Existence and regularity of solutions to optimal partition problems involving Laplacian eigenvalues

Miguel Ramos, Hugo Tavares and Susanna Terracini

Università di Torino
susanna.terracini@unito.it

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain and $m \in \mathbb{N}$. Given $k_1, \ldots, k_m \in \mathbb{N}$, we consider a wide class of optimal partition problems involving Dirichlet eigenvalues of elliptic operators, including the following

$$\inf \left\{ \Phi(\omega_1, \ldots, \omega_m) := \sum_{i=1}^{m} \lambda_{k_i}(\omega_i) : \ (\omega_1, \ldots, \omega_m) \in \mathcal{P}_m(\Omega) \right\},$$

where $\lambda_{k_i}(\omega_i)$ denotes the k_i–th eigenvalue of $(-\Delta, H_0^1(\omega_i))$ counting multiplicities, and $\mathcal{P}_m(\Omega)$ is the set of all open partitions of Ω, namely

$$\mathcal{P}_m(\Omega) = \{(\omega_1, \ldots, \omega_m) : \ \omega_i \subset \Omega \text{ open}, \ \omega_i \cap \omega_j = \emptyset \ \forall i \neq j\}.$$

We prove the existence of an open optimal partition $(\omega_1, \ldots, \omega_m)$, proving as well its regularity in the sense that the free boundary $\cup_{i=1}^{m} \partial\omega_i \cap \Omega$ is, up to a residual set, locally a $C^{1,\alpha}$ hypersurface.

In order to prove this result, we first treat some general optimal partition problems involving all eigenvalues up to a certain order. This class of problems includes the one with cost function $\Phi_p(\omega_1, \ldots, \omega_m) := \sum_{i=1}^{m} \left(\sum_{j=1}^{k_i} (\lambda_j(\omega_i))^p \right)^{1/p}$, whose solutions approach the solutions of the original problem as $p \to +\infty$. The study of this new class of problems is done via a singular perturbation approach with a class of Schrödinger-type systems which models competition between different groups of possibly sign-changing components. An optimal partition appears in relation with the nodal set of the limiting components, as the competition parameter becomes large.