On the Aubin property of solution maps to parameterized variational systems

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Consider the *generalized equation* (GE)

$$0 \in H(p,x) + \hat{N}_\Gamma(x), \tag{1}$$

where $p \in \mathbb{R}^l$ is the *parameter*, $x \in \mathbb{R}^n$ is the *decision variable*, $H : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $\Gamma \subset \mathbb{R}^n$ is a closed set. Denote by $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ the respective *solution map*, i.e.,

$$S(p) = \{x | 0 \in H(p,x) + \hat{N}_\Gamma(x)\} \tag{2}$$

and consider a *reference pair* $(\bar{p}, \bar{x}) \in \operatorname{gph} S$.

Our aim is to derive a workable sharp criterion for the Aubin property of S for the case of non-ample parameterization (i.e. $\nabla_p H(\bar{p}, \bar{x})$ is not surjective).

Such a criterion may be applied, among other things, in

1) post-optimal analysis;
2) treatment of MPECs and EPECs, where (1) arises among the constraints.

Outline

(i) Selected tools of variational analysis;
(ii) General criterion for implicitly defined multifunctions;
(iii) New calculus rules;
(iv) The new criterion for the considered S;
(v) Conclusion.

Definition

Given a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$, we define

(i) the *tangent (Bouligand) cone* to A at $\bar{x}$ by

$$T_A(\bar{x}) := \{h \in \mathbb{R}^n | \exists h_i \to h, \vartheta_i \searrow 0 : \bar{x} + \vartheta_i h_i \in A \forall i\};$$

(ii) the *regular (Fréchet) normal cone* to A at $\bar{x}$ by

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ;$$

(iii) the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$ by

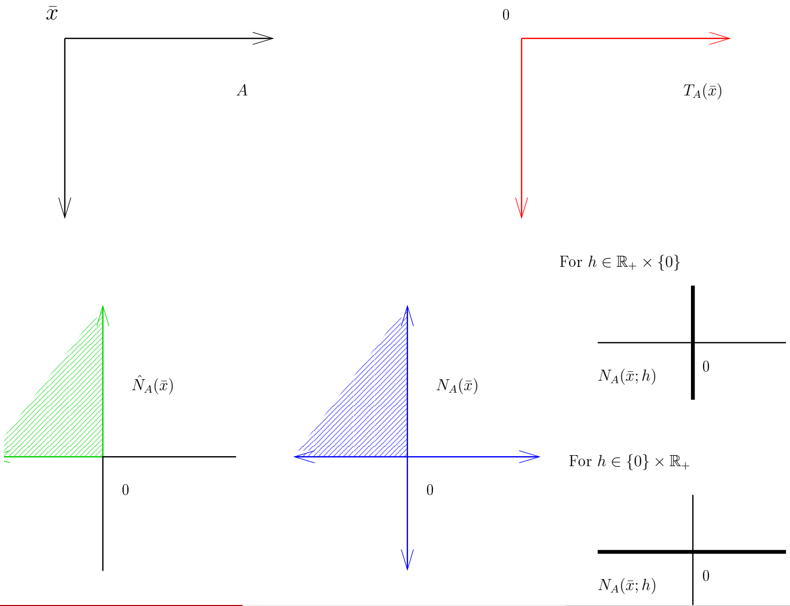
$$N_A(\bar{x}) := \{\xi \in \mathbb{R}^n | \exists x_i \xrightarrow{A} \bar{x}, \xi_i \to \xi : \xi_i \in \widehat{N}_A(x_i) \forall i\}.$$

(iv) Finally, given a direction $h \in \mathbb{R}^n$, the cone

$$N_A(\bar{x}; h) := \{\xi \in \mathbb{R}^n | \exists h_i \to h, \vartheta_i \searrow 0, \xi_i \to \xi : \xi_i \in \widehat{N}_A(\bar{x} + \vartheta_i h_i) \forall i\}$$

is called the *directional limiting normal cone* to A at $\bar{x}$ in the direction h.

Ad (i) Example



$\bar{x}$
A
0
$T_A(\bar{x})$
For $h \in \mathbb{R}_+ \times \{0\}$
$N_A(\bar{x}; h)$
0
$\hat{N}_A(\bar{x})$
0
$N_A(\bar{x})$
0
For $h \in \{0\} \times \mathbb{R}_+$
$N_A(\bar{x}; h)$
0

Definition

Consider a point $(\bar{u}, \bar{v}) \in \operatorname{Gr} F$. Then

(i) the multifunction $DF(\bar{u}, \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^l$, defined by

$$DF(\bar{u}, \bar{v})(h) := \{k \in \mathbb{R}^l | (h, k) \in T_{\mathrm{gph}\, F}(\bar{u}, \bar{v})\}, h \in \mathbb{R}^n,$$

is called the *graphical derivative* of F at $(\bar{u}, \bar{v})$;

(ii) the multifunction $\hat{D}^* F(\bar{u}, \bar{v}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$, defined by

$$\hat{D}^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in \hat{N}_{\mathrm{gph}\, F}(\bar{u}, \bar{v})\}, v^* \in \mathbb{R}^l,$$

is called the *regular (Fréchet) coderivative* of F at $(\bar{u}, \bar{v})$.

(iii) the multifunction $D^* F(\bar{u}, \bar{v}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\mathrm{gph}\, F}(\bar{u}, \bar{v})\}, v^* \in \mathbb{R}^l,$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{u}, \bar{v})$.

(iv) Finally, given a pair of directions $(h, k) \in \mathbb{R}^n \times \mathbb{R}^l$, the multifunction $D^* F((\bar{u}, \bar{v}); (h, k)) : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F((\bar{u}, \bar{v}); (h, k))(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\mathrm{gph}\, F}((\bar{u}, \bar{v}); (h, k))\}, v^* \in \mathbb{R}^l, \tag{3}$$

is called the *directional limiting coderivative* of F at $(\bar{u}, \bar{v})$ in direction (h, k).

Consider a multifunction $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ and a point $(\bar{v}, \bar{u}) \in \operatorname{gph} S$.

1) S has the *Aubin property* around $(\bar{v}, \bar{u})$, provided $\exists$ neighborhoods V, U of $\bar{v}, \bar{u}$, respectively, and a constant $\kappa \geq 0$ such that

$$S(v') \cap U \subset S(v) + \kappa \|v - v'\| \mathbb{B}_{\mathbb{R}^l} \text{ for all } v, v' \in V.$$

2) S is *calm* at $(\bar{v}, \bar{u})$, provided $\exists$ a neighborhood U of $\bar{u}$, and a constant $\kappa \geq 0$ such that

$$S(v) \cap U \subset S(\bar{v}) + \kappa \|v - \bar{v}\| \mathbb{B}_{\mathbb{R}^l} \text{ for all } v \in \mathbb{R}^l.$$

It is well-known that S has the Aubin property around $(\bar{v}, \bar{u})$ iff $F := S^{-1}$ is *metrically regular* at $(\bar{u}, \bar{v})$, i.e., $\exists$ neighborhoods U, V of $\bar{u}, \bar{v}$, respectively, and a constant $\kappa \geq 0$ such that

$$d(u, F^{-1}(v)) \leq \kappa\, d(v, F(u)) \text{ for all } u \in U, v \in V.$$

Likewise S is calm at $(\bar{v}, \bar{u})$ iff $F := S^{-1}$ is *metrically subregular* at $(\bar{u}, \bar{v})$, i.e., $\exists$ a neighborhood U of $\bar{u}$ and a constant $\kappa \geq 0$ such that

$$d(u, F^{-1}(\bar{v})) \leq \kappa\, d(\bar{v}, F(u)) \text{ for all } u \in U.$$

Ad (ii) General criterion

Denote

$$M(p, x) := H(p, x) + \hat{N}_\Gamma(x). \tag{4}$$

Theorem 1 ([GO16]).

Assume that

(i) $\{u | 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\} \neq \emptyset$ for all $q \in \mathbb{R}^l$;

(ii) M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;

(iii) For nonzero $(q, u) \in \mathbb{R}^l \times \mathbb{R}^n$ satisfying $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ one has the implication

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \Rightarrow q^* = 0.$$

Then S has the Aubin property around $(\bar{p}, \bar{x})$ and for any $q \in \mathbb{R}^l$

$$DS(\bar{p}, \bar{x})(q) = \{u | u \in DM(\bar{p}, \bar{x}, 0)(q, u)\}.$$

The above assertion remains true provided the assumptions (ii), (iii) are replaced by

(iv) For every nonzero $(q, u) \in \mathbb{R}^l \times \mathbb{R}^n$ satisfying $0 \in D^*M(\bar{p}, \bar{x}, 0)(q, u)$ one has the implication

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \Rightarrow q^* = 0, v^* = 0.$$

For M given by (4) we obtain directly from the definitions that for any directions $(q,u)\in\mathbb{R}^l\times\mathbb{R}^n$

$$\begin{aligned}&DM(\bar p,\bar x,0)(q,u)\\&=\nabla_pH(\bar p,\bar x)q+\nabla_xH(\bar p,\bar x)u+D\hat N_\Gamma(\bar x,-H(\bar p,\bar x))(u,-\nabla_pH(\bar p,\bar x)q-\nabla_xH(\bar p,\bar x)u),\end{aligned}$$

and for any $v^*\in\mathbb{R}^n$

$$\begin{aligned}&D^*M((\bar p,\bar x,0)(q,u,0))(v^*)\\&=\begin{bmatrix}\nabla_pH(\bar p,\bar x)^Tv^*\\ \nabla_pH(\bar p,\bar x)^Tv^*+D^*\hat N_\Gamma((\bar x,-H(\bar p,\bar x));(u,-\nabla_pH(\bar p,\bar x)q-\nabla_xH(\bar p,\bar x)u))(v^*)\end{bmatrix}.\end{aligned}$$

Assumptions

In what follows we will assume that $\Gamma = g^{-1}(D)$, where $g : \mathbb{R}^n \to \mathbb{R}^s$ is twice continuously differentiable and $D \subset \mathbb{R}^s$ is nonempty and closed. Further we impose the assumptions

A1: $\exists$ a closed set $\Theta \in \mathbb{R}^d$ along with a C^2 mapping $h : \mathbb{R}^s \to \mathbb{R}^d$ and a neighborhood V of $g(\bar{x})$ such that

1) $\nabla h(g(\bar{x}))$ is surjective, and
2) $D \cap V = \{z \in V | h(z) \in \Theta\}$.

A2: $\mathrm{rge} \nabla g(\bar{x}) + \ker \nabla h(g(\bar{x})) = \mathbb{R}^l$.

Remark

Under (A1), (A2) the mapping $\hat{N}_\Gamma(\cdot)$ has a closed graph around $\bar{x}$ and, given $x^* \in \hat{N}_\Gamma(x)$ with $x \in \Gamma$ close to $\bar{x}$, $\exists$ a unique $\lambda \in N_D(g(x))$ such that

$$x^* = \nabla g(x)^T \lambda.$$

Theorem 2.

Let (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying

$$\nabla g(\bar{x})^T \bar{\lambda} = \bar{x}^*, \bar{\lambda} \in \hat{N}_D(g(\bar{x})). \tag{5}$$

Then

$$T_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*) =$$
$$\{(u, u^*) | \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}), u^* = \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u\}.$$

Corollary.

In the setting of Theorem 2 assume that D is convex and the projection operator P_D is directionally differentiable at $g(\bar{x})$. Then

$$(\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}) \Leftrightarrow \nabla g(\bar{x})u = P_D'(g(\bar{x}) + \bar{\lambda}; \nabla g(\bar{x})u + \xi). \tag{6}$$

Definition ([MOR15b]).

A convex set $\Xi \subset \mathbb{R}^s$ satisfies the *projection derivation condition* (PDC) at $\bar{z} \in \Xi$ if we have

$$P'_\Xi(\bar{z}+b;h) = P_K(h)\ \forall\ b \in N_\Xi(\bar{z}) \text{ and } h \in \mathbb{R}^s,$$

where $K = T_\Xi(\bar{z}) \cap [b]^\perp$.

Under PDC condition, posed on D at $g(\bar{x})$, (6) amounts to

$$\nabla g(\bar{x})u = P_{\mathcal{K}}(\nabla g(\bar{x})u + \xi),$$

where $\mathcal{K} = T_D(g(\bar{x})) \cap [\bar{\lambda}]^\perp$.

Theorem 3 ([MOR15a]).

In the setting of Theorem 2

$$\hat{N}_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*) = \left.\begin{array}{r}\{(w^*,w) | \exists v^* : (v^*, \nabla g(\bar{x})w) \in \hat{N}_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda})\} \\ w^* = -\nabla^2\langle\bar{\lambda},g\rangle(\bar{x})w + \nabla g(\bar{x})^T v^*\end{array}\right\} \tag{7}$$

and for any $w \in \mathbb{R}^n$

$$\hat{D}^*\hat{N}_\Gamma(\bar{x},\bar{x}^*)(w) = \nabla^2\langle\bar{\lambda},g\rangle(\bar{x})w + \nabla g(\bar{x})^T\hat{D}^*\hat{N}_D(g(\bar{x}),\bar{\lambda})(\nabla g(\bar{x})w).$$

Theorem 4.

In the setting of Theorem 2 assume that we are given a pair of directions $(u, u^*) \in T_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*)$. Then for any $w \in \mathbb{R}^n$ one has

$$D^* \hat{N}_\Gamma((\bar{x}, \bar{x}^*); (u, u^*))(w) \\ = \nabla^2 \langle \bar{\lambda}, g \rangle(\bar{x}) w + \nabla g(\bar{x})^T D^* \hat{N}_D((g)(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x}) u, \xi))(\nabla g(\bar{x}) w),$$

where $\xi \in \mathbb{R}^s$ is the unique vector satisfying the relations

$$(\nabla g(\bar{x}) u, \bar{\xi}) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}), u^* = \nabla g(\bar{x})^T \bar{\xi} + \nabla^2 \langle \bar{\lambda}, g \rangle(\bar{x}) u.$$

Remark.

Setting $(u, u^*) = (0, 0)$, we recover the formula from [OR11]

$$D^* \hat{N}_\Gamma(\bar{x}, \bar{x}^*)(w) = \nabla^2 \langle \bar{\lambda}, g \rangle(\bar{x}) w + \nabla g(\bar{x})^T D^* N_D(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x}) w).$$

Ad(iv) The new criterion

Theorem 5.

Let $0 \in H(\bar{p},\bar{x}) + \hat{N}_\Gamma(\bar{x})$, assumptions (A1), (A2) be fulfilled and let $\bar{\lambda}$ be the unique multiplier satisfying

$$0 = \mathcal{L}(\bar{p},\bar{x},\bar{\lambda}) := H(\bar{p},\bar{x}) + \nabla g(\bar{x})^T\bar{\lambda}, \quad \bar{\lambda} \in \hat{N}_D(g(\bar{x})).$$

Further assume that

(i) for any $q \in \mathbb{R}^l$ the variational system

$$\begin{aligned} 0 = \quad & \nabla_p H(\bar{p},\bar{x})q + \nabla_x \mathcal{L}(\bar{p},\bar{x},\bar{\lambda})u + \nabla g(\bar{x})^T\xi \\ & (\nabla g(\bar{x})u,\xi) \in T_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda}) \end{aligned} \tag{8}$$

has a solution $(u,\xi) \in \mathbb{R}^n \times \mathbb{R}^s$;

(ii) M is metrically subregular at $(\bar{p},\bar{x})$, and

(iii) for any nonzero (q,u), satisfying (with a corresponding unique ξ) relations (8), one has the implication

$$\begin{aligned} 0 \in \quad & \nabla_x \mathcal{L}(\bar{p},\bar{x},\bar{\lambda})^T u^* + \nabla g(\bar{x})^T D^*\hat{N}_D((g(\bar{x}),\bar{\lambda});(\nabla g(\bar{x})u,\bar{\xi}))(\nabla g(\bar{x})v^*) \\ & \Rightarrow v^* \in \ker \nabla_p H(\bar{p},\bar{x})^T. \end{aligned} \tag{9}$$

Ad(iv) The new criterion

Then S has the Aubin property around $(\bar{p}, \bar{x})$ and for any $q \in \mathbb{R}^l$

$$DS(\bar{p}, \bar{x})(q)$$

$$= \{u | \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}), 0 = \nabla_p H(\bar{p}, \bar{x})q + \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda})^T u + \nabla g(\bar{x})^T \xi\}.$$

The above assertions remain true provided assumptions (ii), (iii) are replaced by

(iv) for any nonzero (q, u), satisfying (with a corresponding unique ξ) relations (8), the left-hand side of (9) implies $v^* = 0$.

Remark

Using the Mordukhovich criterion and the standard calculus with limiting objects, implication (9) is replaced by [cf. M06]

$$0 \in \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda})^T v^* + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x}) v^*) \Rightarrow v^* \in \ker \nabla_p H(\bar{p}, \bar{\lambda})^T. \quad (10)$$

Example.

$$H(p,x)=\begin{bmatrix} x_1-p \\ -x_2 \end{bmatrix}, g(x)=\begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix} \textit{ and}$$

D is the Lorentz cone in $\mathbb{R}^2$. Further, $(\bar{p},\bar{x})=(0,(0,0))$. Here, (A1), (A2), PDC at $g(\bar{x})$ and assumption (ii) are fulfilled. So, we may proceed as follows:

1° Analyze the appropriately simplified variational system (8) to verify assumption (i) and to compute the critical directions (q,u) together with the respective vectors ξ;

2° To examine implication (9) for all triples (q,u,ξ). This step requires the computation of the directional limiting coderivative of the projection onto D.

In this way we conclude that all conditions of Theorem 5 are fulfilled and hence the respective S does have the Aubin property around $(\bar{p},\bar{x})$. However, implication (10) does not hold. △

Ad(v). Conclusion

We investigated the Aubin property of a class of parameterized GEs. To this purpose we derived a new formula for the graphical derivative of $\hat{N}_\Gamma$ for the considered Γ under weakened assumptions. Further, we have established a chain rule for the directional limiting normal cone to Γ. On the basis of these results we obtained eventually a new sharp criterion which may be used for a broad class of constraint sets Γ provided the parameterizations are non-ample. This class includes, for instance, conic programs with Lorentz or Löwner cones.

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