ARGO 2022: WORKSHOP ON ALGEBRAIC GEOMETRY AND OPTIMIZATION

AUGUST 30TH - SEPTEMBER 2ND, SANTIAGO, CHILE

Real Algebraic Geometry deals with the study of sets of real solutions of polynomial equalities (real algebraic varieties) or inequalities (semi-algebraic sets). In optimization, the local or global extrema of a real function on a set is sought; problems in which both the function and the set are semi-algebraic play a fundamental role. Therefore, real algebraic geometry and optimization are closely linked, and progress in each field requires progress in the other. Moreover, the use of semi-algebraic techniques in Optimization, and more generally in Applied Mathematics, has expanded in recent years, since the most common operations that arise in optimization, while destroying smoothness, respect semi-algebraicity.

This Workshop brings together several key problems in real algebraic geometry, polynomial and semi-algebraic optimization, and with structural results in constrained optimization and non-smooth dynamics. Trough the Mathamsud project *ARGO: Algebraic Real Geometry and Optimization*, teams with complementary specialties, including real algebraic geometry, tropical geometry, symbolic computation, computational optimization, variational analysis, and dynamical systems (gradient flows), come together to present new developments at the interface of pure and applied mathematics.

Organizing Committee

- Aris Daniilidis, CMM, Universidad de Chile.
- David Salas, CMM, Universidad de O'Higgins.
- Héctor Ramírez, CMM, Universidad de Chile.
- Rodolfo Gutiérrez, CMM, Universidad de Chile.
- Alejandro Maass, CMM, Universidad de Chile.

Speakers

- Antoine Bereau CMAP, École Polytechnique, France.
- Frédéric Bihan Université Savoie Mont Blanc,France.
- Alicia Dickenstein, Universidad de Buenos Aires, Argentina.
- Stéphane Gaubert INRIA and CMAP, École polytechnique, CNRS, France.
- Cristopher Hermosilla Universidad Técnica Federico Santa María, Chile.
- Gabriela Jeronimo Universidad de Buenos Aires, Argentina.
- Santiago Laplagne, Universidad de Buenos Aires, Argentina.
- Gregorio Malajovich Universidade Federal do Rio de Janeiro, Brazil.
- Pedro Pérez-Aros Universidad de O'Higgins, Chile.
- Daniel Perrucci, Universidad de Buenos Aires, Argentina.
- Emilio Vilches Universidad de O'Higgins, Chile.

Program

	Tuesday 30th	**Wednesday 31th**	**Thursday 1st**	**Friday 2nd**
10:00 - 11:00	Reception and Opening	Gregorio Malajovich	Stéphane Gaubert	Free work sesion
11:00 - 11:20	Coffee Break	Coffee Break	Coffee Break	
11:20 - 12:20	Alicia Dickenstein	Daniel Perrucci	Pedro Pérez-Aros	
12:30 - 14:30	LUNCH			
14:30 - 15:30	Frédéric Bihan	Gabriela Jeronimo	Emilio Vilches	
15:30 - 15:50	Coffee Break	Coffee Break	Coffee Break	
16:00 - 17:00	Cristopher Hermosilla	Antoine Bereau	Santiago Laplagne	
19:00		Workshop's Dinner		

Abstracts

Alicia Dickenstein

Title: Lower and upper bounds for the number of positive roots with applications to the study of multistationarity questions in biochemical reaction network.

Abstract: I will discuss (known and unknown) lower and upper bounds for the number of positive solutions of systems of n sparse polynomials in n variables, together with applications to the determination of the number of positive steady states for families of multivariate polynomials occurring in the standard mass-action kinetics modeling of the dynamics of biochemical reaction networks.

Frédéric Bihan

Title: An overview of fewnomial theory.

Abstract: In this talk, we will review recent results of fewnomial theory. We will try to state open questions or interesting problems on the subject.

Cristopher Hermosilla

Title: Optimal Control Problems on Well-structured Domains.

Abstract: The aim of this talk is to discuss about Optimal control problems with tame state-constraints. In particular, we are concerned with the characterization of the Value Function of an optimal control problem with state-constraints, where the state constraint set is semi-algebraic. We also discuss other contexts where real algebraic geometry intersects optimal control theory.

Gregorio Malajovich

Title: On the expected number of roots of random polynomials and exponential sums.

Abstract: The expected number of real projective roots of orthogonally invariant random homogeneous real polynomial systems is known to be equal to the square root of the Bézout number. A similar result is known for random multi-homogeneous systems, invariant through a product of orthogonal groups. I shall present a generalization to certain families of sparse polynomial systems, no orthogonal invariance assumed.

Daniel Perrucci

Title: Winding number, Cauchy index and subresultant polynomials.

Abstract: In the complex plane, the winding number of a closed curve around a point counts the number of times that the curve turns around the point counterclockwise. This quantity provides a tool to compute the number of zeros minus the number of poles of a meromorphic function in the inerior of the region defined by a closed curve, as long as the curve itself avoids the zeros and poles of the meroporphic function.

In this talk we introduce a new algebraic version of the winding number, which can be used to count the number of zeros of a polynomial in a rectangle. A nice thing about this approach is that is not required that the boundary of the rectangle avoids the zeros of the polynomial.

This is a joint work with Marie-Françoise Roy from Université de Rennes 1.

Gabriela Jeronimo

Title: Certificates of non-negativity on semialgebraic sets contained in cylinders.

Abstract: A certificate of non-negativity (resp. positivity) for a polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ on a semialgebraic set $S \subset \mathbb{R}^n$ is an algebraic identity that makes evident the fact that $f(x) \geq 0$ (resp. $f(x) > 0$) for every $x \in S$. These certificates go back to the classical Positivstellenstaz proved by Krivine in 1964, and they have been widely studied and applied since then.

If $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0\}$ with $g_1, \ldots, g_s \in \mathbb{R}[\mathbf{x}]$, every polynomial in the quadratic module $M = \{\sigma_0 + \sum_{1 \leq i \leq s} \sigma_i g_i \mid \sigma_i$ is a sum of squares in $\mathbb{R}[\mathbf{x}]\}$ is non-negative on S. Putinar's Positivstellensatz states that, if M is Archimedean (which implies that S

is compact), every $f \in \mathbb{R}[\mathbf{x}]$ positive on S lies in M. Furthermore, Nie and Schweighofer proved an upper bound for the degrees of the polynomials in a representation of f as an element of M.

Putinar and Vasilescu gave a generalization of the previous result to non-compact sets, under certain assumptions on $f, g_1, \ldots, g_s$: they showed that there exists a non-negative integer B such that $(1 + \sum_{1 \le j \le n} x_j^2)^B f \in M$. Recently, Escorcielo and Perrucci proved other certificates of non-negativity, extending Putinar's original result to certain non-compact situations.

In this talk, we will present a new certificate of non-negativity for polynomials that are positive on a non-compact semialgebraic set included in a cylinder. Under certain assumptions on the positive polynomial f and the polynomials $g_1, \ldots, g_s$ defining the set, we will show the existence of the proposed certificate of non-negativity of f and explain how we obtain an upper bound for the degrees of the polynomials appearing in the representation. This is joint work with Daniel Perrucci.

Antoine Bereau

Title: The Nullstellensatz for Sparse Tropical Polynomial Systems.

Abstract: Grigoriev and Podolskii (2018) have established a tropical analog of the effective Nullstellensatz, showing that a system of tropical polynomial equations is solvable if and only if a linearized system obtained from a Macaulay matrix truncated up to some degree N is solvable. Their result provides an explicit value of N as a function of the number of variables and of the degrees of the input polynomials. This value is higher than its classical analog, and in fact the question of the optimal truncation degree was left open. We provide a new proof of the tropical Nullstellensatz leading to a smaller value of N. We recover in particular the classical Macaulay degree bound for square homogeneous systems. Our approach is inspired by a construction of Canny and Emiris for sparse polynomial systems, refined by Sturmfels. It leads to new truncation methods adapted to sparse tropical polynomial systems. This is a joint work with Marianne Akian and Stéphane Gaubert.

Stephane Gaubert

Title: Convexity, Mean Payoff Games and Nonarchimedean Convex Programming

Abstract: Convex sets can be defined over ordered fields with a non-archimedean valuation. Then, tropical convex sets arise as images by the valuation of non-archimedean convex sets. The tropicalization of polyhedra and spectrahedra can be described in terms of deterministic and stochastic games with mean payoff, being characterized in terms of sub or super-fixed point sets of Shapley operators, which determine the value of the game. This is motivated by open complexity issues in linear programming, leading in particular to a counter example showing that interior point methods are not strongly polynomial. We shall survey these results, and then discuss a recent extension, called ambitropical convexity as it includes both tropical convexity and its dual. We will relate in particular the notion of ambitropical hull with the one of injective hull and tight span of metric spaces.
The results on ambitropical convexity are from a work with Akian and Vannucci; the ones on the tropicalization of nonarchimedean convex sets and their application to complexity issues in convex programming are from a series of works with Allamigeon, Benchimol, Joswig, Skomra and Vandame.

Pedro Pérez-Aros

Title: Generalized differentiation of probability functions.

Abstract: In this talk we summary some results about (generalized) differentiability and Lipschitz continuity of probability functions given by

$$\varphi(x) = \mathbb{P}(g(x, \xi) \leq 0),$$

where $(\Omega, \Sigma, \mathbb{P})$ is a probability space, ξ is an m-dimensional random vector with elliptical distribution, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz and convex in the second variable. We show some formulae for gradients and subgradients using the so called spherical radial decomposition. Also, results an open questions concerning generalized gradients for probabilistic/robust (probust) probability functions are presented.

Keywords: Probability functions; Stochastic programming; Spherical radial decomposition; Generalized differentiability.

References

[1] Hantoute A., Henrion R. and Pérez-Aros P. Subdifferential characterization of probability functions under Gaussian distribution. Math. Program. 174 (2019), no. 1-2, Ser. B, 167?194.

[2] van Ackooij W. and Pérez-Aros P. Generalized differentiation of probability functions acting on an infinite systems of constraints.SIAM Journal on Optimization (2019).Vol. 29, No. 3.

[3] van Ackooij and Henrion R. and Pérez-Aros P. Generalized gradients for probabilistic/robust (probust) constraints. Optimization (2019).

Emilio Vilches

Title: Recent results on the determination of convex functions.

Abstract: In this talk, we review recent results on the determination of convex functions in Hilbert spaces. First, we show that any two convex proper lower semicontinuous functions bounded from below, for which the norm of their minimal subgradients coincide, coincide up to a constant [1]. Second, we provide comparison principles for convex functions through its proximal mappings. As a consequence, we obtain that the norm of the proximal operator determines a convex function up to a constant [2]. These results show that for convex functions bounded from below, the slopes and the norm of the proximal operator provide sufficient information to determine the function up to a constant.

Keywords: Convex Function; Convex Subdifferential; Proximal Operator; Slope.

References

[1] Pérez-Aros, P.; Salas, D.; Vilches, E., *Determination of convex functions via subgradients of minimal norm*, Mathematical Programming, (2020).
[2] Vilches, E,, *Proximal determination of convex functions*, J. Convex Analysis **28**, (2021).

Santiago Laplagne

Title: Strictly positive polynomials in the border of the cone of sum of squares.

Abstract: The decomposition of a real multivariate polynomial as a sum of squares of real polynomials is a central problem in real algebraic geometry, with many theoretical and practical applications.

In this talk we study the boundary of the cone of real polynomials that can be decomposed as a sum of squares (SOS) of real polynomials. This cone is included in the cone of non-negative polynomials and both cones share a part of their boundary, which corresponds to polynomials that vanish at at least one point. We focus on the part of the boundary which is not shared, corresponding to strictly positive polynomials.

For the cases of polynomials of degree 6 in 3 variables and degree 4 in 4 variables, this boundary has been completely characterized by G. Blekherman. For the cases of more variables or higher degree, results by G. Blekherman, R. Sinn and M. Velasco and other authors based on general conjectures give bounds for the maximum number of polynomials that can appear in a SOS decomposition and the maximum rank of the matrices in the Gram spectrahedron. In joint work with Marcelo Valdettaro we show that these bounds can also be deduced from a conjecture by D. Eisenbud, M. Green and J. Harris, and we prove that these bounds are sharp for polynomials in any number of variables and any degree, providing an explicit construction of polynomials satisfying attaining the bound.