Lines of descent in a deterministic model with mutation and frequency dependent selection

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*joint work (in progress) with E. Baake and S. Hummel

Structure of the talk

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- In the sequel, we focus on the case $\beta > 0, \gamma \leq 0$.

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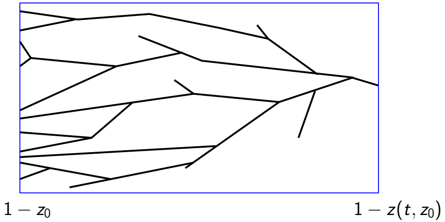

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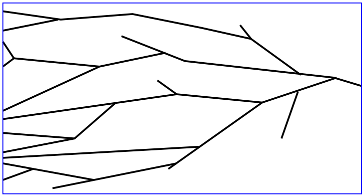

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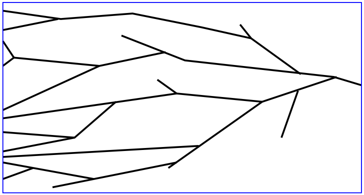
- Can we express this probability using an ancestral process (duality!)?



$1 - z_0$ $\qquad$ $1 - z(t, z_0)$

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- Each individual mutates to type i at rate u_i, $i \in \{0, 1\}$.

The graphical representation ($\beta > 0, \gamma < 0$)

→ = Neutral reproduction, ⇾ = Non-interactive selective reproduction,

□–•–▷ = Interactive selective reproduction,

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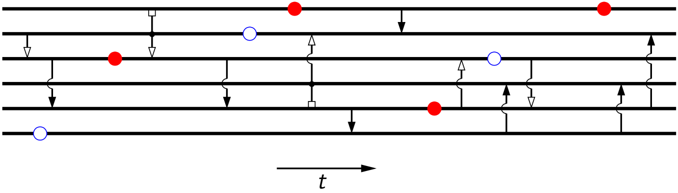

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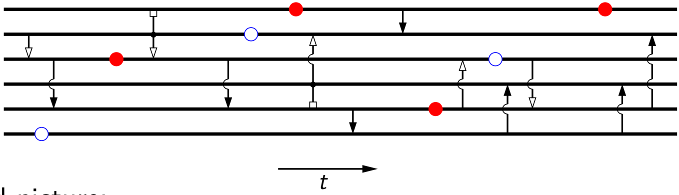

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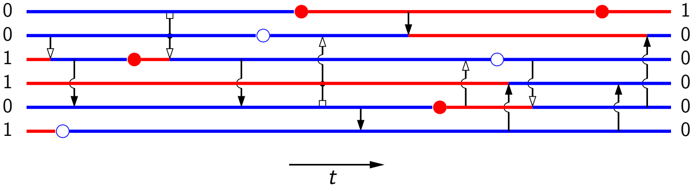


Dynamical law of large numbers

For $t \geq 0$, we denote by X_t^N the number of $0-$individuals at time t in the Moran model of size N.

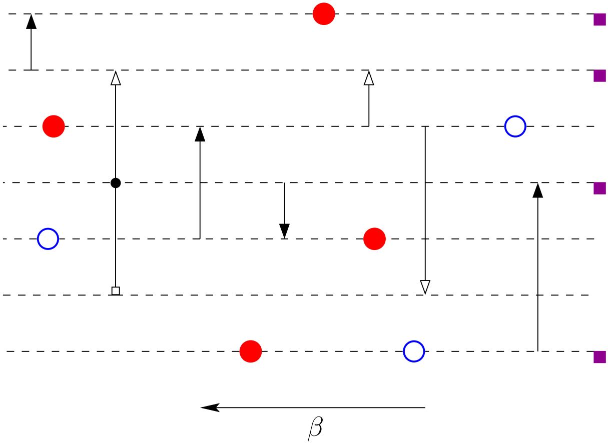
Proposition (Law of large numbers)

For $z_0 \in [0,1]$, let $z(z_0,\cdot)$ be the solution of (1) *with $z(0,z_0) = z_0$. Assume that $\lim_{N\to\infty} \frac{X_0^N}{N} = z_0 \in [0,1]$. Then, for all $\varepsilon > 0$, we have*

$$\lim_{N\to\infty} P\left(\sup_{t\leq T}\left|\frac{X_t^N}{N} - z(t,z_0)\right| > \varepsilon\right) = 0,$$

i.e. X^N/N converges to $z(\cdot,z_0)$ uniformly in compacts in probability.

The ASG (N large)



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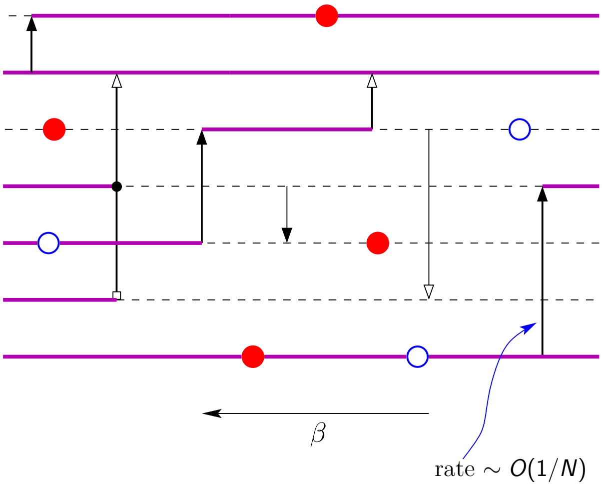

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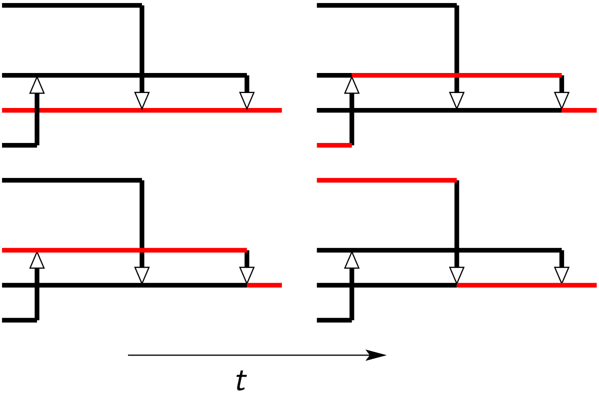

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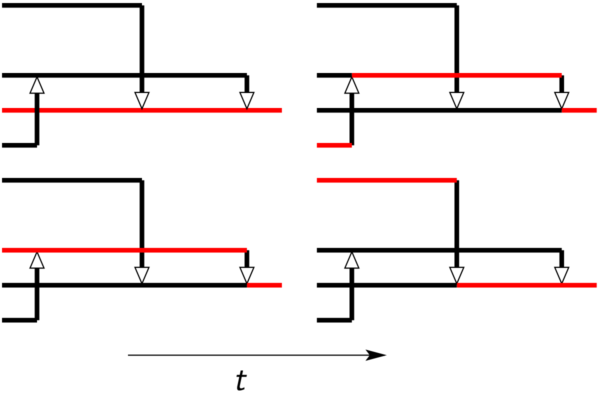
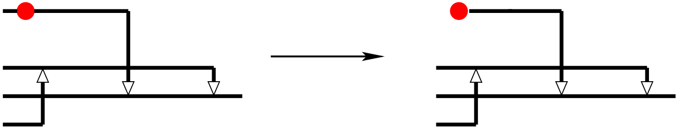
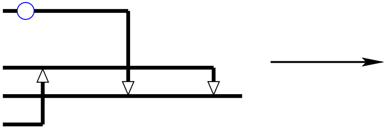


The probability of sampling an individual of type 0 at (forward) time t depends only on the number of lines in the ASG at time 0.

The killed ASG ($\gamma = 0$)

- With mutations:





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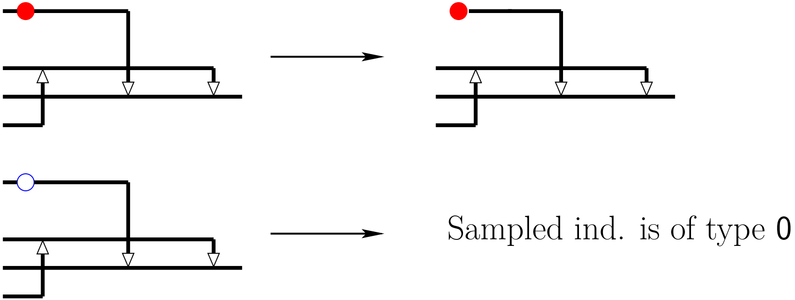

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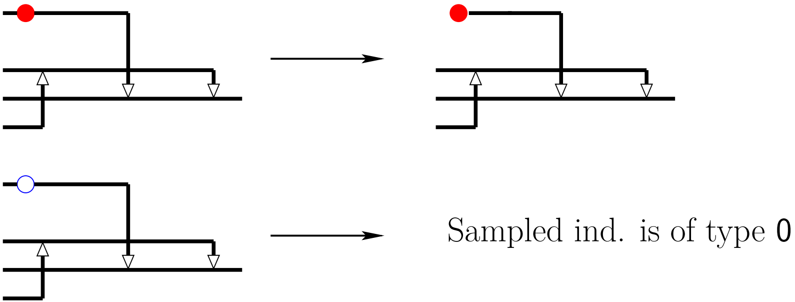

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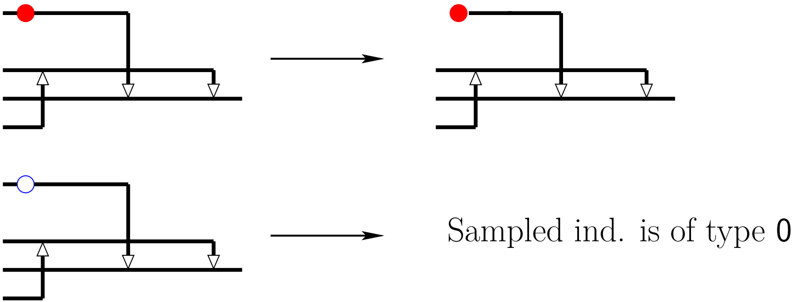

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Killed ASG (E. Baake, U. Lenz, A. Wakolbinger)

Duality

Let $R = (R_t)_{\geq 0}$ be the line-counting process in the killed-ASG, i.e. the continuous time Markov chain with rates

$$q_R(k,j) := \begin{cases} ks & \text{if } j = k+1, \\ ku_1 & \text{if } j = k-1, \\ ku_0 & \text{if } j = \infty. \end{cases}$$

Theorem (Duality)

For $t \geq 0$ and $z_0 \in (0,1)$, we have

$$1 - z(t, z_0) := E_1\left[(1-z_0)^{R_t}\right].$$

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$$y_* := \begin{cases} \frac{u+s-\sqrt{(u+s)^2 - 4u_1 s}}{2s} & \text{if } s > 0, \\ \frac{u_1}{u} & \text{if } s = 0. \end{cases}$$

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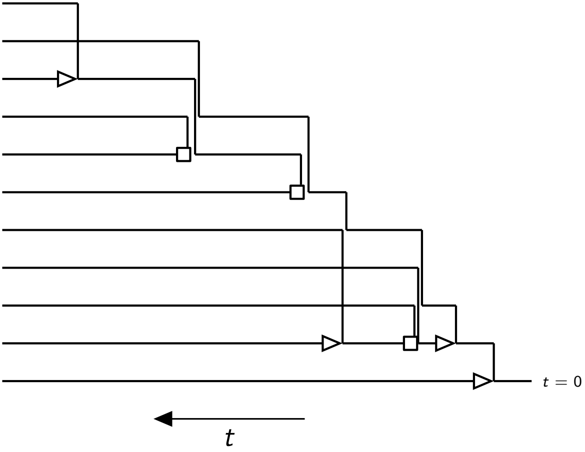
- Duality $\Rightarrow y_*$ is also the proportion of 1's at stationarity.

The Asymptotic ASG with interactions and no mutations

The asymptotic ASG consists of bifurcations and trifurcations.



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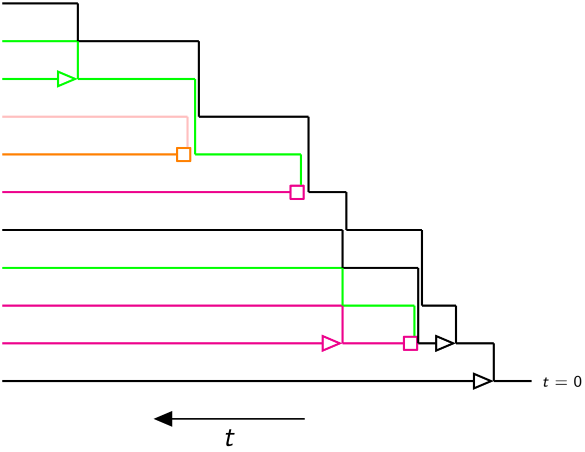
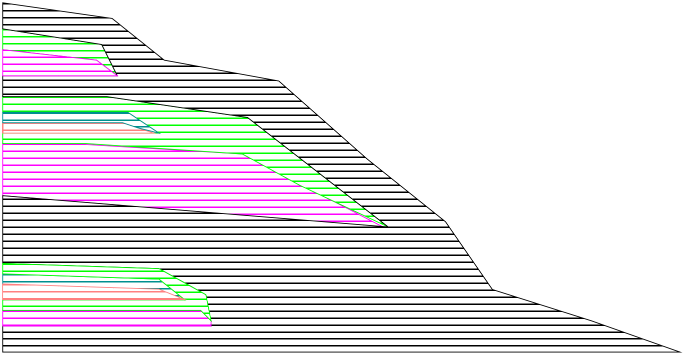


Ancestral selection graph



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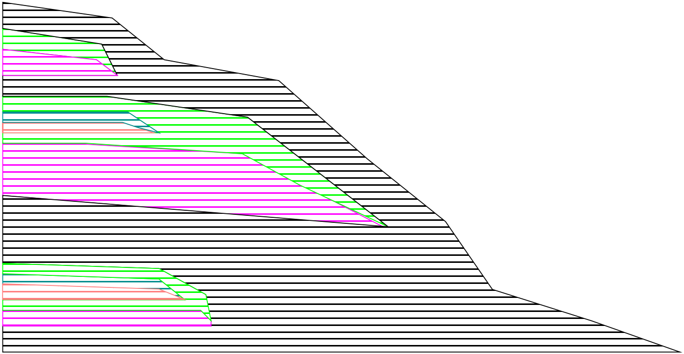

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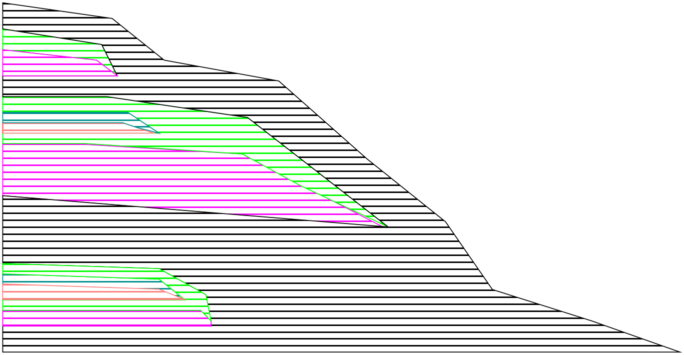

How to encode this information in order to determine the probability of sampling an individual of type 1? Ternary trees with marked leaves!

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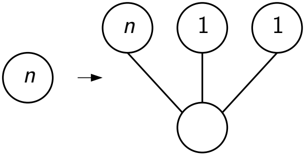

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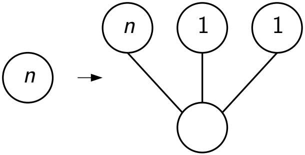

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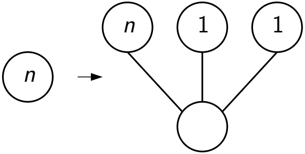

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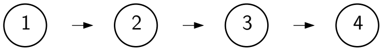

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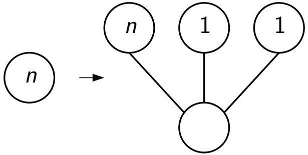

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  - Right-child $\equiv$ the incoming line.

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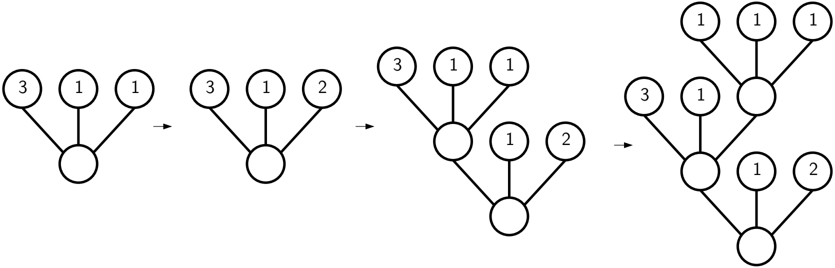

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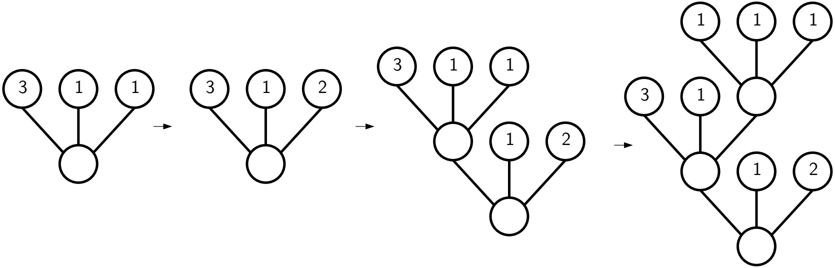

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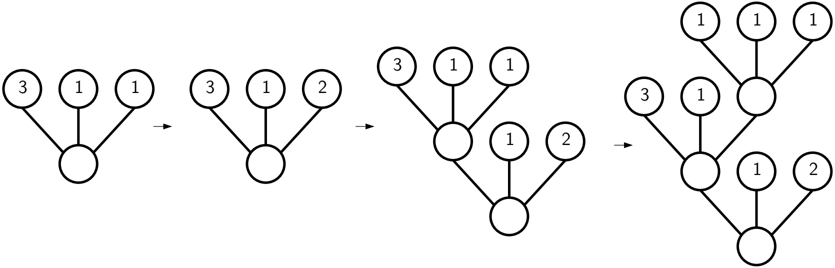

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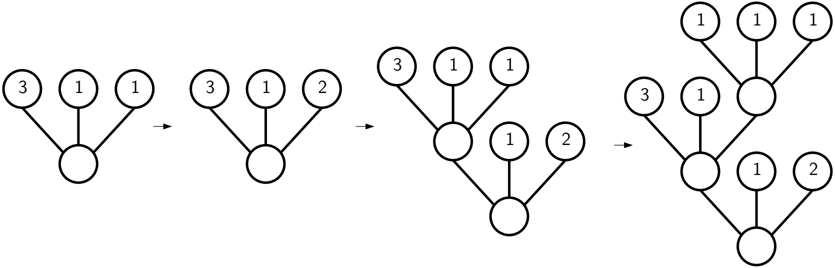

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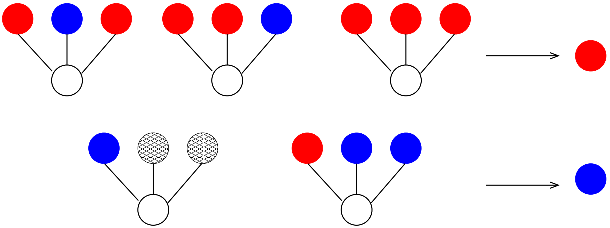

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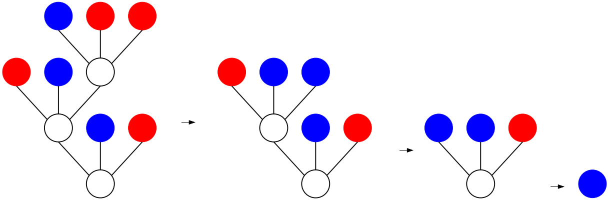

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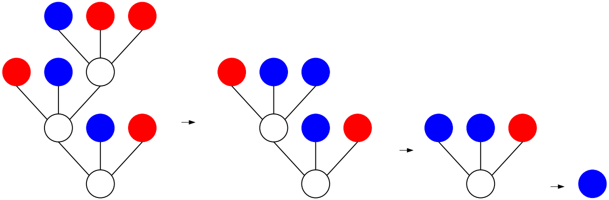

    3. The sampled ind. is of type 1 iff the root is colored red.

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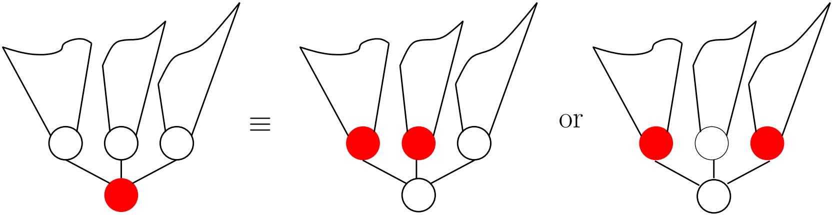
- If v_1, v_2, v_3 are the left, middle and right children of the root:

$$H(\mathcal{T}, y) = H(\mathcal{T}_{v_1}, y)\left[H(\mathcal{T}_{v_2}, y) + H(\mathcal{T}_{v_3}, y) - H(\mathcal{T}_{v_2}, y)H(\mathcal{T}_{v_3}, y)\right].$$



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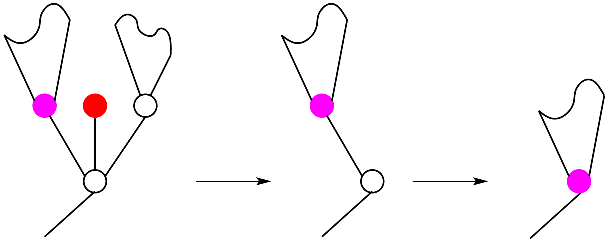

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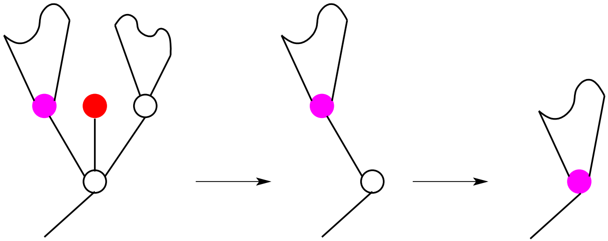

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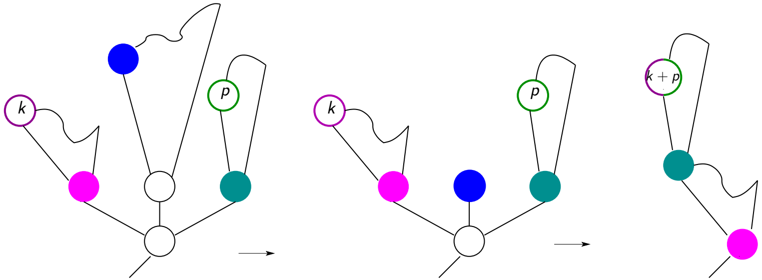

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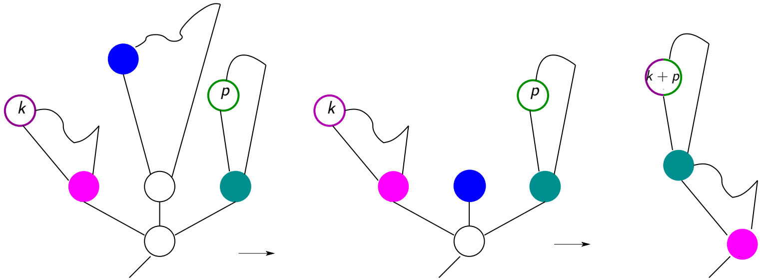

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The line-counting tree

Definition

The line-counting tree process $\mathcal{T}^* := (\mathcal{T}^*(t))_{t \geq 0}$ is the jump process with values on $\Upsilon \cup \{\bowtie\}$ starting at $\textcircled{1}$ and with the following transitions. If $\mathcal{T}^*$ is currently in state $\mathcal{T} := (\tau, m)$: for each $\ell \in L_\tau$,

1. $\mathcal{T} \longrightarrow \mathcal{T}^{\curlyvee,\ell}$ at rate $\alpha\beta\, m(\ell)$.
2. $\mathcal{T} \longrightarrow \mathcal{T}^{\Psi,\ell}$ at rate $\alpha|\gamma|\, m(\ell)$.
3. $\mathcal{T} \longrightarrow \mathcal{T}^{\bullet,\ell}$ at rate $u_1\, m(\ell)$.
4. $\mathcal{T} \longrightarrow \mathcal{T}^{\circ,\ell}$ at rate $u_0\, m(\ell)$.

The states $\textcircled{0}$ and $\bowtie$ are absorbing states of $\mathcal{T}^*$.

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We extend the definition of H to the cemetery point by setting $H(\bowtie, y) := 0$.

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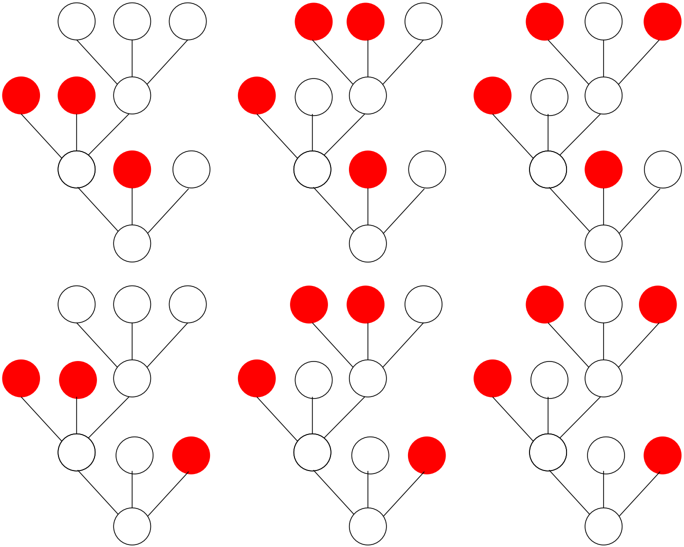
The result follows from classical duality results. □

Minimal red-colorings



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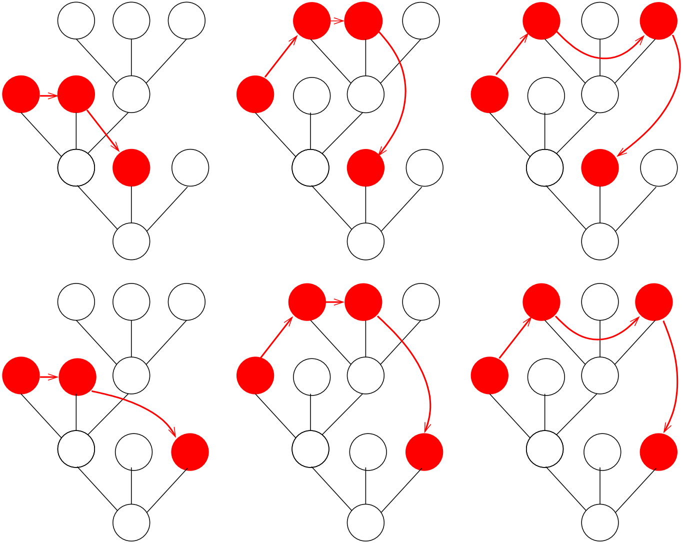
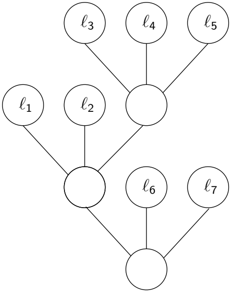


From ternary trees to directed graphs (digraphs)



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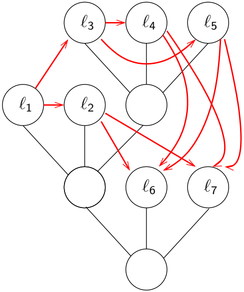

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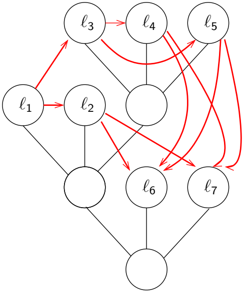

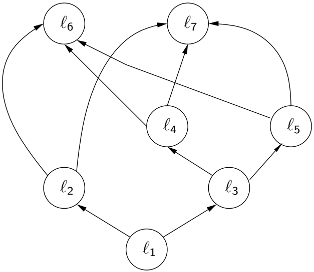

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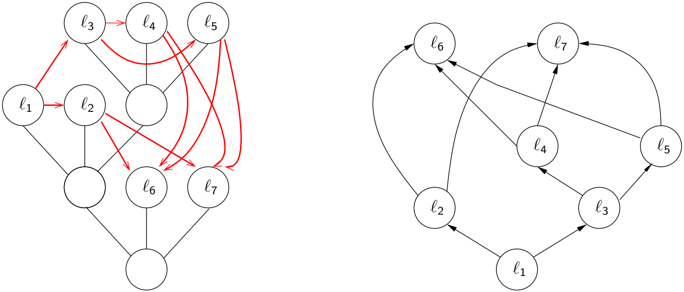

- Most left-leave $\mapsto$ Source.

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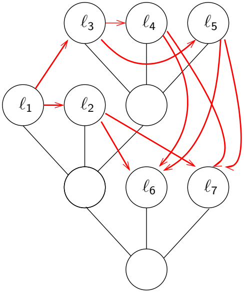

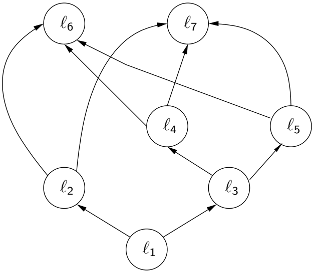

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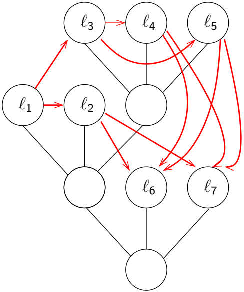

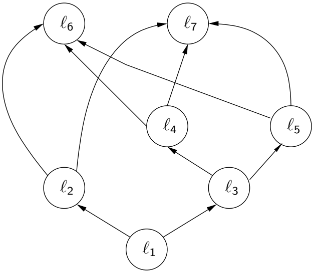

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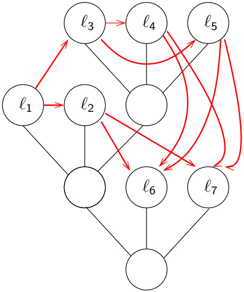

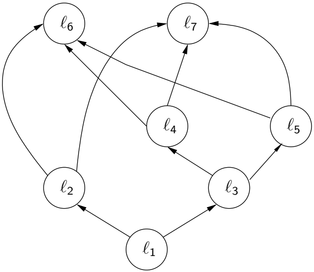

- Most left-leave $\mapsto$ Source.
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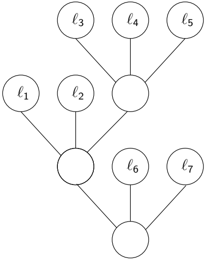

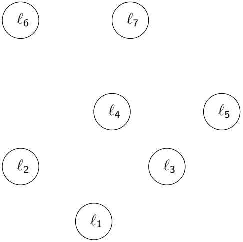

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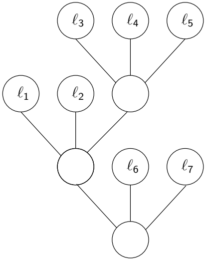

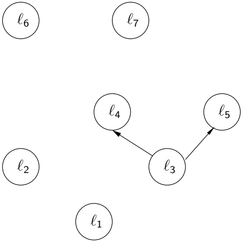

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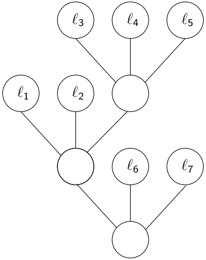

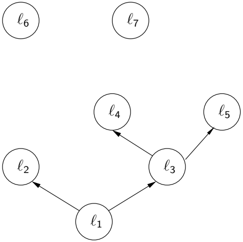

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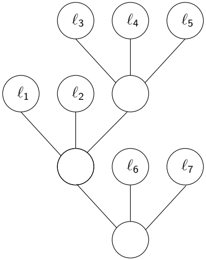

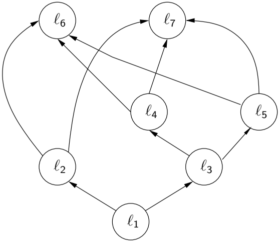

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Proposition

For each $\mathcal{T} \in \Upsilon$ and $y \in [0, 1]$, we have

$$H(\mathcal{T}, y) = \mathcal{H}(\rho(\mathcal{T}), y).$$

The line-counting digraph process

Definition

The line-counting digraph process is the continuous time Markov chain $\mathcal{G} := (\mathcal{G}(t))_{t \geq 0}$ with values in $\Sigma_* \cup \{\bowtie\}$, defined by $\mathcal{G}(t) = \rho(\mathcal{T}^*(t))$, $t \geq 0$.

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  $+(k-1)(\alpha(\beta + 2\gamma))a_{k-1}(t) - (k-2)\alpha\gamma a_{k-2}(t), \quad k \geq 3.$

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- One can show that

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- Set $h_0 = 1$, $h_\infty = 0$ and

$$h_k := P\left(T_{\circledcirc} < \infty | \mathcal{T}^*(0) = \textcircled{k}\right), \quad k \geq 1.$$

- A first step analysis leads to

$$\begin{aligned}(u + \alpha(\beta - \gamma))h_k &= \alpha\beta h_{k+1} + u\nu_1 h_{k-1} \\ &\quad - \alpha\gamma P\left(T_{\circledcirc} < \infty | \mathcal{T}^*(0) = \textcircled{k} \otimes \textcircled{1} \otimes \textcircled{1}\right).\end{aligned}$$

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- Duality $\Rightarrow$ h_1 = stationary proportion of $1'$s in the forward model.

Thank you for your attention!