Geometry and Dynamics on Flat Surfaces

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Lecture 1

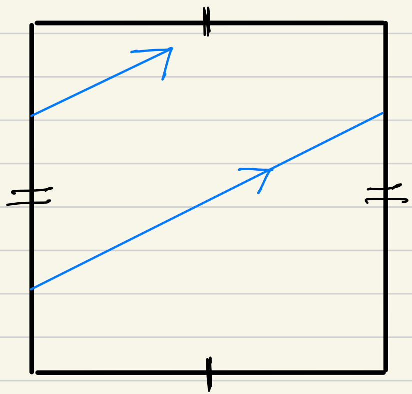
References

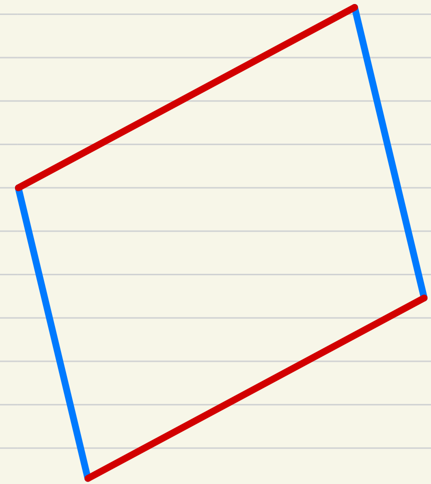
1 Zorich : Flat surfaces, Springer 2006.

2 Yoccoz : Interval exchange transformations and translation surfaces, Pisa lecture notes, Clay Math 2010

3 Wright : From rational billiards to dynamics on moduli spaces, Bull. Amer. Math. Soc. 2016 .

I Flat Surfaces

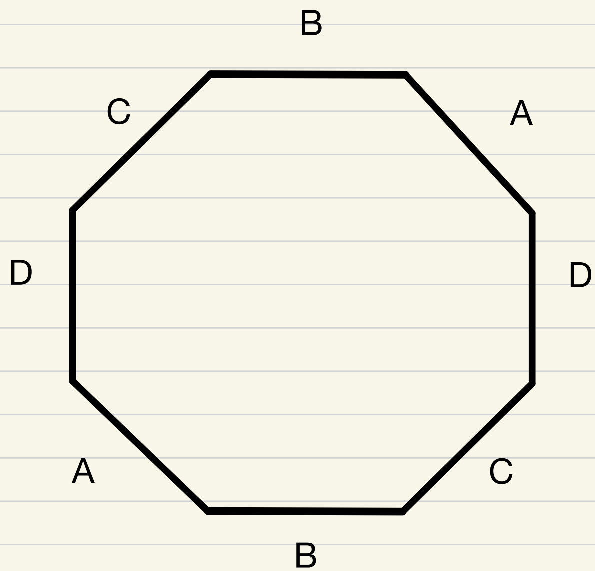




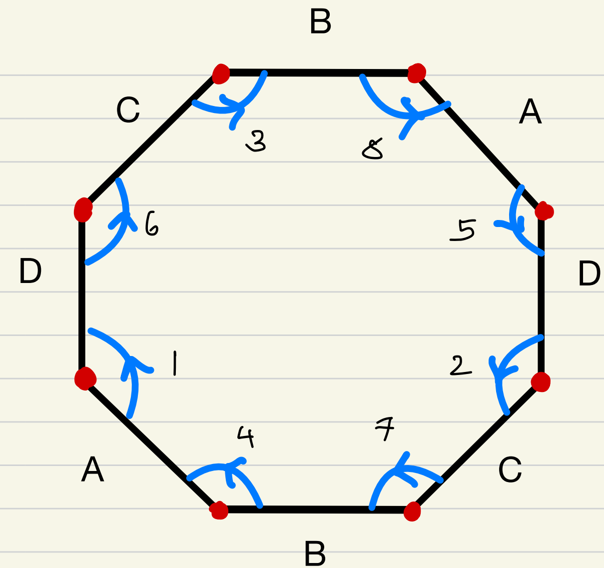
Tori

I

Flat Surfaces



Genus 2

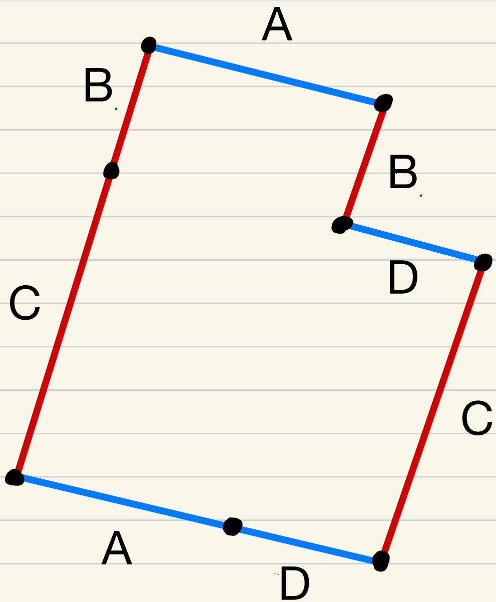


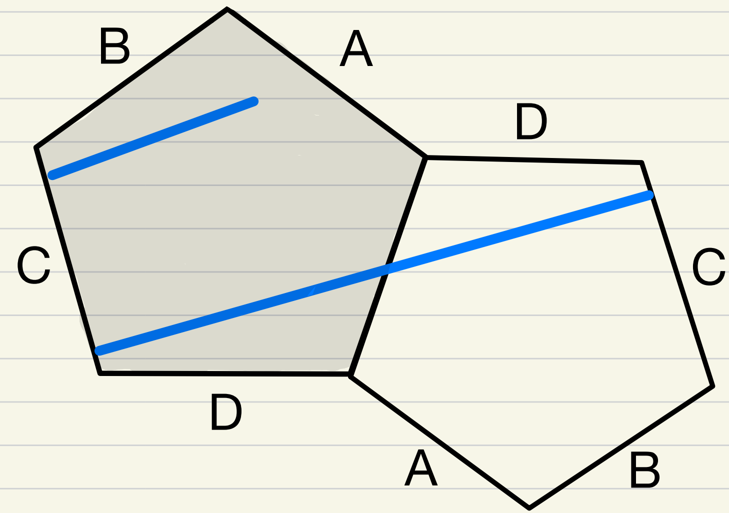
One singularity in red

Excess angle at singularity is $2(2\pi)$

Consistent with Gauss-Bonnet

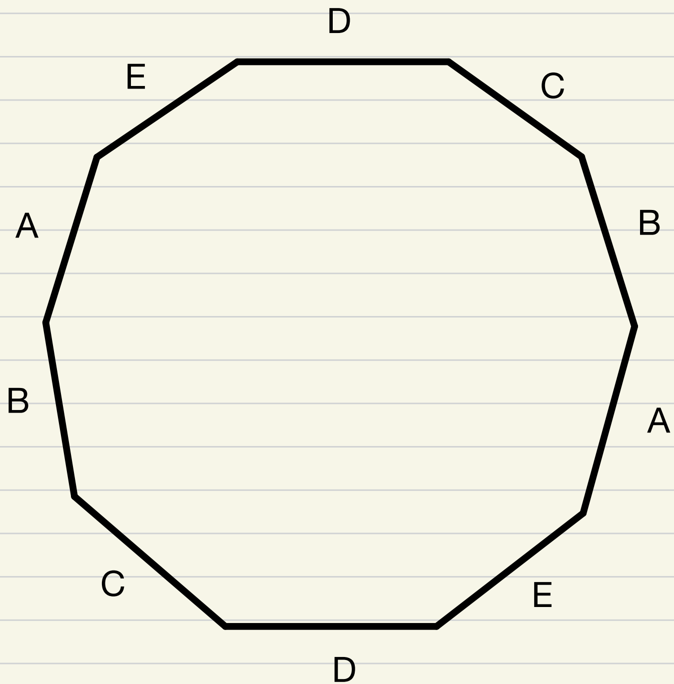
I Flat Surfaces



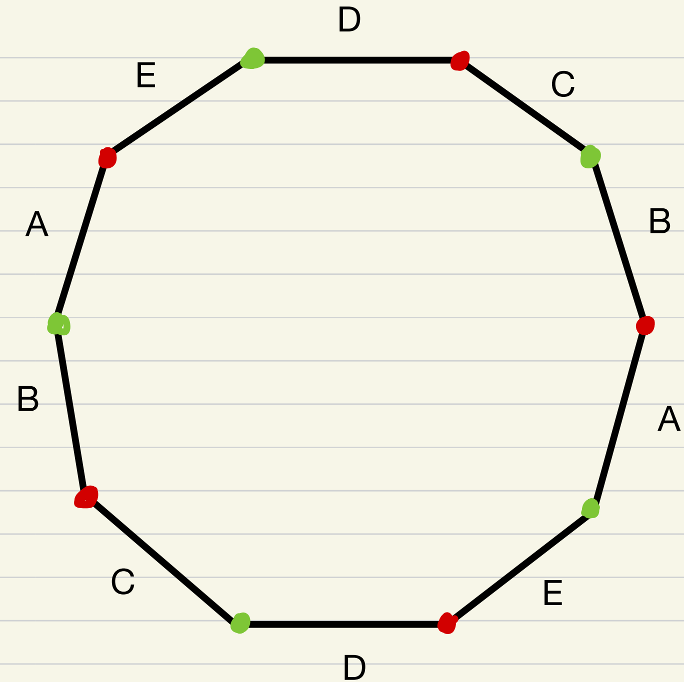


Genus 2

I Flat Surfaces

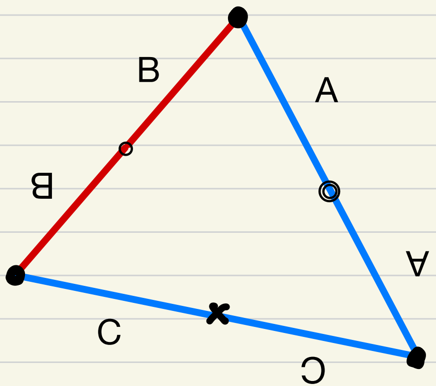


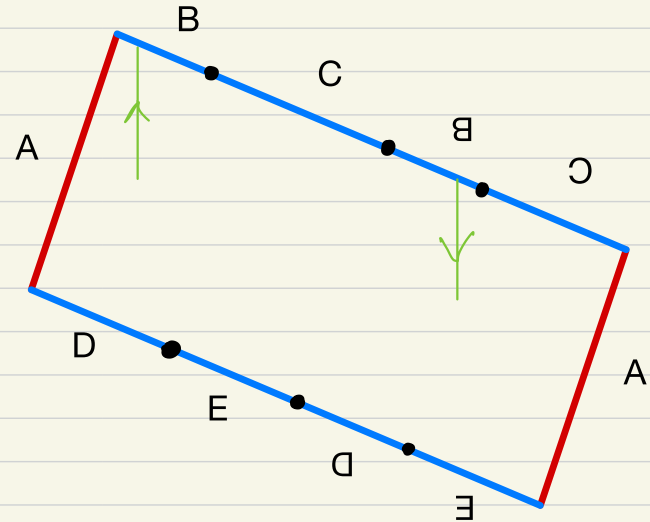
Genus 2



Two singularities; excess angle at each is 2π

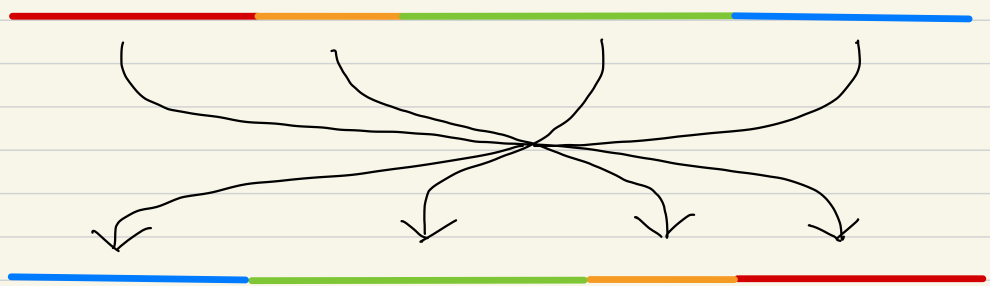
I Flat Surfaces





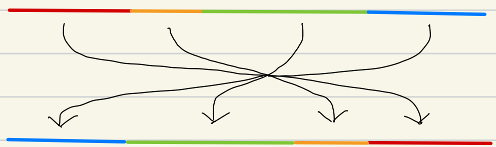
II Motivations from Dynamics

Interval exchange maps

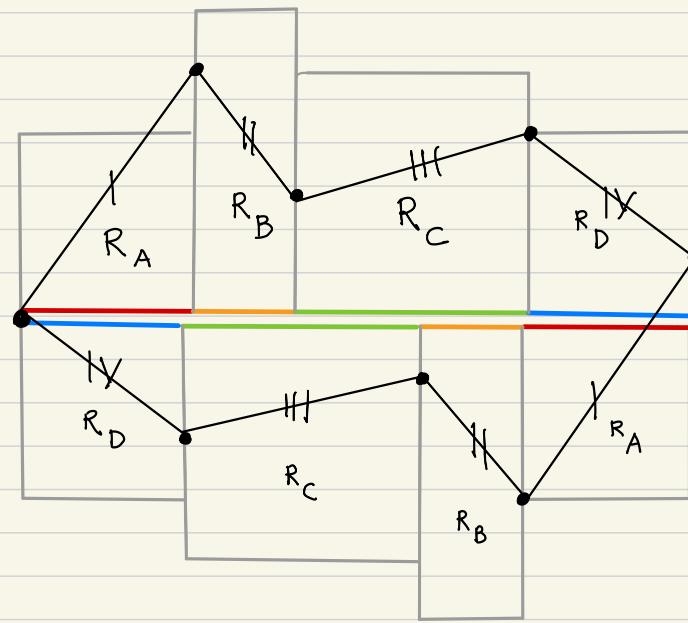


Piecewise isometry of the interval.

II Motivations from Dynamics

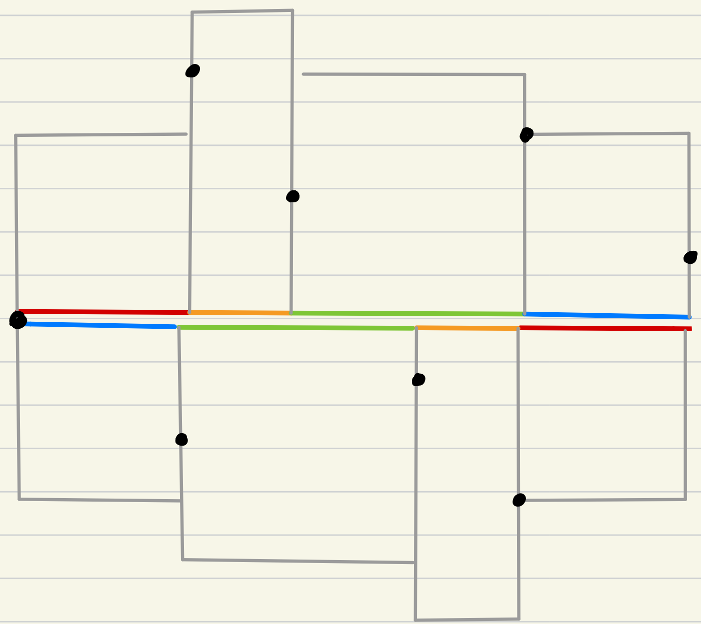


Suspensions of interval exchange maps give flat surfaces.



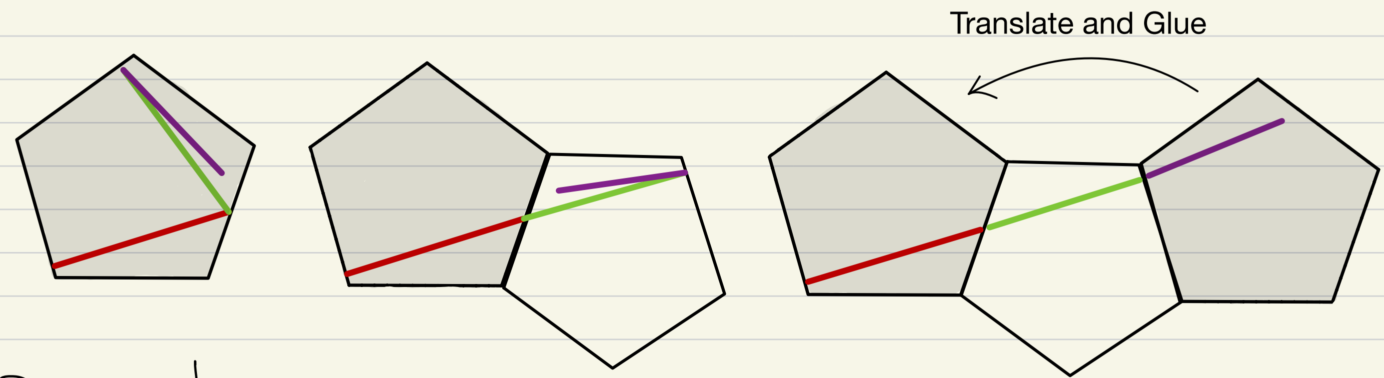
II Motivations from Dynamics

Veech : zippered rectangles

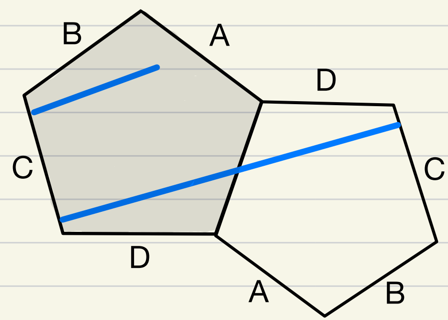


II Motivations from Dynamics

Rational billiards



Polygonal angles rational multiples of π



II Motivations from Dynamics

1 Straight line flows on flat surfaces do not have (semi)- hyperbolic dynamics

2 Teichmüller flow on moduli spaces of flat surfaces renormalises straight line flow

3 Teichmüller flow is semi-hyperbolic.

Zorich (ETDS 1997): deviations for ergodic averages of interval exchange maps are related to the Lyapunov spectrum of the absolute homology co-cycle for Teichmüller flow.

III Holomorphic viewpoint

Let S be a surface of $\underbrace{\text{finite type}}_{\substack{\text{finite genus} \\ \text{finitely many} \\ \text{marked points}}}$.

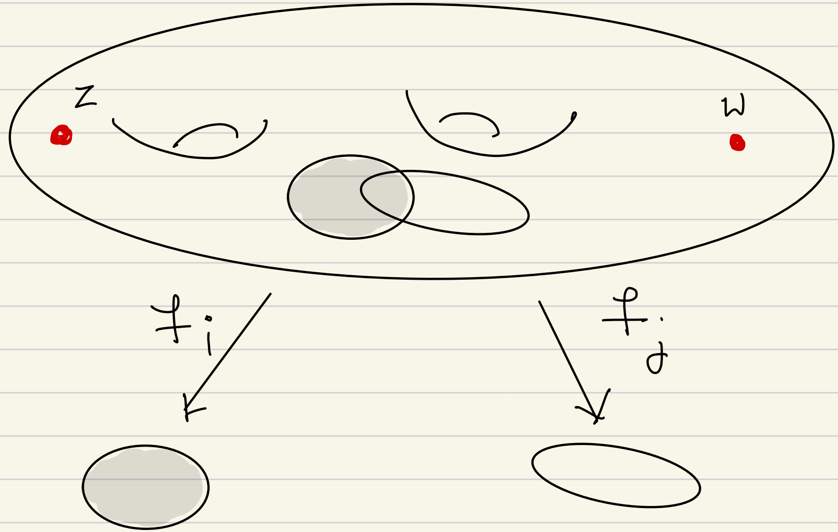
A flat surface homeomorphic to S is equivalent to charts from $S - Z$ to $\mathbb{C}$, where Z is a finite set containing the marked points with transition functions of the form

$$z \longrightarrow \pm z + c$$

III Holomorphic viewpoint

transition functions

$f_j \circ f_i^{-1}$



By Removable Singularity Theorem, conformal structure extends to all of S when transitions are translations / half translations.

III Holomorphic viewpoint

The structure of charts to $\mathbb{C}$ with half-translation/translation transitions is equivalent to a meromorphic quadratic differential with at most simple poles at the marked points.

In a local co-ordinate on $S - Z$ $\quad q = f(z)\,dz^2$

$$\underset{\text{meromorphic}}{\downarrow}$$

IV

Teichmüller theory

(S, Z) orientable surface S with finite genus and a finite set Z of marked points; assume $|Z| = n$

Uniformisation theorem : universal cover is either

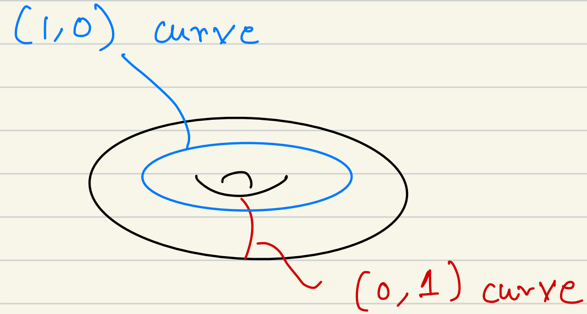
$$\mathbb{CP}^1, \quad \mathbb{C}, \quad \mathbb{D}$$

$\mathbb{CP}^1$	$\mathbb{C}$	$\mathbb{D}$
spherical metric	(non-singular) flat metric	hyperbolic metric
Euler charac	Euler charac	Euler charac
$2-2g-n > 0$	$2-2g-n = 0$	$2-2g-n < 0$
$\Rightarrow g=0, n \leq 1$;	$\Rightarrow g=0, n=2$ $g=1, n=0$	most surfaces

IV Teichmüller theory

The torus $g = 1$, $n = 0$

A marked conformal torus.



$$\mathrm{Aut}(\mathbb{C}) = \{ z \to az + b \}$$

We want $\pi_1(S_{1,0}) \simeq \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\rho} \mathrm{Aut}(\mathbb{C})$ to act on $\mathbb{C}$ properly discontinuously, so has to be by translations.

Post-composing by rotations and dilations, assume

$$\rho(1,0) = \{ z \to z + 1 \}$$

Then $\rho(0,1) = \{ z \to z + \tau \}$ where $\mathrm{Im}(\tau) > 0$

that is $\tau \in \mathbb{H}$

IV

Teichmüller theory

Moduli space of conformal tori:

For convenience, suppose $\tau = i$; $\Lambda = \mathbb{Z}\cdot 1 \oplus \mathbb{Z} i$

Then $\underbrace{SL(2,\mathbb{Z})}_{\text{2x2-matrices over } \mathbb{Z} \text{ with determinant } 1} \cdot \Lambda = \Lambda$.

Note that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ changes the marking to (a,c) and (b,d) curves.

Moduli space $\mathcal{M}_0 = \underbrace{\mathbb{H} / SL(2,\mathbb{Z})}$ ← modular surface $(2,3,\infty)$ hyp. orbifold.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

IV

Teichmüller Theory

$\mathcal{M}_0$ = moduli space of elliptic curves.

$$y^2 = x^3 + ax + b \quad \hookleftarrow$$

The connection between the two descriptions of $\mathcal{M}_0$ is through the theory of Weierstrass $\wp$-functions.

Similarly

$\mathcal{M}_{g,n}$ is also the moduli space of algebraic curves that are conformally a Riemann surface of type $S_{g,n}$.

IV

Teichmüller theory

Moduli space of flat tori $= \{(X, \omega) : X$ a torus with conformal str, ω is a holomorphic 1-form$\}$

$= T^1 M_0$

Exercise: Figure out what happens at orbifold points.

Important point: $\omega = \mathrm{Re}\,\omega + \mathrm{Im}\,\omega$

$\mathrm{Re}\,\omega$: vanishes on the vertical foliation

$\mathrm{Im}\,\omega$: vanishes on the horizontal foliation.

IV

Teichmüller theory

Nielsen-Thurston classification : $A \in SL(2,\mathbb{Z})$

Fixed point equation on $\mathbb{H}$: $\dfrac{a\tau+b}{c\tau+d} = \tau$ use quadratic formula

Complex roots : $\mathrm{Tr}\, A = 0 \Longrightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|\mathrm{Tr}(A)| < 2$ conjugate to

symmetry of the square torus

$\mathrm{Tr}\, A = 1 \Longrightarrow A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

conjugate to

symmetry of the hexagonal torus

In general: finite order mapping class arises as a Riemann surface automorphism.

IV

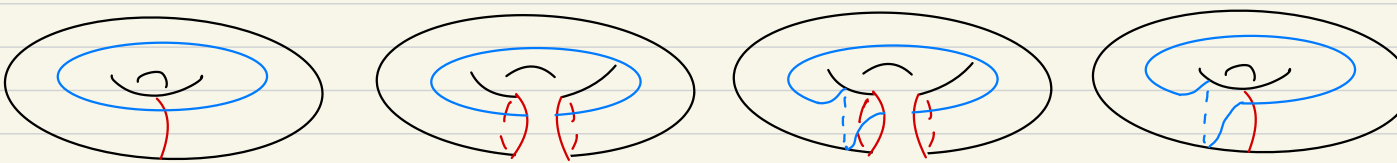
Teichmüller theory

Repeated real roots : $|\mathrm{Tr}(A)| = 2$

A is a parabolic matrix, conjugate to

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

✓ acts on the torus by a Dehn twist in the $(1,0)$ curve



In general: reducible mapping classes, some power fixes a multi-curve.

IV

Teichmüller theory

Distinct real roots : A is a hyperbolic matrix

$|\mathrm{Tr}(A)| > 2$ for example $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$\swarrow$

acts on the torus as a hyperbolic map with stable and unstable foliations given by eigendirections

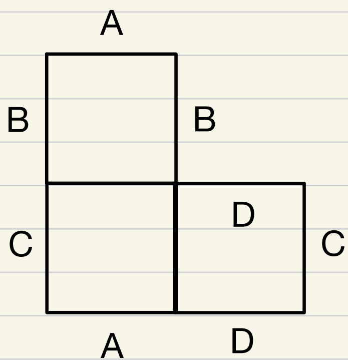
- admit nice Markov partitions
- translate along a hyperbolic geodesic in Teich_0

In general : pseudo-Anosov mapping classes expand and contract measured foliations; translate along a Teichmüller geodesic.

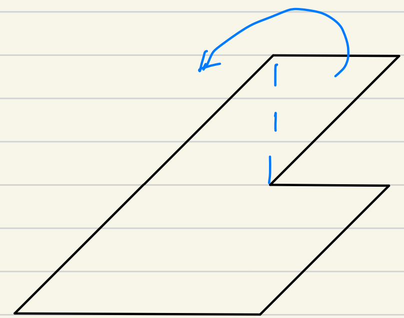
IV

Teichmüller theory

Example of a pseudo-Anosov square-tiled L shaped table



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



but not a scissors move on cylinder C

Exercise: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ gives a scissors move on cylinders B and C to get back original surface;

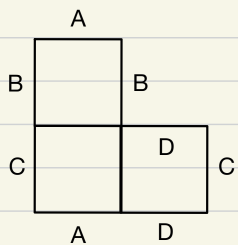
mapping class twists in core curves of horizontal cylinders

IV

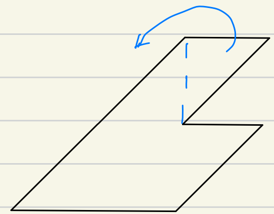
Teichmüller theory

Example of a pseudo-Anosov

square-tiled L shaped table



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



not a scissors

move on cylinder C

Similarly $\begin{matrix} 1 & 0 \\ 2 & 1 \end{matrix}$ twists in core curves of vertical cylinders

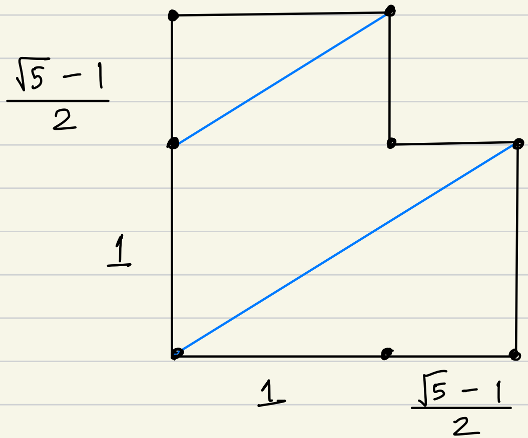
The product $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ is a pseudo-Anosov.

Eigendirections give stable and unstable measured foliations.

IV

Teichmüller theory

pseudo-Anosov on a golden L



Exercise: show that

$$\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix}$$ gives a twist

in horizontal cylinders

Hint: the diagonals in blue have same slopes.

Then $\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1+\sqrt{5}}{2} & 1 \end{bmatrix}$ gives a pseudo-Anosov.

IV

Teichmüller theory

(S, Z) orientable finite type surface with finitely many marked points Z

Assume Euler characteristic is negative.

$\text{Teich}(S, Z)$ = space of marked conformal structures on (S, Z)

= space of marked complete hyperbolic metrics on $S - Z$

Mod = group of orientation preserving diffeo / isotopy.

IV

Teichmüller theory

Mod = group of orientation preserving diffeo / isotopy.

Moduli space $\mathcal{M}$ = Teich / Mod

Similarly define the Teichmüller space of meromorphic quadratic differentials with at most simple poles.

This is stratified by the orders of the zeroes

$\underline{\kappa} = (\kappa_1, \ldots, \kappa_j)$ such that $\sum \kappa_i = 4g-4$

IV

Teichmüller theory

Strata of (unmarked) quadratic differentials

$$= \quad Q\,\mathrm{Teich}(\underline{k}) / \mathrm{Mod}$$

Kontsevich-Zorich, Boissy-Lanneau, Chen-Möller:

Classification of connected components of strata.

Invariants: quadratic or Abelian, hyper-elliptic or not, odd or even spin, regular or irregular.

Study stratum components in detail tomorrow.

V

Does anyone have questions ?