Moral Hazard and mean field type interactions: A tale of a Principal and many Agents

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Workshop on Variational and Stochastic Analysis.
CMM-Universidad de Chile, March 2017.

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Problem: The Principal is potentially imperfectly informed about the actions of the Agent which impact her wealth (the output).

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Problem: The Principal is potentially imperfectly informed about the actions of the Agent which impact her wealth (the output).

Goal: Design a contract that maximises the utility of the Principal under constraints.

Examples

- Optimal remuneration of an employee,
- How regulators with imperfect information and limited policy instruments can motivate firms to reduce pollution,
- How a company can optimally compensate its executives,
- How banks achieve optimal securitization of mortgage loans
- How investors should pay their portfolio managers
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Here we focus on moral hazard: the Principal does not control the action provides by her Agent.

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A Stackelberg-like equilibrium between the Principal and the Agent:

- compute the best-reaction function of the Agent given a contract
- determine his corresponding optimal effort
- use this in the utility function of the Principal to maximise over all contracts.

The Holmström-Milgrom problem

Holmström-Milgrom (1985). Weak formulation of the problem.

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- Fix a contract ξ. The Agent compute his best reaction effort given ξ. He solves (exponential utilities)

$$U_0^A(\xi) := \sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \underbrace{\left(\xi - \int_0^T k(a_s) ds \right)}_{\text{salary - cost of his effort}} \right]. \tag{1}$$

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- Martingale representation Theorem:

"(1) $\Longleftrightarrow$ solving a **Backward SDE** with a unique solution (Y, Z) ",

$$Y_t = \xi + \int_t^T \left(-\frac{R_A}{2} |Z_s|^2 + \sup_a \{ b(s, X_s, a_s) Z_s - k(a_s) \} \right) ds - \int_t^T Z_s dW_s$$

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$$U_0^A(\xi) = -e^{-R_A Y_0}, \quad \text{optimal effort: } a^\star(Z).$$

We get the following representation for admissible contract ξ

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The Principal's Problem:

$$U_0^P = \sup_{\xi,\ U_0^A(\xi) \geqslant R_0} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(X_T - \xi) \right],$$

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A stochastic control problem with

- State variables: the output X and the value function of the Agent,
- controlled variable: Z and Y_0.

$\hookrightarrow$ HJB equation associated with it

see Sannikov (07'), Cvitanić, Possamaï, Touzi (14', 17').

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Multi Agents models.

- **One period model**: Holmström; Mookherjee; Green and Stokey; Harris, Kriebel and Raviv; Nalebuff and Stiglitz (among others)
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The Principal problem: a standard stochastic control problem.

$2N$ state variables: the outputs controlled by the Agents and their continuation utilities.

The problem under interest

What happens when N goes to $+\infty$?

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What happens when N goes to $+\infty$?

- Related to Mean Field Game theory. Introduced by Lasry and Lions; Huang, Caines and Malhamé (06',07').
- Typical situations: how a firm should provide electricity to a large population, how city planners should regulate a heavy traffic or a crowd of people.
- Systemic risk: study large number of banks and the underlying contagion phenomenon. See for instance Carmona, Fouque and Sun; Garnier, Papanicolaou and Yan; Fouque and Langsam...

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

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$$\frac{d\mathbb{P}^{\mu,q,\alpha}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \sigma_t^{-1}(X) b(t, X, \mu, q_t, \alpha_t) dW_t\right).$$

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The Agent problem as a MFG problem

- Stackelberg equilibrium: For given ξ, and μ and q, the representative Agent has to solve

$$U_0^A(\mu, q, \xi) := \sup_{a \in \mathcal{A}} \underbrace{\mathbb{E}^{\mathbb{P}^{\mu,q,a}} \left[\xi - \int_0^T k_s(X, \mu, q_s, a_s) ds \right]}_{=: u_0^A(\mu, q, \xi, a)} .$$

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$\hookrightarrow$ Find a Mean field equilibrium.

- Solve the Mean Field Game problem: $(a^\star, \mu, q)$ such that

$$\textbf{(MFG)}(\xi) \begin{cases} u_0^A(\mu, q, \xi, a^\star) = U_0^A(\mu, q, \xi), \\ \mathbb{P}^{a^\star, \mu, q} \circ (X)^{-1} = \mu \\ \mathbb{P}^{a^\star, \mu, q} \circ (a_t^\star)^{-1} = q_t. \end{cases}$$

See the works of Carmona and Lacker; Lacker; Carmona, Delarue and Lacker...

We now consider the following system which is intimately related to mean-field FBSDE

$$
\textbf{(MF-BSDE)}(\xi)\begin{cases} Y_t = \xi + \int_t^T \sup_{\alpha}\left(b(s,X,\mu,q_s,\alpha)Z_s - k_s(X,\mu,q_s,\alpha)\right)ds \\ \qquad - \int_t^T Z_s dX_s, \\ \mathbb{P}^{\alpha^\star(X,Z,\mu,q),\mu,q} \circ (X)^{-1} = \mu, \\ \mathbb{P}^{\alpha^\star(X,Z,\mu,q),\mu,q} \circ (\alpha^\star_t)^{-1} = q_t. \end{cases}
$$

Similar studies on MF-BSDEs: Carmona and Delarue; Buckdahn, Djehiche, Li, and Peng; Li and Luo...

$$\text{" Solve } (\textbf{MFG})(\xi) \Longleftrightarrow \text{ Solve } (\textbf{MF-BSDE})(\xi)\text{"}$$

The Agent problem: an other story of BSDEs

" Solve (**MFG**)(ξ) $\Longleftrightarrow$ Solve (**MF-BSDE**)(ξ)"

Theorem (Elie, M., Possamaï (16'))

- *Let ξ be such that* (**MFG**)(ξ) *admits a solution* $(\mu^\star, q^\star, a^\star)$. *Then there exists a solution* $(Y^\star, Z^\star, \mu, q)$ *to* (**MF-BSDE**)(ξ) *and* $a^\star$ *is a maximiser which provides an optimal effort. We thus have*

$$\xi = Y_0^\star - \int_0^T \left(b(s, X, \mu, q_s, a_s^\star) Z_s^\star - k_s(X, \mu, q_s, a_s^\star)\right) ds + \int_0^T Z_s^\star dX_s.$$

- *Conversely, if there exists a solution* $(Y^\star, Z^\star, \mu, q)$ *to* (**MF-BSDE**)(ξ) *then* (**MFG**)(ξ) *has a solution.*

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Let us denote Ξ the set of admissible contracts ξ such that (**MFG**)(ξ) has a solution.

A fundamental characterization of Ξ

Let $Y_0 \in \mathbb{R}$ and Z predictable + integrability conditions. Let $\alpha^{\star,Z}$ be any maximiser of the generator of (**MF-BSDE**)(ξ). Consider the controlled McKean-Vlasov system:

$$\textbf{(SDE)}_{MV} \begin{cases} X_t = x + \int_0^t b(s, X, \mu, q_s, \alpha_s^{\star,Z}) ds + \int_0^t \sigma_s(X) dW_s^{\mu,q,\alpha_\cdot^{\star,Z}}, \\ Y_t^{Y_0,Z} = Y_0 + \int_0^t k_s(X, \mu, q_s, \alpha_s^{\star,Z}) ds + \int_0^t Z_s \sigma_s(X) dW_s^{\mu,q,\alpha^{\star,Z}}, \\ \mu = \mathbb{P}^{\mu,q,\alpha^\star(\cdot,X,Z_\cdot,\mu,q_\cdot)} \circ X^{-1}, \\ q_t = \mathbb{P}^{\mu,q,\alpha^{\star,Z}} \circ (\alpha_t^{\star,Z})^{-1}. \end{cases}$$

Theorem (Elie, M., Possamaï (16'))

$$\Xi = \left\{ Y_T^{Y_0,Z},\ Y_0 \geqslant R_0,\ Z \textit{ sufficiently integrable...} \right\}.$$

The Principal problem: a non standard stochastic control problem

$$U_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} \left[X_T - \xi \right]$$

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On the admissibility of the contract (motivated by examples):

- Assume that the HJB equation has a solution with an optimal $z^\star$ for instance.
- We check that for this $z^\star$, the system $(\mathbf{SDE})_{MV}$ has indeed a solution and then $\xi^\star := Y_T^{R_0, z^\star}$ will be an optimal admissible contract.

Application: mean dependency and variance penalisation

$$b(s,x,\mu,q,a) := a + \alpha x + \beta_1 \int_{\mathbb{R}} x d\mu_s(x) + \beta_2 \int_{\mathbb{R}} x dq_s(x) - \gamma V_\mu(s),$$

$$V_\mu(s) := \int_{\mathbb{R}} |x|^2 d\mu_s(x) + \left| \int_{\mathbb{R}} x d\mu_s(x) \right|^2, \quad k(a) = c\frac{|a|^n}{n}.$$

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Theorem (Elie, M. , Possamaï (16'))

The optimal contract for the problem of the Principal is

$$\xi^\star := \delta + \beta_1(1+\beta_2) \int_0^T e^{(\alpha+\beta_1)(T-t)} X_t dt + (1+\beta_2)\left(X_T - e^{(\alpha+\beta_1)T} X_0 \right)$$

for some constant δ explicitly given and the associated optimal effort of the Agent is

$$a_u^\star := (1+\beta_2)^{\frac{1}{n-1}} \left(\frac{e^{(\alpha+\beta_1)(T-u)}}{c} \right)^{\frac{1}{n-1}}, u \in [0,T].$$

Economic interpretations and extension.

$$dX_t = (a_t + \alpha X_t + \beta_1 \mathbb{E}^\star[X_t] + \beta_2 \mathbb{E}^\star[a_t] - \gamma \mathrm{Var}^\star[X_t])\, dt + \sigma dW_t^\star.$$

	c	α	β_1	β_2	γ
Expectation of ξ^*	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$	$=$
Variance of ξ^*	$=$	$\nearrow$	$\nearrow$	$\nearrow$	$=$
Fixed salary part δ	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$
Optimal effort of A	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$	$=$

- $a^\star$ is increasing with β_2. No free-rider type behaviour.
- the volatility of the project, as well as the volatility penalisation with γ have no impact on $a^\star$ (not realistic, Agent is risk neutral).
- If $\beta_1 > 0$ the optimal contract is not Markovian.
- Compensation between fixed part of the salary and its average value.
- δ increases with γ in order to compensate the negative effect on the dynamics of the project of the dispersion of the results of all the projects of the company.

Extension to risk averse Principal.

$$U_0^P = \sup_Z \left(\mathbb{E}^{\mathbb{P}^\star}\left[X_T - \xi \right] - \lambda_X \mathrm{Var}_{\mathbb{P}^\star}(X_T) - \lambda_\xi \mathrm{Var}_{\mathbb{P}^\star}(\xi) - \lambda_{X\xi} \mathrm{Var}_{\mathbb{P}^\star}(X_T - \xi) \right),$$

We take $c(a) := c\frac{|a|^2}{2}$.

Theorem (Elie, M., Possamaï (17'))

The optimal contract for the problem of the Principal is

$$\xi^\star = C + \beta_1 \frac{1+\beta_2}{1+2(\lambda_\xi + \lambda_{X\xi})c\sigma^2} \int_0^T e^{\kappa(T-t)} X_t dt + \frac{1+\beta_2 + 2c\lambda_{X\xi}\sigma^2}{1+2(\lambda_\xi + \lambda_{X\xi})c\sigma^2} X_T,$$

and the associated optimal effort of the Agent is

$$a_t^\star = \frac{1+\beta_2}{c\left(1+2(\lambda_\xi + \lambda_{X\xi})c\sigma^2\right)} e^{(\alpha+\beta_1)(T-t)} + \frac{2\lambda_{X\xi}\sigma^2}{1+2(\lambda_\xi + \lambda_{X\xi})c\sigma^2} e^{\alpha(T-t)}$$

Interpretation.

	λ_X	λ_ξ	$\lambda_{X\xi}$
Optimal effort of the Agent	$=$	$\searrow$	$\searrow$
Expectation of ξ^*	$=$	$\searrow$	$\searrow$
Variance of ξ^*	$=$	$\searrow$	$\searrow$

- No effects with λ_X since $a^\star$ is deterministic.
- The optimal contract provides incentives for Agents to provide less efforts, to reduce the variance of the output.

Link with the N-agents model.

Let $(t, x, a) \in [0, T] \times \mathbb{R}^N \times A^N$,

$$b^N\big(t, x, \mu^N(x), a\big) := a + \alpha x + \beta_1 \int_{\mathbb{R}^N} w \mu^N(dw),$$

with $\mu^N(x)$ the empirical distribution of x.

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Theorem (Elie, M., Possamaï (16'))

$$a_t^{N,\star} = \exp((\alpha + \beta_1)(T - t))\mathbf{1}_N.$$

In particular, the optimal effort of the ith Agent in the N players model coincides with the optimal effort of the Agent in the mean–field model. The optimal contract $\xi^{N,\star}$ proposed by the Principal is

$$\xi^{N,\star} = R_0^N - \int_0^T \frac{e^{2\kappa(T-t)}}{2}\mathbf{1}_N dt - \int_0^T e^{\kappa(T-t)} B_N X_t^N dt + \int_0^T e^{\kappa(T-t)} dX_t^N,$$

and for any $i \in \{1, \ldots, N\}$ we have

$$\mathbb{P}_N^{a^{N,\star}} \circ \left((\xi^{N,\star})^i\right)^{-1} \xrightarrow[N\to\infty]{\text{weakly}} \mathbb{P}^{a^\star} \circ (\xi^\star)^{-1}.$$

Thank you.