infty} P(Z_n = 0) = \lim_{n \to \infty} f_n(0).
$$
Idea of the proof: Observe $f_n(0) = \mathbb{P}(Z_n = 0)$. Since
\[ \{Z_n = 0\} \subseteq \{Z_{n+1} = 0\} \] and therefore
\[ \eta = \mathbb{P}\left( \bigcup_{n \geq 1} \{Z_n = 0\}\right) = \lim_{n \to \infty} \mathbb{P}(Z_n = 0) = \lim_{n \to \infty} f_n(0). \]

Since $f_n(0) = f( f_{n-1}(0))$, and $\lim_{n \to \infty} f_n(0) = \eta$, from the dominated convergence Theorem, we deduce
\[ \eta = \lim_{n \to \infty} f( f_{n-1}(0)) = \lim_{n \to \infty} \mathbb{E}\left[ (f_{n-1}(0))^{Z_1}\right] = \mathbb{E}\left[ \eta^{Z_1}\right] = f(\eta). \]
Idea of the proof: Observe $f_n(0) = \mathbb{P}(Z_n = 0)$. Since \( \{Z_n = 0\} \subseteq \{Z_{n+1} = 0\} \) and therefore

\[
\eta = \mathbb{P}\left(\bigcup_{n \geq 1} \{Z_n = 0\}\right) = \lim_{n \to \infty} \mathbb{P}(Z_n = 0) = \lim_{n \to \infty} f_n(0).
\]

Since $f_n(0) = f(f_{n-1}(0))$, and $\lim_{n \to \infty} f_n(0) = \eta$, from the dominated convergence Theorem, we deduce

\[
\eta = \lim_{n \to \infty} f(f_{n-1}(0)) = \lim_{n \to \infty} \mathbb{E}\left[(f_{n-1}(0))^{Z_1}\right] = \mathbb{E}[\eta^{Z_1}] = f(\eta).
\]

The rest of the proof follows from the shape of $f_1(s)$. 
Assume that $\mu > 1$ and let us introduce

$$W_n = \frac{Z_n}{\mu^n}, \quad n \geq 0,$$

which is well defined as long as $\mu < \infty$. 
Assume that $\mu > 1$ and let us introduce

$$W_n = \frac{Z_n}{\mu^n}, \quad n \geq 0,$$

which is well defined as long as $\mu < \infty$.

It is known that $W_n$ is a non-negative martingale, therefore

$$W_n \to W, \quad \text{a.s.},$$

for some non-negative r.v. $W$. 

Theorem

Assume that $1 < m < \infty$. Then,

$$\mathbb{E}[W] = 1 \iff P(W > 0 | \text{non-extinction}) = 1 \iff \mathbb{E}[Z_1 \log[Z_1]] < \infty.$$
Assume that $\mu > 1$ and let us introduce

$$W_n = \frac{Z_n}{\mu^n}, \quad n \geq 0,$$

which is well defined as long as $\mu < \infty$.

It is known that $W_n$ is a non-negative martingale, therefore

$$W_n \to W, \quad \text{a.s.,}$$

for some non-negative r.v. $W$.

**Theorem**

Assume that $1 < m < \infty$. Then,

$$\mathbb{E}[W] = 1 \iff \mathbb{P}(W > 0|\text{non-extinction}) = 1 \iff \mathbb{E}[Z_1 \log^+ Z_1] < \infty.$$
GW processes in random environment

Let $\Delta$ be the Polish space of probability measures on $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ equipped with the metric of the total variation.
GW processes in random environment

Let $\Delta$ be the Polish space of probability measures on $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ equipped with the metric of the total variation.

Let $e = (Q_i)_{i \geq 0}$ be a sequences of i.i.d. r.v.’s taking values in $\Delta$. 
GW processes in random environment

Let $\Delta$ be the Polish space of probability measures on $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ equipped with the metric of the total variation.

Let $e = (Q_i)_{i \geq 0}$ be a sequences of i.i.d. r.v.’s taking values in $\Delta$. Conditioned on $e$ the GWRE $(Z_i)_{i \geq 0}$ is defined as

$$Z_{i+1} = \sum_{j=1}^{Z_i} X_{j,i}, \quad i \geq 0,$$

where $Z_0$ is independent of $e$ and where $\{X_{j,i}; j, i \geq 0\}$ conditioned on $e$ are i.i.d. with common distribution

$$\mathbb{P}(X_{j,i} = k \mid e) = Q_i(k), \quad j, i, k \geq 0.$$
Let

$$m_i = \sum_{k \geq 0} k Q_i(k).$$
Let

\[ m_i = \sum_{k \geq 0} kQ_i(k). \]

We introduce the following associated random walk. Let \( S_0 = 0 \) and

\[ S_{k+1} = S_k + \log m_k, \quad k \geq 0. \]
Let
\[ m_i = \sum_{k \geq 0} k Q_i(k). \]

We introduce the following associated random walk. Let \( S_0 = 0 \) and
\[ S_{k+1} = S_k + \log m_k, \quad k \geq 0. \]

We assume that \( \log m_0 \) is a.s. finite and see that the conditional expectation of \( Z_i \), given the environment \( e \), satisfies
\[ \mu_i := \mathbb{E}\left[ Z_i \mid Z_0, e \right] = Z_0 \prod_{k=0}^{i} m_k = Z_0 e^{S_i}. \]
Recall from fluctuation theory of random walks that there are three different regimes for the behaviour of $S$.

i) If the random walk $S$ drift to $+\infty$, we have $\mu_i \to \infty$ a.s., provided $Z_0 \geq 1$, and we call $Z$ supercritical GWRE.

ii) If the random walk $S$ drift to $-\infty$, we have $\mu_i \to 0$ a.s., and we call $Z$ subcritical GWRE.

iii) If the random walk $S$ oscillates, i.e. $\lim \sup_{k \to \infty} S_k = \infty$ and $\lim \inf_{k \to \infty} S_k = -\infty$, a.s., we have $\lim \sup_{k \to \infty} \mu_k = \infty$ and $\lim \inf_{k \to \infty} \mu_k = 0$, a.s., and we call $Z$ critical GWRE.
Recall from fluctuation theory of random walks that there are three different regimes for the behaviour of $S$.

\begin{itemize}
\item[i)] If the random walk $S$ drift to $+\infty$, we have $\mu_i \to \infty$ a.s., provided $Z_0 \geq 1$, and we call $Z$ supercritical GWRE.
\end{itemize}
Recall from fluctuation theory of random walks that there are three different regimes for the behaviour of $S$.

i) If the random walk $S$ drift to $+\infty$, we have $\mu_i \to \infty$ a.s., provided $Z_0 \geq 1$, and we call $Z$ supercritical GWRE.

ii) If the random walk $S$ drift to $-\infty$, we have $\mu_i \to 0$ a.s., and we call $Z$ subcritical GWRE.
Recall from fluctuation theory of random walks that there are three different regimes for the behaviour of $S$.

i) If the random walk $S$ drift to $+\infty$, we have $\mu_i \to \infty$ a.s., provided $Z_0 \geq 1$, and we call $Z$ supercritical GWRE.

ii) If the random walk $S$ drift to $-\infty$, we have $\mu_i \to 0$ a.s., and we call $Z$ subcritical GWRE.

iii) If the random walk $S$ oscillates, i.e.

$$\limsup_{k \to \infty} S_k = \infty \quad \text{and} \quad \liminf_{k \to \infty} S_k = -\infty, \quad \text{a.s.}$$

we have

$$\limsup_{k \to \infty} \mu_k = \infty \quad \text{and} \quad \liminf_{k \to \infty} \mu_k = 0, \quad \text{a.s.}$$

a.s., and we call $Z$ critical GWRE.
Observe that the estimate

\[ \mathbb{P}(Z_i > 0 \mid Z_0, e) = \min_{0 \leq k \leq i} \mathbb{P}(Z_k > 0 \mid Z_0, e) \]

\[ \leq \min_{0 \leq k \leq i} \mathbb{E}[Z_k \mid Z_0, e] \]

\[ = Z_0 e^{\min_{0 \leq k \leq i} S_k}, \]

implies that \( \mathbb{P}(Z_i > 0 \mid Z_0, e) \) goes to 0 in the critical and subcritical cases, and consequently

\[ \mathbb{P}(Z_i > 0) \to 0, \quad \text{as} \quad i \to \infty. \]
Observe that the estimate

\[ \mathbb{P}(Z_i > 0|Z_0, e) = \min_{0 \leq k \leq i} \mathbb{P}(Z_k > 0|Z_0, e) \]

\[ \leq \min_{0 \leq k \leq i} \mathbb{E}[Z_k|Z_0, e] \]

\[ = Z_0 e^{\min_{0 \leq k \leq i} S_k}, \]

implies that \( \mathbb{P}(Z_i > 0|Z_0, e) \) goes to 0 in the critical and subcritical cases, and consequently

\[ \mathbb{P}(Z_i > 0) \to 0, \quad \text{as} \quad i \to \infty. \]

As was observed by Afanasyev (80), and later independently by Dekking (88), there are three possibilities for the asymptotic behavior of subcritical branching processes. These regimes are called as weakly subcritical, intermediately subcritical and strongly subcritical.
Theorem (Kozlov, 76)

For a critical GWRE whose random environment satisfies some moment conditions, then there are some constants $0 < c_1 \leq c_2 < \infty$ such that $n \geq 1$

$$c_1 n^{-1/2} \leq \mathbb{P}(Z_n > 0) \leq c_2 n^{-1/2}.$$
Theorem (Kozlov, 76)

For a critical GWRE whose random environment satisfies some moment conditions, then there are some constants $0 < c_1 \leq c_2 < \infty$ such that $n \geq 1$

$$c_1 n^{-1/2} \leq \mathbb{P}(Z_n > 0) \leq c_2 n^{-1/2}.$$

Theorem (Afanasyev et al., 05)

For a critical GWRE whose random environment satisfy some moment conditions and that there exist $0 < \rho < 1$ such that

$$\frac{1}{n} \sum_{m=1}^{n} \mathbb{P}(S_m > 0) \to \rho, \quad \text{as} \quad n \to \infty.$$

Then there is a positive constant $\theta$ such that

$$\mathbb{P}(Z_n > 0) \sim \theta n^{-(1-\rho)} l(n), \quad \text{as} \quad n \to \infty,$$

where $(l(n))_{n \geq 1}$ is a slowly varying sequence at infinity.
Theorem (Guivarc’h and Liu, 01)

For a subcritical GWRE, we have

i) If $\mathbb{E}[m_0 \log m_0] < 0$ and $\mathbb{E}[Z_1 \log^+ Z_1] < \infty$, then for some constant $c \in (0, \infty)$

$$\mathbb{P}(Z_n > 0) \sim c(\mathbb{E}[Z_1])^n,$$

as $n \to \infty$,

ii) If $\mathbb{E}[m_0 \log m_0] = 0$, $\mathbb{P}(m_0 = 1) < 1$ and $\mathbb{E}[m_0^2] < \infty$, then for some constant $0 < c_1 \leq c_2 < \infty$

$$c_1 n^{-1/2}(\mathbb{E}[Z_1])^n \leq \mathbb{P}(Z_n > 0) \leq c_2 n^{-1/2}(\mathbb{E}[Z_1])^n$$

as $n \to \infty$,

iii) If $\mathbb{E}[m_0 \log m_0] > 0$ and $\mathbb{E}[m_0^2] < \infty$, then for some constant $0 < c_3 \leq c_4 < \infty$

$$c_3 n^{-3/2} \rho^n \leq \mathbb{P}(Z_n > 0) \leq c_4 n^{-3/2} \rho^n$$

as $n \to \infty$,

where $\rho = \inf_{0 \leq t \leq 1} \mathbb{E}[m_0^t]$
Theorem (Guivarc’h and Liu, 01)

Let $p > 1$, fixed and $(Z_n, n \geq 0)$ be a supercritical GWRE. Then, the following assertions are equivalent:

i) $0 < \mathbb{E}[W^p] < \infty$

ii) $\mathbb{E}[m_0^{-(p-1)}] < 1$ and $\mathbb{E}[(Z_1/m_0)^p] < \infty$.

iii)

$$W_n = \frac{Z_n}{\prod_{i=0}^{n} m_i} \to W \quad \text{in } L^p.$$
Now fix $n \geq 1$ and let $e^{(n)} = (Q_{i}^{(n)})_{i \geq 1}$ be a sequence of i.i.d. r.v.'s.
Now fix $n \geq 1$ and let $e^{(n)} = (Q_i^{(n)})_{i \geq 1}$ be a sequence of i.i.d. r.v.’s. Conditioned on $e^{(n)}$ we introduce a GWRE $(Z_i^{(n)})_{i \geq 0}$ satisfying

$$Z_i^{(n)} = \sum_{j=1}^{Z_{i+1}^{(n)}} X_{j,i}, \quad i \geq 0,$$

where $Z_0^{(n)}$ is independent of $e^{(n)}$ and where $(X_{j,i}^{(n)})_{j,i \geq 0}$ conditioned on $e^{(n)}$ are i.i.d. with common distribution

$$\mathbb{P}(X_{j,i}^{(n)} = k \mid e^{(n)}) = Q_i^{(n)}(k), \quad j, i, k \geq 0.$$
Now fix $n \geq 1$ and let $e^{(n)} = (Q_i^{(n)})_{i \geq 1}$ be a sequence of i.i.d. r.v.'s. Conditioned on $e^{(n)}$ we introduce a GWRE $(Z_i^{(n)})_{i \geq 0}$ satisfying

$$Z_i^{(n)} = \sum_{j=1}^{Z_i^{(n)}} X_{j,i}, \quad i \geq 0,$$

where $Z_0^{(n)}$ is independent of $e^{(n)}$ and where $(X_{j,i}^{(n)})_{j,i \geq 0}$ conditioned on $e^{(n)}$ are i.i.d. with common distribution

$$\mathbb{P}(X_{j,i}^{(n)} = k \mid e^{(n)}) = Q_i^{(n)}(k), \quad j, i, k \geq 0.$$

Again, let

$$m_i^{(n)} = \sum_{k \geq 0} kQ_i^{(n)}(k).$$
Let $Z_t^{(n)} = Z_{[t]}^{(n)}$ be a continuous time version of the GWRE and $(S_t^{(n)})_{t \geq 0}$ be its associated random walk which is defined by

$$S_t^{(n)} = \sqrt{n} \sum_{i=1}^{[t]-1} \log \left( m_i^{(n)} \right), \quad t \geq 0.$$
Let $Z^{(n)}_t = Z^{(n)}_{[t]}$ be a continuous time version of the GWRE and $(S^{(n)}_t)_{t \geq 0}$ be its associated random walk which is defined by

$$S^{(n)}_t = \sqrt{n} \sum_{i=i}^{[t]-1} \log \left( m^{(n)}_i \right), \quad t \geq 0.$$ 

If $Z^{(n)}_0 / n \to Z_0$ in law. Hence under the following assumptions
Let $Z_t^{(n)} = Z_{[t]}^{(n)}$ be a continuous time version of the GWRE and $(S_t^{(n)})_{t \geq 0}$ be its associated random walk which is defined by

$$S_t^{(n)} = \sqrt{n} \sum_{i=i}^{[t]-1} \log \left( m_i^{(n)} \right), \quad t \geq 0.$$ 

If $Z_0^{(n)}/n \to Z_0$ in law. Hence under the following assumptions

i) \[
\lim_{n \to \infty} n \mathbb{E}[(m_1^{(n)} - 1)] = \beta \in \mathbb{R}.
\]
Let \( Z^{(n)}_t = Z^{(n)}_{[t]} \) be a continuous time version of the GWRE and \((S^{(n)}_t)_{t \geq 0}\) be its associated random walk which is defined by

\[
S^{(n)}_t = \sqrt{n} \sum_{i=i}^{[t]-1} \log \left( m^{(n)}_i \right), \quad t \geq 0.
\]

If \( Z^{(n)}_0/n \to Z_0 \) in law. Hence under the following assumptions

i) \[
\lim_{n \to \infty} n \mathbb{E}[(m^{(n)}_1 - 1)] = \beta \in \mathbb{R}.
\]

ii) \[
\lim_{n \to \infty} n \mathbb{E}[(m^{(n)}_1 - 1)^2] = \sigma \geq 0.
\]
Let $Z^{(n)}_t = Z^{(n)}_{[t]}$ be a continuous time version of the GWRE and $(S^{(n)}_t)_{t \geq 0}$ be its associated random walk which is defined by

$$S^{(n)}_t = \sqrt{n} \sum_{i=1}^{[t]-1} \log (m^{(n)}_i), \quad t \geq 0.$$ 

If $Z^{(n)}_0/n \to Z_0$ in law. Hence under the following assumptions

i) $\lim_{n \to \infty} n\mathbb{E}[ (m^{(n)}_1 - 1)] = \beta \in \mathbb{R}.$

ii) $\lim_{n \to \infty} n\mathbb{E}[ (m^{(n)}_1 - 1)^2] = \sigma \geq 0.$

iii) $\sup_{n \geq 0} \mathbb{E} \left[ \sum_{k \geq 0} \left| \frac{k}{m^{(n)}_1} - 1 \right|^2 Q^{(n)}_{1}(k) \right] < \infty.$
\[ \lim_{n \to \infty} \mathbb{E} \left[ \sum_{k \geq 0} \left| \frac{k}{m_1^{(n)}} - 1 \right|^2 Q_1^{(n)}(k) \right] = \gamma, \]
iv)  

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sum_{k \geq 0} \left| \frac{k}{m_1(n)} - 1 \right|^{2} Q_1^{(n)}(k) \right] = \gamma,
\]

we have (Kurtz, 78)

\[
\left( \frac{Z_{tn}}{\sqrt{n}}, \frac{S_{tn}}{\sqrt{n}} \right)_{t \geq 0} \to (Z_t, S_t)_{t \geq 0}
\]

in the sense of Skorokhod.
iv)\[
\lim_{n \to \infty} \mathbb{E} \left[ \sum_{k \geq 0} \left| \frac{k}{m_1^{(n)}} - 1 \right|^2 Q_1^{(n)}(k) \right] = \gamma,
\]

we have (Kurtz, 78)

\[
\left( \frac{Z_{tn}^{(n)}}{n}, \frac{S_{tn}^{(n)}}{\sqrt{n}} \right)_{t \geq 0} \to (Z_t, S_t)_{t \geq 0}
\]
in the sense of Skorokhod. Moreover

\[
Z_t = Z_0 + \frac{\sigma^2}{2} \int_0^t Z_s \, ds + \int_0^t \sqrt{2\gamma^2 Z_s} \, dB_s + \int_0^t Z_s \, dS_s,
\]

where \( S_t = \beta t + \sigma W_t, t \geq 0, \beta \in \mathbb{R} \) and \( B \) and \( W \) are two independent standard Brownian motions.
Bibliography

- Afanasyev, V. I. On the survival probability of a subcritical branching process in a random environment. Dep. VINITI (1979), No. M1794–79


Bibliography
