On the dynamic representation of some time-inconsistent risk measures in a Brownian filtration

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Plan

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.
L^∞ the space of essentially bounded random variables $\Rightarrow$ Losses.

A functional $\rho : L^\infty \to \mathbb{R}$ is a (monetary-convex) **risk measure** if

- (Monotonicity) $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$;
- (Convexity) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$;
- (Cash-invariance) $\rho(X + c) = \rho(X) + c$, for $c \in \mathbb{R}$.

Interpretation: diversification reduces risk of big losses; subtracting $\rho(X)$ units of cash from X yields an "acceptable" (risk-less) position.

Examples:
$\rho(X) = \mathbb{E}[X]\,,\ \text{esssup}(X)\,,\ \alpha^{-1}\int_{1-\alpha}^{1} F_X^{-1}(t)dt\,,\ \log\mathbb{E}[\exp(X)]\,,\ \ldots$

Artzner, Delbaen, Eber, Heath, Föllmer, Schied, Rüschendorf, Ruszczyński, Shapiro, Rockafellar, Uryasev, Acerbi, Weber, ...

Introduction

It may be desirable that the risk of $X \in L^\infty$ depends only on its distribution; often Ω is just a mathematical gadget!

- (Law-invariance) $\rho(X) = \rho(Y)$ if $\mathrm{Law}(X) = \mathrm{Law}(Y)$.

.... Kusuoka, Schachermayer, Touzi, Jouini ....

A large family of particularly tractable law-invariant convex risk measures is given by the **Optimized certainty equivalents (OCE)**:

Definition (Ben-Tal & Teboulle)

Let $\ell : \mathbb{R} \to \mathbb{R}$ convex and increasing. The OCE with *loss function* ℓ is

$$\rho(X) \;=\; \inf\{\, \mathbb{E}[\ell(X - r)] + r :\; r \in \mathbb{R}\,\}.$$

Examples: Entropic case: $\ell \sim e^x$, Monotone Mean-Variance: $\ell \sim [x]_+^2$, Conditional Value-at-Risk: $\ell \sim [x]_+$ ....

Introduction

This is a static picture. For a dynamic version, let

$$(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{P}) \text{ be a filtered probability space.}$$

A functional $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ is a **conditional risk measure** if for every $\lambda \in L^\infty(\mathcal{F}_t) \cap [0,1], c \in L^\infty(\mathcal{F}_t)$:

- (Monotonicity) $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$;
- (Convexity) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$;
- (Cash-invariance) $\rho(X + c) = \rho(X) + c$.

For a family $\{\rho_t\}_{t=0}^T$ of such operators, one defines:

- (Time-consistency) $\rho_{t+h}(X) \geq \rho_{t+h}(Y) \Rightarrow \rho_t(X) \geq \rho_t(Y)$.
  Equivalently, $\rho_t(\rho_{t+h}(X)) = \rho_t(X)$.

.... Detlefsen, Scandolo, Cheridito, Kupper, Acciaio, Penner ....

Introduction

A crucial result of Kupper & Schachermayer:

The only families of time-consistent, law invariant convex risk measures: expected values, essential suprema, and entropic risk measures.

$\Rightarrow$ OCE risk measures are most often not time-consistent.

Time inconsistency in stochastic optimization: .... Zhou, Li, Ekeland, Lazrak, Bäuerle, Ott, Shapiro, Pflug, Pichler, Chow, Tamar, Mannor, Pavone, Miller, Yang ....

Our Goal: to nevertheless understand the dynamic behaviour of OCEs.

From now on, the setting is:

- continuous time, $t \in [0, T]$;
- Brownian filtration, $\mathcal{F}_t = \sigma(W_s : s \leq t)$, where W is a B.M.

Introduction

The time-consistent case is well understood in this framework!
.... Delbaen, Peng, Rosazza Gianin, Coquet, Hu, Mémin ....

There is a correspondence between time-consistent convex risk measures and certain Backward Stochastic Differential Equations (BSDE)[1], in the sense that $Y_t := \rho_t(X)$ solves, along some process Z:

$$Y_t = X + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s \,,$$

for suitable *generator* g.

- This provides a dynamic way of computing $\rho(X) = \rho_0(X)$.
- If X is a "Markovian claim" on a diffusion process $\Rightarrow$ HJB equation.

We ask: For Markovian claims, and OCE risk measures, is there a HJB? Can this *characterize* $\rho(X)$?.

[1] A.K.A. g-expectations or non-linear expectations.

Setup

We want to compute the risk of a claim written on a diffusion process[2]:

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dW_t.$$

We make standard Lipschitz and linear growth assumptions
$\Rightarrow$ existence and uniqueness of strong solution.

We shall consider positions/claims written on Y, such as[3]

$$X = f(Y_T) + \int_0^T g(t, Y_t)\, dt;$$

we assume f, g bounded and "Lipschitz in space variable."

Let ρ be an OCE risk measure with reasonable loss function ℓ.

[2] W is a d-dimensional B.M. and Y is an m-dimensional process.
[3] one says X is a *Markovian/static/additive* function of Y.

Stochastic control interpretation

Fact: by convex analysis, we have the *dual representation*:

$$\rho(X) = \sup\left\{E[XZ] - E[\ell^*(Z)] : Z \in L^1_+, Z \in \mathrm{dom}(\ell^*),\ E[Z]=1\right\}.$$

Crucial for rest of the talk:

- Z satisfies $dZ_t = Z_t\beta_t dW_t$, for some β;
- we may view β as a *control*;
- the pair (Y, Z) is then a controlled diffusion;

$\Rightarrow$ the dual representation is the value of a stochastic control problem.
.... Mataramvura, Oksendal ....

For $t \geq s$, we denote by $Y_t^{s,y}, Z_t^{s,z,\beta}$ the state of the controlled system when started from y, z at time s. Consider the *value function*:

$$V(s,y,z) := \sup_\beta \mathbb{E}\left[f(Y_T^{s,y})Z_T^{s,z,\beta} - \ell^*(Z_T^{s,z,\beta}) + \int_s^T g(t, Y_t^{s,y})Z_t^{s,z,\beta}dt\right].$$

Consistency out of inconsistency

Our first result:

Proposition (B., Tangpi)

The following "Bellman principle of sorts" holds, for any $X \in L^\infty$:

$$\rho(X) = \sup_{\substack{Z \in dom(\ell^*) \\ Z \in L^1_+(\mathcal{F}_t), E[Z]=1}} E\left[essinf_{r\in\mathbb{R}}\big(E[\ell(X-r)|\mathcal{F}_t] + rZ\big)\right].$$

In this Markovian setting, the "Bellman principle of sorts" specializes:

Proposition (B., Tangpi)

The value function satisfies

$$V(s,y,z) = \rho^{\ell_z}\left(z\left[f(Y_T^{s,y}) + \int_s^T g(t, Y_t^{s,y})dt\right]\right),$$

where ρ^{ℓ_z} is the OCE with loss function $\ell_z(x) := \ell(x/z)$. Furthermore

$$V(s,y,z) = \sup_\beta \mathbb{E}\left[\int_s^{s+u} g(t, Y_t^{s,y})Z_t^{s,z,\beta}dt + V(s+u, Y_{s+u}^{s,y}, Z_{s+u}^{s,z,\beta})\right].$$

The HJB for the problem

One guesses that V should be related to the following HJB equation:

$$\partial_t V + b(s,y)\partial_y V + \frac{1}{2} tr\left(\sigma(s,y)\sigma(s,y)'\partial^2_{yy} V\right) + \sup_{\beta\in\mathbb{R}^d}\left[\frac{1}{2}z^2\beta^2\partial^2_{zz}V + ztr\left(\sigma(s,y)\beta\partial^2_{yz}V\right)\right] + zg(s,y) = 0,$$

with

$$V(T,y,z) = f(y)z - \ell^*(z).$$

We thus expect:

characterizing $\rho(X) =$ proving that the *value function* of the stochastic control problem solves a HJB equation.

Not so fast!

However:

- the diffusion coefficient

$$(s, y, z) \mapsto \Sigma(s, y, z) := (\sigma(s, y), \beta z),$$

  is not Lipschitz, and it degenerates;
- the controls β do not take values in a compact set a priori;
- the *Hamiltonian* H is singular:

$$H(s, y, z, \gamma, \Gamma) := \gamma b(s, y) + \frac{1}{2} \sup_{\beta \in \mathbb{R}^d} \mathrm{tr}([\Sigma \Sigma'](s, y, z, \beta)\Gamma) + zg(s, y).$$

Consequence:
The candidate HJB equation for the problem is very intractable!

Examples

- Let $Y_t^y := y \in \mathbb{R}$ for all t and $X^y := f(Y_T^y)$. Then $\rho(X^y) = f(y)$ may not be differentiable.
- Let $Y_T^y := \text{sign}(W_T) + y$ and $X^y = [Y_T^y]_+$. There is a martingale diffusion "hitting" Y_T^y at time T. Its volatility is

$$\sigma(t, \cdot) = \sqrt{\frac{2}{\pi(T-t)}} \exp\left\{-\frac{1}{2}\left[\Phi^{-1}\left(\frac{y+1-\cdot}{2}\right)\right]^2\right\},$$

  so in particular σ is not uniformly parabolic. We have

$$\log \mathbb{E}[\exp(X^y)] = \log\left(\frac{1}{2}\left(\exp\{[y-1]_+\} + \exp\{[y+1]_+\}\right)\right),$$

  which is continuous but is not differentiable at $y = \pm 1$.
- For $Y_t^y = W_t - y$, which is uniformly parabolic, and $X^y := [Y_T^y]_+$, we have $\log(E[e^{Y_T^+}]) = \log \int e^{[c]_+} h(y+c)dc$, where h is the density of a centred Gaussian with variance T. Thus $\rho(X^y)$ is smooth in y.

Viscosity semisolutions

Definition

A l.s.c. function v defined on $[0, T] \times \mathbb{R}^m \times \mathcal{O}$ is said to be a viscosity supersolution of the HJB equation

$$\begin{cases} \partial_t V + H(s, y, z, DV, D^2 V) = 0, \\ V(T, y, z) = \psi(y, z), \end{cases}$$

if for all $(s_0, y_0, z_0) \in [0, T] \times \mathbb{R}^m \times \mathcal{O}$ and $\varphi \in C^2([0, T] \times \mathbb{R}^m \times \mathcal{O})$ such that $v - \varphi$ has a local minimum at (s_0, y_0, z_0), if $s_0 = T$ we have $v(s_0, y_0, z_0) \geq \psi(y_0, z_0)$ and if $s_0 < T$ we have

$$\partial_t \varphi(s_0, y_0, z_0) + H(s_0, y_0, z_0, D\varphi(s_0, y_0, z_0), D^2\varphi(s_0, y_0, z_0)) \leq 0.$$

We define viscosity subsolutions analogously, changing “local minumum” by “local maximum,” changing the order of inequalities, AND requiring above that $(s_0, y_0, z_0, D\varphi(s_0, y_0, z_0), D^2\varphi(s_0, y_0, z_0)) \in int(dom(H))$.

Remark

Our notion of subsolution is not standard, but seems to be better suited than the more usual “variational inequality” version of it.

Main result

The candidate HJB does indeed characterize the value function:

Theorem (B., Tangpi)

Let $X^y := f(Y_T^{0,y}) + \int_0^T g(t, Y_t^{0,y})\,dt$*. Then*

$$\rho(X^y) = V(0, y, 1),$$

where the value function V *is a viscosity solution of the HJB, and is further characterized as the minimal viscosity supersolution of it.*

Core of the proof:

- supersolution property follows from Bellman principle;
- for subsolution property, we start by several approximation arguments, reducing to the case $|\beta| \leq K$;
- the obtained truncated value function is a viscosity solution of a truncated HJB;
- stability of viscosity subsolutions w.r.t. monotone convergence.

Remark

We do not know if a comparison principle holds, so we neither know if V *is the unique viscosity solution.*

Examples & classical solutions: Entropic case

In the entropic case $\ell \sim e^x$ so $\rho(X) = \log \mathbb{E}[e^X]$. This is our time-consistent benchmark.

In the literature $\Rightarrow$ *Risk sensitive criteria* (... Fleming, Soner ...)

We obtain

$$V(s, y, z) \ = \ -z \log z + z \log u(s, y),$$

where $u = u(s, y)$ is a viscosity solution of the backward Kolmogorov PDE associated to b, σ^2, with discount rate g and final condition $\exp(f)$, namely:

$$\left(\partial_t + b\partial_y + \frac{1}{2}\sigma^2\partial^2_{yy} + g\right) u = 0.$$

If $\sigma^2 > \epsilon$ and (b, σ^2) is bounded, that PDE has a unique classical solution $\Rightarrow$ our V is the unique classical sol. of the HJB equation.

Examples & classical solutions: Monotone Mean Variance[4]

Here $\ell \sim [x]_+^2$, so $\rho(X) \sim \inf_r\{\mathbb{E}[(X-r)_+^2] + r\}$.
This is our time-inconsistent benchmark.

We write $V(s,y,z) = \phi(s,y) + z\tilde{V}(s,y) + \ell^*(z)$, where *necessarily*

$$\begin{cases} \left(\partial_t + b\partial_y + \frac{1}{2}\sigma^2\partial_{yy}^2\right)\tilde{V} & = & -g \\ \tilde{V}(T,\cdot) & = & f(\cdot), \end{cases}$$

and

$$\begin{cases} \left(\partial_t + b\partial_y + \frac{1}{2}\sigma^2\partial_{yy}^2\right)\phi & = & -\frac{1}{2}\sigma^2[\partial_y\tilde{V}]^2 \\ \phi(T,\cdot) & = & 0. \end{cases}$$

Proposition (B., Tangpi)

Assume $\sigma^2 > \epsilon$ and that (b, σ^2, f, g) are sufficiently smooth and with bounded derivatives. Then the above PDEs have unique classical solutions, and so does our HJB. Further, $V = \phi + z\tilde{V} + \ell^*$ indeed.

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[4] Maccheroni, Marinacci, Rustichini.

Examples & classical solutions: discussion

The Conditional Value-at-Risk[5] at level $\alpha \in (0,1)$, given by $\rho(X) = \alpha^{-1} \int_{1-\alpha}^{1} F_X^{-1}(t)dt$, is an OCE with loss function

$$\ell(x) = \alpha^{-1}[x]_+ \,.$$

Our results say that

$$V(s,y,z) = \mathrm{CVaR}_{\alpha z}\left(z f(Y_T^{s,y}) + z \int_s^T g(t, Y_t^{s,y}) dt \right),$$

is the minimal supersolution of the corresponding HJB. In discrete time, this goes to Pflug and Pichler.

- Can the dimensionality of the HJB be reduced, as above?,
- can we hope to have classical solutions in the regular case?.

We actually conjecture that combinations of the entropic and MMV cases are the only ones for which the first question has positive answer.

[5] average/tail VaR, worst conditional expectation, expected shortfall...

Some extensions

- A proof of concept: non-Markovian claims.

Any bounded claim X is, in distribution, the final value of a diffusion, e.g.

$$dY_s = \partial_x u(t, u^{-1}(t, Y_s))dW_s, \text{ with } u(s,x) := \mathbb{E}[F_X^{-1} \circ F_N(W_T) | W_s = x],$$
$$dY_s = \partial_x \log v(s, Y_s)ds + dW_s, \text{ with } (\partial_t + \partial^2_{xx}/2)v = 0, \text{ and } v(T, \cdot) = F'_X.$$

- Beyond OCE risk measures: the utility-based expected shortfall case:

$$\rho^{ES}(X) := \inf\{r \in \mathbb{R} : \mathbb{E}[\ell(X - r)] \leq x_0\}.$$

The new value function $\mathcal{V}$ is the positive homogeneous envelope of $V - x_0$; we prove that it is also a subsolution of the HJB with

$$\mathcal{V}(T, y, z) = zf(y) - \inf_{\lambda>0}\{\lambda^{-1}[\ell^*(\lambda z) - x_0]\}.$$

- Work in progress: stochastic control with OCE cost criteria
  $\Rightarrow$ HJBI equation ... much more technical!

Some open questions

Regarding our work:

- Comparison principle $\Rightarrow$ uniqueness of viscosity solution.
- Numerical analysis of our HJB.
- More examples, classical solutions, etc.
- Unbounded positions/claims.
- Introduce jumps $\Rightarrow$ HJB has non-local terms.

In general:

- Non-Markovian claims: path-dependent PDEs, Semigroups, BSDEs?
- Non-Markovian claims: what if claims have still specific structure?
- General risk measures: there is next to nothing done.

Thank you all!