Furstenberg’s conjecture on intersections of Cantor sets, and self-similar measures

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Base $p$ expansions

Let $p \in \mathbb{N}_{\geq 2}$. Every point $x$ has an expansion to base $p$:

$$x = 0.x_1 x_2 \ldots = \sum_{n=1}^{\infty} x_n p^{-n}, \quad x_i \in \{0, 1, \ldots, p - 1\}. $$

Basic facts:

1. All but countably many (rational) points have a unique expansion; the remaining ones have two expansions.
2. A point is rational if and only if the expansion is eventually periodic.
3. Expansions in bases $p^n$ and $p^k$ are “almost the same” (look at base $p$ in blocks of length $n$ and $k$).
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Multiplication by \( p \)

**Definition**

For \( p \in \mathbb{N}_{\geq 2} \), let

\[
T_p = px \mod 1
\]

be multiplication by \( p \) on the circle.

Symbolically, \( T_p x \) corresponds to shifting the \( p \)-ary expansion \( x \): there is a factor map, which is one-to-one outside of the countably many points with two \( p \)-ary expansions.
Multiplying by 2 and by 3: the founding father
Some of the areas that Furstenberg initiated

1. Ergodic theoretic methods in combinatorics (ergodic proof of Szemerédi’s Theorem,...).
2. Products of random matrices, non-commutative ergodic theory (simplicity of Lyapunov exponents, ...).
3. Unique ergodicity of horocycle flow, toral maps, ...
4. Disjointness of dynamical systems.
5. $\times 2, \times 3$, rigidity of higher order actions.
6. Fractal geometry $\cap$ ergodic theory (CP-processes, ...).
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Expansions in different bases

Principle (Furstenberg)

Expansions in bases 2 and 3 have no common structure. More generally, this holds for bases $p$ and $q$ which are not powers of a common integer or, equivalently, $\log p / \log q$ is irrational.

Remark

Furstenberg proved some results, and proposed many conjectures, which make precise (in different ways) the concept of “no common structure”.
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Invariant sets

**Definition**

A set $A \subset [0, 1)$ is $T_p$-invariant if $T_p(A) \subset A$. That is, shifting the $p$-ary
expansion of a point in $A$ gives another point in $A$.

- If $p$ and $q$ are coprime, then $\{0, 1/q, \ldots, (q-1)/q\}$ is $T_p$-invariant.
- $[0, 1)$ is $T_p$-invariant.
- Let $D \subset \{0, 1, \ldots, p-1\}$. The set $A = A_{p,D}$ of points whose base
  $p$-expansion has only digits from $D$ is $T_p$-invariant. We call it a
  $p$-Cantor set. Example: the middle-thirds Cantor set.
- There is a **wild abundance** of invariant sets and no classification or
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Invariant sets and shared structure

**Principle (Furstenberg, slightly more concrete version)**

*If* $A, B$ are closed and invariant under $T_2, T_3$ respectively, then *A and B have no common structure.*

**Theorem (Furstenberg (1967))**

*If* $A$ is jointly invariant under $T_2$ and $T_3$, then $A$ is either finite or the whole circle $[0, 1)$.

**Remarks**

- The theorem is a weak confirmation of the principle since the set $A$ and itself certainly have a lot of common structure!
- One should think of finite sets and the whole circle as sets “without structure”.
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A corollary in terms of orbits

Observation

- If $x$ is rational, then the orbit $\{ T_2^n T_3^m x \}_{n,m=1}^{\infty}$ is infinite.
- If $x$ is irrational, then the orbit $\{ T_2^n T_3^m x \}_{n,m=1}^{\infty}$ is infinite (and its closure is invariant under $T_2$ and $T_3$).

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“The” $\times 2$, $\times 3$ Furstenberg conjecture

Definition
A Borel probability measure $\mu$ on $[0, 1)$ is $T_p$-invariant if

$$\mu(B) = \mu(T_p^{-1}B) \quad \text{for all Borel sets } B.$$ 

Conjecture (Furstenberg 1967)
If $\mu$ is $T_2$ and $T_3$ invariant, then $\mu$ is a convex combination of Lebesgue measure and an atomic measure supported on rationals.
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How to quantify “shared structure”

1 Furstenberg’s Theorem says that non-trivial $T_2$ and $T_3$ invariant sets do not have too much shared structure in the most basic sense: they cannot be equal.

2 How can we quantify shared structure in finer/more quantitative ways? The sets we are interested in are fractal: they are uncountable but of zero Lebesgue measure, and have some form of (sub)-self-similarity.

3 Geometry helps quantify common structure. For example, if two sets $A, B \subset \mathbb{R}$ have no shared structure one expects the sumset $A + B = \{a + b : a \in A, b \in B\}$ to be “as large as possible” and the intersection $A \cap B$ and $A \cap (\lambda B + t)$ to be “as small as possible”.
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Hausdorff Dimension

• Best exponent for coverings of the set by balls of arbitrary (possibly different) radii:

\[ \dim_H(A) = \inf \left\{ s : \inf \left\{ \sum_i r_i^s : A \subset \bigcup_i B(x_i, r_i) \right\} = 0 \right\} \]

• Gives a notion of “size” for sets in \( \mathbb{R}^d \), varies between 0 and \( d \), gives the right size to smooth objects, is invariant under bi-Lipschitz maps, is countably stable, assigns size \( \log 2 / \log 3 \) to the middle-thirds Cantor set, ...

• If \( A \subset \mathbb{T} \) is \( T_p \)-invariant, then \( \dim_H A = h_{\text{top}}(A) / \log p \).

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Furstenberg’s sumset conjecture
In all conjectures, \( p, q \) are rationally independent (not powers of a common integer). E.g. 2 and 3, or 6 and 12 (but not 8 and 16).

**Conjecture 1**
Let \( A, B \) be closed and \( T_p, T_q \) invariant. Then

\[
\dim_H(A + B) = \max(\dim_H(A) + \dim_H(B), 1).
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**Motivation**
- One “typically” expects the formula above to hold. For example, for arbitrary sets \( A, B \) it holds that

\[
\dim_H(A + \lambda B) = \max(\dim_H(A) + \dim_H(B), 1) \ 	ext{for almost all} \ \lambda \in \mathbb{R}.
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- Moreover, the right-hand side is always a (trivial) upper bound.
- For a strict inequality to occur, \( A \) and \( B \) must have “shared structure at many scales”.
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* This is an example to illustrate the concept, not a formal proof.
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If $A, B$ are a $p$-Cantor set and a $q$-Cantor set, then

$$\dim_H(A + \lambda B) = \min(\dim_H(A) + \dim_H(B), 1) \text{ for all } \lambda \in \mathbb{R} \setminus \{0\}.$$
Theorem (M.Hochman-P.S. 2012)

If $A, B$ are closed and $T_p, T_q$-invariant, then

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Product, projection, fiber
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"×2, ×3"
Product, projection, fiber
More general notions of shared structure?

I argued that if

$$\dim_H(A + B) < \min(\dim_H(A) + \dim_H(B), 1),$$

then $A$ and $B$ have “common structure” at many scales.

But the opposite is far from true! For many (“most”) sets $A$, even of dimension $\leq 1/2$, even $T_p$-invariant ones,

$$\dim_H(A + A) = \min(2 \dim_H(A), 1).$$

A stronger notion of shared structure is given by the size of intersections. For example, $A \cap A$ is always larger than “expected” (if $\dim_H(A) > 0$).
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Conjecture 2 (Furstenberg 1969)

Let $A, B$ be closed and invariant under $T_p, T_q$ (seen as subsets of $\mathbb{R}$). Then for every affine bijection $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\dim_H(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0).$$

Motivation

- It is known that for arbitrary sets $A, B$ one cannot do better than the right-hand side. Counting heuristics show that the RHS is the “average size” of an intersection.
- Conjecture 2 is far stronger than Conjecture 1. Heuristically, the sumset $A + B$ is “large” if “many” fibers are “small”. The conjecture asserts that all fibers are small.
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Previous results on Furstenberg’s conjecture

**Theorem (Furstenberg 1969, Wolff 2000)**

The conjecture holds if $\dim_H(A) + \dim_H(B) \leq 1/2$. More generally, one always has

$$\dim_H(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1/2, 0).$$

**Remark**

No example of invariant sets $A, B$ for which the conjecture holds with $\dim_H(A) + \dim_H(B) > 1/2$ were known.
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Solution to Furstenberg’s conjecture 2

Theorem (P.S. 2016)

Furstenberg’s conjecture 2 holds.

Remark

Meng Wu (University of Oulu, Finland) independently found another proof. The proofs are completely different. Wu’s proof is purely ergodic theoretical, using CP-processes (introduced by Furstenberg in the paper where he stated the conjecture) and Sinai’s factor theorem.
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Our old friend again: $A \times B$. 

More pictures!
More pictures!

$A \times B \cap \text{diagonal} = A \cap B$. 
$A \times B \cap \text{any line} = A \cap \text{affine image of } B.$
A corollary on subsets of integers

Corollary
Let $A$ be the natural numbers with digits $0, 3$ in base $4$, and $B$ the natural numbers with digits $1, 2, 7$ in base $10$. Then

$$\lim_{n \to \infty} \frac{\log |A \cap B \cap \{1, \ldots, n\}|}{\log n} = 0,$$

in other words, given $\varepsilon > 0$,

$$|A \cap B \cap \{1, \ldots, n\}| \leq n^\varepsilon \quad \text{for } n \text{ large enough}.$$
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**Question (Related to another conjecture of Furstenberg)**

Is $A \cap B$ finite?
Tools involved in the proof

1. **Additive combinatorics**: an inverse theorem for the $L^q$ norm of the convolution of two finitely supported measures (Balog-Szemerédi-Gowers Theorem, Bourgain’s additive part of discretized sum-product results).

2. **Ergodic theory**: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).

3. **Multifractal analysis** ($L^q$ spectrum, regularity at points of differentiability).

4. General scheme of proof follows Mike Hochman’s strategy in his recent landmark paper on the dimensions of self-similar measures.
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Reduction to a problem in multifractal analysis

- By a standard argument, it is enough to prove the theorem when $A, B$ are a $p$-Cantor set and a $q$-Cantor set respectively (with digit sets $D_1 \subset \{0, 1, \ldots, p - 1\}$, $D_2 \subset \{0, 1, \ldots, q - 1\}$).
- There are natural measures $\mu, \nu$ on $A, B$ (Hausdorff measure, measure of maximal entropy, they all agree).
- Let
  \[
  \eta_t = \mu \ast S_t \nu
  \]
  where $S_t x = tx$ scales by $x$. Alternatively, $\eta_t$ is the push-down measure of $\mu \times \nu$ under the linear projection $(x, y) \mapsto x + ty$.
- Given a probability measure $\eta$ on $[0, 1]$, let
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Theorem (P.S. 2016)

For all $t \neq 0$,

$$D_q(\eta_t) = \min\left(\dim_H(A) + \dim_H(B), 1\right) \quad \text{for all } q > 1.$$ 

Remark

The theorem says that $\eta_t$ is very uniformly distributed in its support $A + tB$ with no points of “larger than expected” mass.
Multifractal analysis → intersections I

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Proof of Furstenberg’s conjecture assuming theorem.

Let
\[ s = \dim_H(A) + \dim_H(B) = \dim_H(A \times B). \]

Suppose
\[ d = \dim_H(A \cap (tB + u)) > \min(s - 1, 0) \]

Let \( u \in I \in \mathcal{D}_n \) with \( n \gg 1 \). Then, writing \( P(x, y) = x + ty \), we have \( A \cap (tB + u) \subset P^{-1}(I) \) so that
\[ \eta_t(I) = (\mu \times \nu)(P^{-1}(I)) \gtrsim 2^{dn}2^{-sn}. \]

It follows that
\[ 2^{\min(s,1)(1-q)n} \geq \sum_{I \in \mathcal{D}_n} \eta_x(I)^q \gtrsim \left(2^{dn}2^{-sn}\right)^q. \]

This is a contradiction if \( q \) is large enough.
Self-similarity

\[ \mu \sim \sum_{n=1}^{\infty} X_n p^{-n}, \quad \nu \sim \sum_{n=1}^{\infty} Y_n q^{-n} \]

with \( X_n, Y_n \) independent and uniform in \( D_1, D_2 \) respectively.

\[ \eta_t = \mu * S_t \nu \sim \sum_{n=1}^{\infty} X_n p^{-n} + \sum_{m=1}^{\infty} t Y_m q^{-m}. \]

One can rearrange terms to find out \( \eta_t \) has a dynamical self-similar structure:

\[ \eta_t = \Delta(t) \ast \eta_{\sigma(t)} \]

where: \( \Delta(t) \) is a finitely supported measure, and \( \sigma \) is a uniquely ergodic transformation of some interval.
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Theorem (P.S. 2016)

If $(\eta_t)$ is a family of “dynamical self-similar measures” where the driving dynamics is uniquely ergodic + some regularity hypotheses, then

$$D_q(\eta_t) = \text{what you expect for all } t \text{ and } q > 1.$$

Remark

As corollaries of this theorem, beyond Furstenberg’s conjecture I get applications to:

1. The dimensions and densities of self-similar measures, including Bernoulli convolutions,
2. The dimensions of slices of many self-similar fractals in the plane including the 1-dimensional Sierpiński gasket (improving another conjecture of Furstenberg).

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Slices of the 1-dim Sierpiński Gasket

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All orthogonal projections with irrational slope have dimension 1.

Theorem (P.S. 2016)

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¡¡¡Muchas gracias!!!