Regular processes and duality

Teemu Pennanen
King's College London

Ari-Pekka Perkkiö
LMU Munich

Introduction

- We give a functional analytic proof of the main theorem of [Bismut, 1978], which identifies regular process as the optional projections of continuous processes.
- Besides being simpler, our proof
  - identifies the Banach dual of regular processes with the space of optional random measures of bounded variation
  - generalizes to situations the total variation needs not be essentially bounded.
- The above provides the basis for duality theory of singular stochastic control developed in a follow-up paper.
- The functional analytic setup brings many problems in stochastic analysis within the reach of variational analysis.

Introduction

- Rockafellar, *Conjugate duality and optimization*, SIAM, 1974.
- Rockafellar, *Conjugate convex functions in optimal control and the calculus of variations*, JMAA, 1970.
- Bismut, *Conjugate convex functions in optimal stochastic control*, JMAA, 1973.
- Rockafellar, *Dual problems of Lagrange for arcs of bounded variation*, in Calculus of variations and control theory, 1975
- Bismut, *Régularité et continuité des processus*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete (Probability theory and related fields), 1978.
- Pennanen and Perkkiö, *Convex integral functionals of regular processes*, submitted.

Raw cadlag processes

- Equipped with the supremum norm, the space of cadlag functions D on $[0,T]$ is a Banach space.
- The dual of D can be identified with $M \times \tilde{M}$, where M is the space of Radon measures on $[0,T]$ and $\tilde{M} \subset M$ is the space of purely atomic measures.
- The dual norm is $\|u\|_{TV} + \|\tilde{u}\|_{TV}$.
- Indeed, for every continuous linear functional on D there is a unique $(u,\tilde{u}) \in M \times \tilde{M}$ with

$$\langle y, (u,\tilde{u})\rangle := \int y du + \int y_- d\tilde{u}.$$

- The space C of continuous functions on $[0,T]$ is paired similarly with M (Riesz representation theorem).

Raw cadlag processes

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{Y}$ be a Banach space of random variables.
- We assume that the dual of $\mathcal{Y}$ can be identified with another space $\mathcal{U}$ of random variables under the pairing

$$\langle y, u \rangle := E(y \cdot u).$$

- Examples include $\mathcal{Y} = L^p$ with $p < \infty$ or $\mathcal{Y} =$ an Orlicz space with a Young whose conjugate satisfies the Δ_2-condition.
- More generally, one can allow for Fréchet spaces of random variables but, for simplicity, we won't consider that here.

Raw cadlag processes

- Let $\mathcal{Y}(D) := \{y \in L^0(D) \mid \|y\|_D \in \mathcal{Y}\}$ and

  $$\mathcal{U}(M \times \tilde{M}) := \{(u, \tilde{u}) \in L^0(M \times \tilde{M}) \mid \|u\|_{TV} + \|\tilde{u}\|_{TV} \in \mathcal{U}\}.$$

  where $L^0(D)$ and $L^0(M \times \tilde{M})$ are the spaces of weakly measurable random variables with values in D and $M \times \tilde{M}$.

Theorem 1 *$\mathcal{Y}(D)$ with $\|y\|_{\mathcal{Y}(D)} := \|\|y\|_D\|_{\mathcal{Y}}$ is a Banach space and its dual can be identified with $\mathcal{U}(M \times \tilde{M})$ through the bilinear form*

$$\langle y, (u, \tilde{u})\rangle := E\left[\int y du + \int y_- d\tilde{u}\right].$$

The dual norm is $\|(u, \tilde{u})\|_{\mathcal{U}(M \times \tilde{M})} = \|\|u\|_{TV} + \|\tilde{u}\|_{TV}\|_{\mathcal{U}}$.

Raw cadlag processes

Let

$$\mathcal{Y}(C) := \{y \in L^0(C) \mid \|y\|_C \in \mathcal{Y}\},$$
$$\mathcal{U}(M) := \{u \in L^0(M) \mid \|u\|_{TV} \in \mathcal{U}\}.$$

Corollary 2 *$\mathcal{Y}(C)$ is a Banach space and its dual can be identified with $\mathcal{U}(M)$ through the bilinear form*

$$\langle y, u\rangle := E\int y du.$$

The dual norm is

$$\|u\|_{\mathcal{U}(M)} = \|\|u\|_{TV}\|_{\mathcal{U}}.$$

Adapted cadlag processes

- Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration with $\mathcal{F}_0$ complete and $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$.
- The optional [predictable] sigma-algebra on $\Omega \times [0, T]$ is that generated by adapted right-[left]-continuous processes.
- Given $y \in \mathcal{Y}(D)$, there exist an optional process ${}^o y$ with

$$ {}^o y_\tau \mathbb{1}_{\{\tau<\infty\}} = E[y_\tau \mathbb{1}_{\{\tau<\infty\}} \mid \mathcal{F}_\tau] \quad P\text{-a.s.} $$

  for every stopping time τ and a predictable process ${}^p y$ with

$$ {}^p y_\tau \mathbb{1}_{\{\tau<\infty\}} = E\left[y_\tau \mathbb{1}_{\{\tau<\infty\}} \mid \mathcal{F}_{\tau-}\right] \quad P\text{-a.s.} $$

  for every predictable time τ.
- The processes ${}^o y$ and ${}^p y$ are called the optional and predictable projections, respectively, of y.

Lemma 3 *For any $y \in \mathcal{Y}(D)$, we have $({}^o y)_- = {}^p(y_-)$.*

Adapted cadlag processes

- The optional projection is a linear mapping from $\mathcal{Y}(D)$ to the space of adapted cadlag processes.
- $y \in \mathcal{Y}(D)$ does not imply ${}^o y \in \mathcal{Y}(D)$, in general, but the Jensen's inequality $|{}^o y_\tau| \le E[\|y\|_D \,|\, \mathcal{F}_\tau]$ gives

$$\sup_{\tau\in\mathcal{T}} \|{}^o y_\tau\|_{\mathcal{Y}} \le \|y\|_{\mathcal{Y}(D)} \quad \forall y \in \mathcal{Y}(D),$$

  where $\mathcal{T}$ is the set of all stopping times.
- We will denote by $\tilde{\mathcal{D}}^{\mathcal{Y}}$ the (Banach) space of optional cadlag processes such that $\{y_\tau \,|\, \tau \in \mathcal{T}\}$ is bounded in $\mathcal{Y}$:

$$\sup_{\tau\in\mathcal{T}} \|y_\tau\|_{\mathcal{Y}} < \infty.$$

Adapted cadlag processes

A random measure u is optional or predictable, respectively, if, for all bounded measurable processes y,

$$E\int {}^{o}y du = E\int y du,$$
$$E\int {}^{p}y du = E\int y du.$$

Let

$$\hat{\mathcal{M}}^{\mathcal{U}} := \{(u,\tilde{u}) \in \mathcal{U}(M \times \tilde{M}) \mid u \text{ optional, } \tilde{u} \text{ predictable}\}.$$

Adapted cadlag processes

Assumption 1 *The optional projection is a continuous mapping from $\mathcal{Y}(D)$ to a Banach space $\mathcal{D}^{\mathcal{Y}} \subseteq \tilde{\mathcal{D}}^{\mathcal{Y}}$ whose dual can be identified with $\hat{\mathcal{M}}^{\mathcal{U}}$ under the bilinear form*

$$\langle y, (u, \tilde{u})\rangle := E\left[\int y du + \int y_- d\tilde{u}\right].$$

Theorem 4 *Under Assumption 1, the optional projection is a surjection from $\mathcal{Y}(D)$ to $\mathcal{D}^{\mathcal{Y}}$, its adjoint is the embedding of $\hat{\mathcal{M}}^{\mathcal{U}}$ to $\mathcal{U}(M \times \tilde{M})$ and the norm of $\mathcal{D}^{\mathcal{Y}}$ is equivalent to*

$$\|y\|_{\mathcal{D}^{\mathcal{Y}}} := \inf_{z \in \mathcal{Y}(D)} \{\|z\|_{\mathcal{Y}(D)} \mid {}^o z = y\}$$

the polar of which is

$$\|(u, \tilde{u})\|_{\hat{\mathcal{M}}^{\mathcal{U}}} = \|(u, \tilde{u})\|_{\mathcal{U}(M \times \tilde{M})} := \|\|u\|_{TV} + \|\tilde{u}\|_{TV}\|_{\mathcal{U}}.$$

Adapted cadlag processes

Example 5 *Let* $\mathcal{Y} = L^1$*. Then the closure* $\mathcal{D}^{\mathcal{Y}}$ *of* $L^\infty(D)$ *in* $\tilde{\mathcal{D}}^{\mathcal{Y}}$ *with respect to the norm*

$$\|y\|_{\tilde{\mathcal{D}}^{\mathcal{Y}}} := \sup_{\tau \in \mathcal{T}} \|y_\tau\|_{\mathcal{Y}}$$

satisfies Assumption 1. Moreover, $\mathcal{D}^{\mathcal{Y}}$ *contains optional cadlag processes with* $\{y_\tau \,|\, \tau \in \mathcal{T}\}$ *uniformly integrable, and the optional projection from* $\mathcal{Y}(D)$ *to* $\mathcal{D}^{\mathcal{Y}}$ *has norm one.*

Example 6 *If* $\mathcal{Y} = L^p$ *with* $p > 1$*, then the space* $\mathcal{D}^{\mathcal{Y}} := \{y \in \tilde{\mathcal{D}}^{\mathcal{Y}} \,|\, \|y\|_D \in \mathcal{Y}\}$ *endowed with the norm* $\|\|y\|_D\|_{L^p}$ *satisfies Assumption 1.*

Regular processes

Let $\mathcal{R}^{\mathcal{Y}} := \{y \in \mathcal{D}^{\mathcal{Y}} \mid {}^p y = y_-\}$. Following [Bismut, 1978] (where $\mathcal{Y} = L^1$), we call the elements of $\mathcal{R}^{\mathcal{Y}}$ regular processes.

Theorem 7 *Under Assumption 1, the space $\mathcal{R}^{\mathcal{Y}}$ is Banach, its dual can be identified with $\mathcal{M}^{\mathcal{U}}$ through the bilinear form*

$$\langle y, u\rangle = E\int y du,$$

the optional projection is a continuous surjection from $\mathcal{Y}(C)$ to $\mathcal{R}^{\mathcal{Y}}$, its adjoint is the embedding of $\mathcal{M}^{\mathcal{U}}$ to $\mathcal{U}(M)$ and the norm of $\mathcal{R}^{\mathcal{Y}}$ is equivalent to

$$\|y\|_{\mathcal{R}^{\mathcal{Y}}} := \inf_{z\in\mathcal{Y}(C)}\{\|z\|_D) \mid {}^o z = y\},$$

the polars of which are given by $\|u\|_{\mathcal{M}^{\mathcal{U}}} := \|\|u\|_{TV}\|_{\mathcal{U}}$.

Regular processes

Example 8 *If $\mathcal{Y} = L^1$, the space $\mathcal{R}^{\mathcal{Y}}$ coincides with the space of cadlag processes y of class (D) with $y_- = {}^p y$ and its dual is $\mathcal{M}^{L^\infty}$.*

Example 9 *If $\mathcal{Y} = L^p$, the space $\mathcal{R}^{\mathcal{Y}}$ coincides with the space of optional cadlag processes y with $\|y\| \in L^p$ and $y_- = {}^p y$ and its dual is $\mathcal{M}^{L^q}$.*

Example 10 *Let $\mathcal{Y}$ be the Morse-heart of the Orlicz space L^Ψ and $\mathcal{U}$ be the Orlicz space L^Φ for conjugate Young functions Ψ and Φ such that $\Psi(\alpha) > 0$ for $\alpha > 0$ and Φ is Δ_2. Then $\mathcal{R}^{\mathcal{Y}}$ coincides with the space of optional cadlag processes y with $\|y\| \in \mathcal{Y}$ and $y_- = {}^p y$ and its dual is $\mathcal{M}^{\mathcal{U}}$.*

Applications: Optimal stopping

- Let $R \in \mathcal{R}^1$ and consider the optimal stopping problem

$$\text{maximize} \quad ER_\tau \quad \text{over} \quad \tau \in \mathcal{T}.$$

- Defining $\mathcal{C}_e := \{u \in \mathcal{M}^\infty_+ \mid \operatorname{rge} u \in \{0,1\}\}$, we can write this as

$$\text{maximize} \quad \langle R, u\rangle \quad \text{over} \quad u \in \mathcal{C}_e,$$

- Clearly, $\mathcal{C}_e \subset \mathcal{C} := \{u \in \mathcal{M}^\infty_+ \mid \operatorname{rge} u \in [0,1]\}$.

Lemma 11 *The set $\mathcal{C}$ is convex, $\sigma(\mathcal{M}^\infty, \mathcal{R}^1)$-compact and $\mathcal{C}_e$ is the set of its extreme points.*

Applications: Optimal stopping

- Krein–Millman and Banach–Alaoglu then give the existence of optimal stopping for any $R \in \mathcal{R}^1$.
- This extends [Bismut and Skalli, 1977] and [El Karoui, 1981] who considered bounded regular processes.
- Note that u solves the relaxed problem iff $R \in \partial\delta_{\mathcal{C}}(u)$ or equivalently $u \in \partial\sigma_{\mathcal{C}}(R)$, where

$$\sigma_{\mathcal{C}}(R) = \sup_{u \in \mathcal{C}} \langle R, u \rangle.$$

- If R is nonnegative, we have $\sigma_{\mathcal{C}}(R) = \|R\|_{\mathcal{R}^1}$ (by Krein–Milman) so the optimal solutions of the relaxed problem are simply the subgradients of $\|\cdot\|_{\mathcal{R}^1}$ at R.
- The relaxed problem and Banach–Alaoglu extend directly to $R \in \mathcal{R}^{\mathcal{Y}}$ and $\mathcal{C} := \{u \in \mathcal{M}^{\mathcal{U}}_+ \mid \|u\|_{\mathcal{M}^{\mathcal{U}}} \leq 1\}$.

Applications: Integral functionals

- Under conditions given in [Rockafellar, 1971] and [Perkkiö, 2014, 2017] the conjugate of the integral functional

  $$I_h(y) := \int_{[0,T]} h(y) d\mu$$

  on C has representation

  $$J_{h^*}(u) := \int_{[0,T]} h^*(du^a/d\mu) + \int_{[0,T]} (h^*)^\infty (du^s/d|u^s|) d|u^s|,$$

  where u^a and u^s are the absolutely continuous and the singular part, respectively, of u with respect to μ.
- We extend this to integral functionals on $\mathcal{R}^{\mathcal{Y}}$.

Applications: Integral functionals

- Given a convex normal integrand h on $(\Omega \times [0,T]) \times \mathbb{R}^d$, and consider the integral functional $EI_h : \mathcal{R}^{\mathcal{Y}} \to \overline{\mathbb{R}}$,

$$EI_h(y) := E \int_{[0,T]} h(v) d\mu.$$

- The space $\mathcal{R}^{\mathcal{Y}}$ is not decomposable nor are the paths of $y \in \mathcal{R}^{\mathcal{Y}}$ continuous, in general.
- We will assume that μ is optional and that h is "regular" in the sense that its conjugate is the "optional projection" of a normal integrand that satisfies pathwise the conditions of [Rockafellar, 1971] or [Perkkiö, 2014, 2017].

Applications: Integral functionals

- A normal integrand g on $\mathbb{R}^d \times \Omega \times [0,T]$ is said to be optional if its epigraph is measurable with respect to the optional sigma algebra on $\Omega \times [0,T]$.
- A stochastic process v is $\mathcal{T}$-integrable if v_τ is integrable for every $\tau \in \mathcal{T}$.
- If g is a convex normal integrand such that $g^*(v)^+$ is $\mathcal{T}$-integrable for some $\mathcal{T}$-integrable v then, by [Kiiski and Perkkiö, 2016], there exists a unique optional convex normal integrand ${}^o g$ such that for every bounded optional x.

$$ {}^o g(x) = {}^o[g(x)] $$

- ${}^o g$ is called the optional projection of g.

Applications: Integral functionals

Definition 12 *An optional convex normal integrand h on $\mathbb{R}^d$ is regular if $h^* = {}^o\tilde{h}^*$ for a convex normal integrand $\tilde{h}$ such that $\tilde{h}(\omega)$ satisfies, for P-almost every ω, the conditions of [Rockafellar, 1971] or [Perkkiö, 2014, 2017] and*

$$\tilde{h}(v) \geq v \cdot \bar{x} - \alpha \quad \text{a.s.e.}$$
$$\tilde{h}^*(x) \geq \bar{v} \cdot x - \alpha \quad \text{a.s.e.}$$

for some $\bar{v} \in \mathcal{Y}(C)$ with $\bar{v} \in C(D)$ almost surely, optional $\bar{x}$ with $\int |\bar{x}| d\mu \in \mathcal{U}$ and some $\mathcal{T}$-integrable α with $\int |\alpha| d\mu \in L^1$.

Applications: Integral functionals

Theorem 13 *If h is a regular convex normal integrand, then $EI_h : \mathcal{R}^{\mathcal{Y}} \to \overline{\mathbb{R}}$ and $EJ_{h^*} : \mathcal{M}^{\mathcal{U}} \to \overline{\mathbb{R}}$ are proper and conjugate to each other and, moreover, $\theta \in \partial EI_h(v)$ iff*

$$\begin{aligned} d\theta^a/d\mu &\in \partial h(v) \quad \mu\text{-a.e.}, \\ d\theta^s/d|\theta^s| &\in \partial^s h(v) \quad |\theta^s|\text{-a.e.} \end{aligned}$$

almost surely.

Here $\partial^s h := \partial \delta_{\mathrm{cl\,dom}\,h}$

Applications: Singular stochastic control

- In the general formulation of (deterministic) singular control, one minimizes functionals of the form J_h over Radon measures or, equivalently, functions of bounded variation; see [Rockafellar, 1978].
- In the general formulation of stochastic control, one minimizes functionals of the form EI_h over semimartingales whose BV part if absolutely continuous; see [Bismut, 1973]
- In a general formulation of singular stochastic control, one minimizes functionals of the form EJ_h over general semimartingales; see [Pennanen and Perkkiö, manuscript].