Bank monitoring under adverse selection

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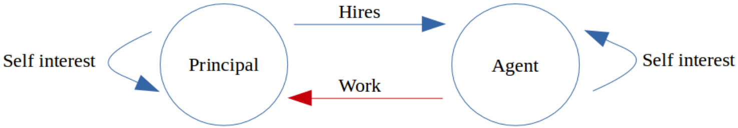
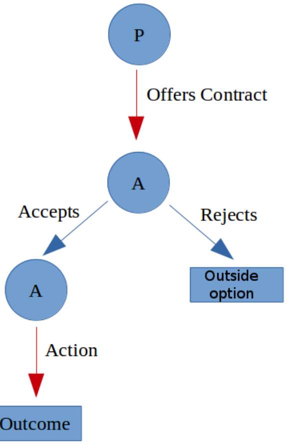
The **Principal**-**Agent** problem arises when one person or entity (the **Principal**) hires another one (the **Agent**) to work on behalf of him.



The work/action performed by the **Agent** is <u>not observable</u> by the **Principal**.

From the game theory point of view, the **Principal** and the **Agent** play a Non-zero sum Stackelberg game.



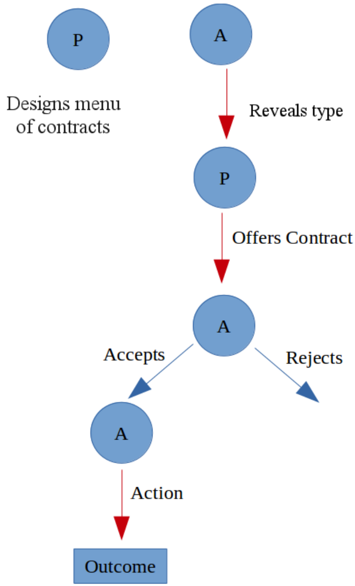
Examples of **Principal**-**Agent** situations:

- **Boss**-**Employee**.
- **Voters**-**Candidates**.
- Electricity: **Planner**-**Consumers**.
- Pollution: **Regulator**-**Companies**.

In the literature three main types of Principal-Agent problems are studied

1. First-best (risk sharing).
2. Second-best (moral hazard).
3. Third-best (adverse selection).

Interaction under adverse selection.



- A **bank** monitors a pool of I identical loans indexed by $j = 1, \ldots, I$.
- Each loan yields cash flow μ per unit of time until it defaults.
- The **bank** raise funds from an **investor**.
- There are two types of **banks** in the market: the **"good"** bank ρ_g and the **"bad"** bank ρ_b. The **investor** does not know the type of the **bank**, only the proportions p_g and p_b.

- Denote by $N_t := \sum_{j=1}^{I} \mathbf{1}_{\{\tau^j \leq t\}}$, the current size of the pool at time t is $I - N_t$.
- The action of the **bank** of type ρ_i is to decide how many loans he will **not monitor**
$$k_t^i \in \{0, \dots, I - N_t\}.$$
- Every non-monitored loan renders a private **benefit** B to the **bank**.
- The associated **default intensity** is given by
$$\lambda_t^{k^i} := \sum_{j=1}^{I-N_t} \alpha_t^{j,i} = \alpha_{I-N_t}\left(I - N_t + \varepsilon k_t^i\right).$$

The **bank** controls the distribution of N_t.

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which N is a Poisson process with intensity λ_t^0.

We call τ the **liquidation time** of the whole pool and let $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ be the minimal filtration containing $(\mathcal{F}_t^N)_{t\geq 0}$ and that makes τ a $\mathbb{G}$-stopping time.

Define $\mathbb{P}^k$ on $\mathcal{G}_t$ by

$$\frac{\mathrm{d}\mathbb{P}^k}{\mathrm{d}\mathbb{P}} = Z_t^k,$$

where Z^k is the unique solution of the following SDE

$$Z_t^k = 1 + \int_0^t Z_{s^-}^k \left(\frac{\lambda_s^k}{\lambda_s^0} - 1 \right) \left(\mathrm{d}N_s - \lambda_s^0 \mathrm{d}s \right), \ 0 \leq t \leq \tau, \ \mathbb{P} - a.s.$$

Then $N_t - \int_0^t \lambda_t^k \mathrm{d}s$, is a $\mathbb{P}^k-$martingale.

The **investor** designs a menu of contracts $(\Psi_i)_{i\in\{g,b\}} := (k^i, \theta^i, D^i)_{i\in\{g,b\}}$ consisting in:

- Predictable, non-decreasing payments D^i.

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Denote $H_t := \mathbf{1}_{t\geq\tau}$, then

$$\mathrm{d}H_t = \begin{cases} 0 \text{ with probability } \theta_t^i, \\ \mathrm{d}N_t \text{ with probability } 1-\theta_t^i. \end{cases}$$

Utility of the **bank** of type ρ_i

$$u_0^i(k^i, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^{k^i}} \left[\int_0^\tau \mathrm{e}^{-rs} (\rho_i \mathrm{d}D_s^i + Bk_s^i \, \mathrm{d}s) \right],$$

Utility of the **investor**

$$v_0\left((\Psi_i)_{i\in\{g,b\}}\right) := \sum_{i\in\{g,b\}} p_i \mathbb{E}^{\mathbb{P}^{k^i}} \left[\int_0^\tau (I - N_s)\, \mu \mathrm{d}s - \mathrm{d}D_s^i \right].$$

Agent's problem:

$$\underset{k\in\mathfrak{K}}{\text{maximize}}\ u_0^i(k,\theta^i,D^i)$$

Principal's problem:

$$\begin{aligned}
\text{maximize} \quad & v_0(\Psi_g,\Psi_b) \\
\text{s.t.} \quad & u_0^i(k^i,\theta^i,D^i)\geq R_0,\ i\in\{g,b\}, \\
& u_0^i(k^i,\theta^i,D^i)=\sup_{k\in\mathfrak{K}} u_0^i(k,\theta^i,D^i),\ i\in\{g,b\}, \\
& u_0^i(k^i,\theta^i,D^i)\geq \sup_{k\in\mathfrak{K}} u_0^i(k,\theta^j,D^j),\ i\neq j,\ (i,j)\in\{g,b\}^2.
\end{aligned}$$

Continuation utility approach:

- Discrete time: Spear and Srivastava (1987).
- Continuous time: Sannikov (2008).

Define the continuation utility of the **bank** at time $t \geq 0$

$$u_t^i(k, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^k}\left[\int_{t\wedge\tau}^{\tau} \mathrm{e}^{-r(s-t)}\left(\rho_i \mathrm{d}D_s^i + k_s B \mathrm{d}s\right)\bigg|\,\mathcal{G}_t\right].$$

Define also the value function of the **bank** for any $t \geq 0$

$$U_t^i(\theta^i, D^i) := \operatorname*{ess\,sup}_{k\in\mathfrak{K}} u_t^i(k, \theta^i, D^i).$$

The process $\mathrm{e}^{-rt}u^i_t(k,\theta^i,D^i)+\int_0^t \mathrm{e}^{-rs}\left(\rho_i \mathrm{d}D^i_s + k_s B \mathrm{d}s\right)$ is a $\mathbb{P}^k$−martingale.

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There exist $\mathbb{G}-$predictable processes $h^{1,i,k}$ and $h^{2,i,k}$ such that

$$\mathrm{d}u_t^i(k,\theta^i,D^i) = \left(ru_t^i(k,D^i,\theta^i) - Bk_t\right)\mathrm{d}t - \rho_i \mathrm{d}D_t^i - h_t^{1,i,k}\left(\mathrm{d}N_t - \lambda_t^k \mathrm{d}t\right) \\ - h_t^{2,i,k}\left(\mathrm{d}H_t - (1-\theta_t^i)\lambda_t^k \mathrm{d}t\right),\ 0 \leq t < \tau,\ \mathbb{P}-a.s.$$

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The continuation utility is a super-solution to a BSDE with jumps.

Let us then define

$$Y_t^{i,k} := u_t^i(k,\theta^i,D^i),\ Z_t^{i,k} := (h_t^{1,i,k}, h_t^{2,i,k})^\top,\ M_t := (N_t, H_t)^\top,$$
$$\widetilde{M}_t^i := M_t - \int_0^t \lambda_s^0(1, 1-\theta_s^i)^\top \mathrm{d}s,\ K_t^i := \rho_i D_t^i,$$

so that we can rewrite the previous equation as follows $\mathbb{P}-a.s.$,

$$Y_t^{i,k} = 0 - \int_t^\tau f^i(s,k_s,Y_s^{i,k},Z_s^{i,k})\mathrm{d}s + \int_t^\tau Z_s^{i,k}\cdot \mathrm{d}\widetilde{M}_s^i + \int_t^\tau \mathrm{d}K_s^i,\ 0\le t\le \tau,$$

where

$$f^i(t,k,y,z) := ry - Bk + k\alpha_{I-N_t}\varepsilon z\cdot(1,1-\theta_t^i)^\top.$$

Denote by (Y^i, Z^i) the unique (super–)solution to the following BSDE

$$Y_t^i = 0 - \int_t^\tau g^i(s, Y_s^i, Z_s^i)\mathrm{d}s + \int_t^\tau Z_s^i \cdot \mathrm{d}\widetilde{M}_s^i + \int_t^\tau \mathrm{d}K_s^i,\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s.,$$

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By the comparison theorems, Royer (2008), the value function of the **bank** has the dynamics, for $t \in [0,\tau]$, $\mathbb{P}-a.s.$

$$\mathrm{d}U_t^i(\theta^i, D^i) = \left(rU_t^i(\theta^i, D^i) - Bk_t^{\star,i} + \lambda_t^{k^{\star,i}} Z_t^i \cdot (1, 1-\theta_t^i)^\top\right)\mathrm{d}t - \rho_i \mathrm{d}D_t^i - Z_t^i \cdot \mathrm{d}\widetilde{M}_t^i, \quad (1)$$

and the optimal monitoring choice of the **bank** is given by

$$\boxed{k_t^{\star,i} = (I - N_t)\mathbf{1}_{\{Z_t^i\cdot(1,1-\theta_t^i)^\top < b_t\}}.}$$

The value function of the **investor** in the pure moral hazard case is

$$V_t^{\mathrm{pm}}(R_0) := \underset{(D^i,\theta^i,Z^i)\in\mathcal{A}^i(R_0)}{\text{ess sup}} \; \mathbb{E}^{\mathbb{P}^{k^\star}}\left[\int_{t\wedge\tau}^{\tau}(I-N_s)\mu\mathrm{d}s - \mathrm{d}D_s^i\middle|\mathcal{G}_t\right],$$

where the set of admissible contracts $\mathcal{A}^i(R_0)$ is defined by

$$\mathcal{A}^i(R_0) := \left\{(\theta^i, D^i, Z^i),\ \text{s.t.}\ \ U_0^i(\theta^i, D^i) \geq R_0\right\}.$$

1 state variable $\Longrightarrow$ associated **system** of HJB equations (**ODE's**).

Define the set $\widehat{\mathcal{V}}_j := [Bj/(r + \widehat{\lambda}_j^{SH}), \infty)$.

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For **every** admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ and $t \geq 0$

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Do all the points from $\widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ correspond to a pair of value functions of the banks under some admissible contract?

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Credible set approach:

- Cvitanić, Wan and Yang (2013).

Definition 1

For any time $t \geq 0$, we define the **credible set** $\mathcal{C}_{I-N_t}$ as the set of $(u^b, u^g) \in \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that there exists some admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ satisfying $U_t^b(\theta, D) = u^b$, $U_t^g(\theta, D) = u^g$ and $(U_s^b(\theta, D), U_s^g(\theta, D)) \in \widehat{\mathcal{V}}_{I-N_s} \times \widehat{\mathcal{V}}_{I-N_s}$ for every $s \in [t, \tau)$, $\mathbb{P} - a.s.$

We denote by $\mathfrak{U}_t(u^b)$ the largest value $U_t^g(\theta, D)$ that the good bank can obtain from all the contracts (θ, D) under which the value of the bad bank is u^b. We also denote the lowest one by $\mathfrak{L}_t(u^b)$.

- $\mathfrak{L}_t(u^b)$ can be simply obtained.
- $\mathfrak{U}_t(u^b)$ solves a stochastic control problem
  $\Longrightarrow$ associated **system** of HJB equations (**ODE's**).

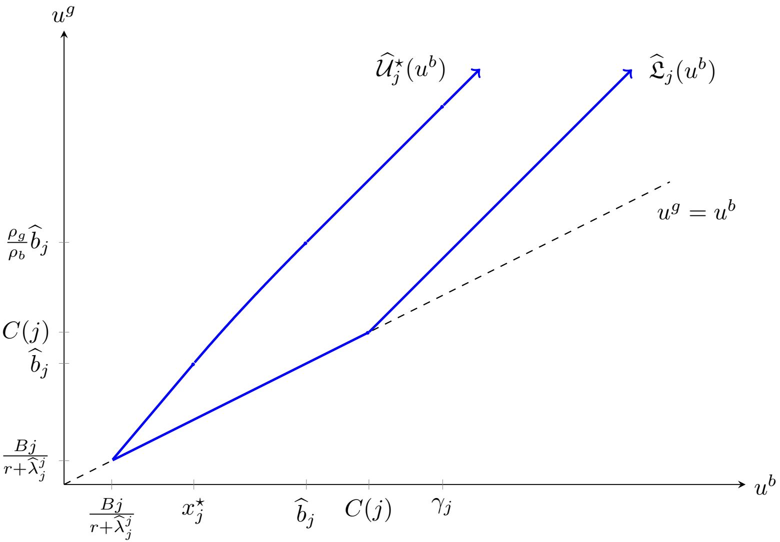

Figure : Credible set with j loans left.

The **investor** designs two contracts:

$$\Psi_g = (D^g, \theta^g),\ \ \Psi_b = (D^b, \theta^b).$$

Each one of these contracts will depend on two state variables, due to the **temptation processes** of each agent.

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Contract designed for the good agent:

$$\mathrm{d}U_t^g(\theta^g, D^g) = \Big(rU_t^g(\theta^g, D^g) - Bk_t^{\star,g} + \lambda_t^{k^{\star,g}} Z_t^g \cdot (1, 1-\theta_t^g)^\top \Big) \,\mathrm{d}t \\ - \rho_g \mathrm{d}D_t^g - Z_t^g \cdot \mathrm{d}\widetilde{M}_t^g, \tag{2}$$

$$\mathrm{d}U_t^{b,c}(\theta^g, D^g) = \Big(rU_t^{b,c}(\theta^g, D^g) - Bk_t^{\star,b,c} + \lambda_t^{k^{\star,b,c}} Z_t^{b,c} \cdot (1, 1-\theta_t^g)^\top \Big) \,\mathrm{d}t \\ - \rho_g \mathrm{d}D_t^g - Z_t^{b,c} \cdot \mathrm{d}\widetilde{M}_t^g. \tag{3}$$

- Value function of the **investor** on the lower boundary:
  $\Longrightarrow$ Obtained from the properties of the lower boundary.
- Value function of the **investor** on the upper boundary:
  $\Longrightarrow$ associated **system** of HJB equations (**ODE's**).
- Value function of the **investor** on the interior of the credible set:
  $\Longrightarrow$ associated **system** of HJB equations (**PDE's**).

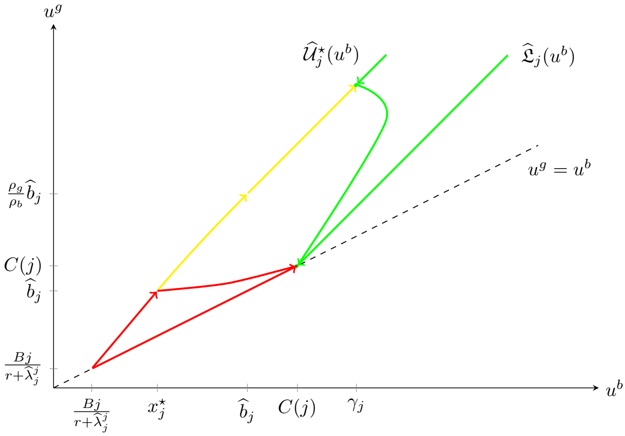

Figure : Optimal contract for the good agent.

- the bank is paid and the project is maintained.
- the bank is not paid and the project is liquidated.
- intermediate situations.

The value of the **investor** is given by

$$v_0 = \sup_{\{R_0 \leq u^b, u^{b,c} \leq u^b, u^{g,c} \leq u^g\}} p_g \widehat{V}_I^g(u^{b,c}, u^g) + p_b \widehat{V}_I^b(u^b, u^{g,c}).$$

$$\alpha_t^{j,i} := \alpha_{I-N_t}\left(1 + \left(1 - \mathrm{e}_t^{j,i}\right)\varepsilon\right),$$

$$\lambda_t^{k^i} := \sum_{j=1}^{I-N_t} \alpha_t^{j,i} = \alpha_{I-N_t}\left(I - N_t + \varepsilon k_t^i\right).$$

Let us then define some family of concave functions, unique solutions to the following system of ODEs

$$\begin{cases} \left(ru + \widehat{\lambda}_j^0 \widehat{b}_j\right)(v_j^i)'(u) + j\mu - \widehat{\lambda}_j^0 \left(v_j^i(u) - \dfrac{u - \widehat{b}_j}{\widehat{b}_{j-1}} v_{j-1}^i(\widehat{b}_{j-1})\right) = 0,\ u \in \left(\widehat{b}_j, \widehat{b}_j + \widehat{b}_{j-1}\right], \\ \left(ru + \widehat{\lambda}_j^0 \widehat{b}_j\right)(v_j^i)'(u) + j\mu - \widehat{\lambda}_j^0 \left(v_j^i(u) - v_{j-1}^i(u - \widehat{b}_j)\right) = 0, u \in \left(\widehat{b}_j + \widehat{b}_{j-1}, \gamma_j^i\right], \\ \rho_i (v_j^i)'(u) + 1 = 0, u > \gamma_j^i, \end{cases} \tag{4}$$

with initial values $\gamma_1^i := \widehat{b}_1$ and

$$v_1^i(u) := \overline{v}_1^i - \frac{1}{\rho_i}(u - \widehat{b}_1), u \geq \widehat{b}_1,\ \overline{v}_1^i := \frac{\mu}{\widehat{\lambda}_1^0} - \frac{\widehat{b}_1(r + \widehat{\lambda}_1^0)}{\rho_i \widehat{\lambda}_1^0},$$

and where for $j \geq 2$, γ_j^i is defined recursively by $r/\widehat{\lambda}_j^0 - 1 \in \partial v_{j-1}^i(\gamma_j^i - \widehat{b}_j)$, where ∂v_{j-1}^i is the super–differential of the concave function v_{j-1}^i.