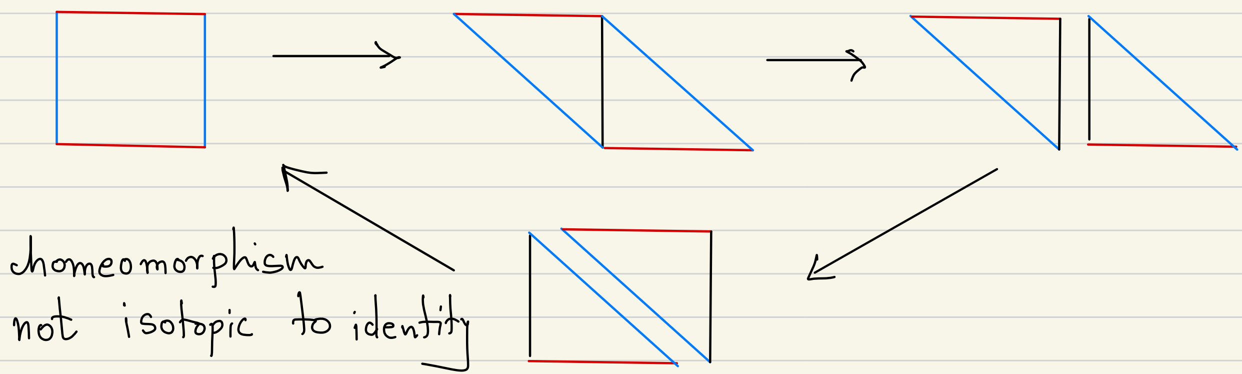
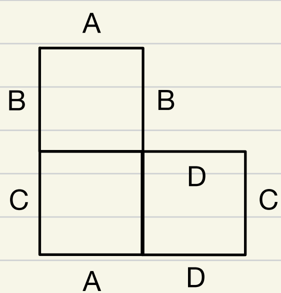
Geometry and Dynamics on Flat Surfaces

Vaibhav Gadre (Glasgow)

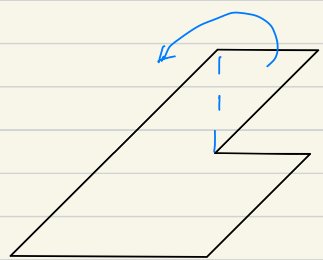
Lecture 2

I Recap Slide : Scissors congruence





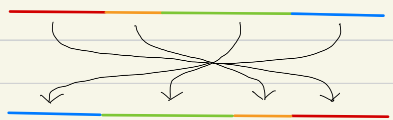
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



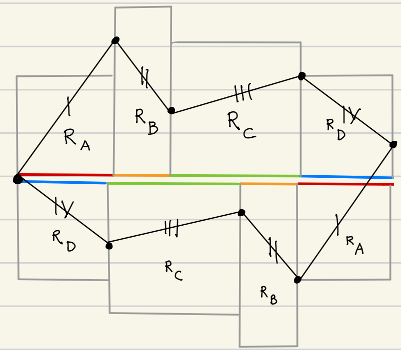
but not a scissors congruence on cylinder C

I Recap slide : interval exchange maps

Suspensions of interval exchange maps give flat surfaces



" width parameters "



" height parameters "

Traditional interval exchanges give holomorphic 1 - forms.

To get quadratic quadratic differentials one has to suspend generalised permutations that give linear involutions.

first studied by Danthony - Nogueira.

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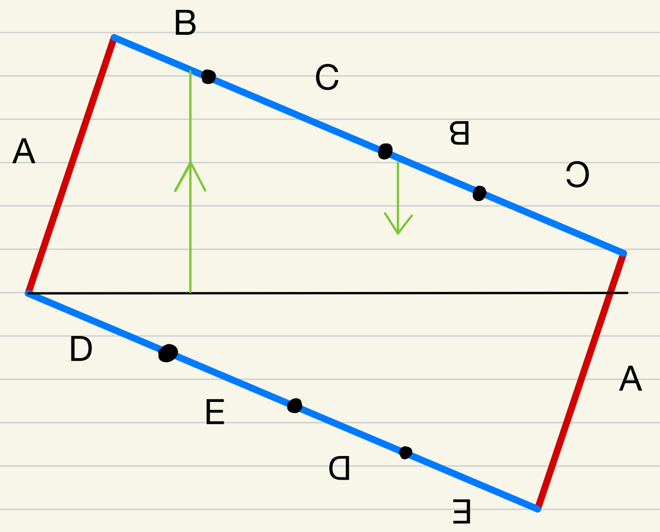
Recap slide : non-classical interval exchanges



$$\frac{A \quad B \quad C \quad B \quad C}{D \quad E \quad D \quad E \quad A}$$

Compare $H(1,1)$ which has simple zeroes locally

$\omega = z\,dz$

Stratum $Q(2,2)$

zero of order 2 ; $z^2 dz^2$ in local co-ordinates

I

Recap slide : Moduli spaces of quadratic differentials

Strata of (unmarked) quadratic differentials

$= \quad Q\,\mathrm{Teich}(\underline{k}) / \mathrm{Mod}$

Kontsevich-Zorich, Boissy-Lanneau, Chen-Möller:

Classification of connected components of strata.

Invariants: quadratic or Abelian, hyper-elliptic or not, odd or even spin, regular or irregular.

II Periods

Let q be a flat surface.

An arc γ is called a saddle connection if

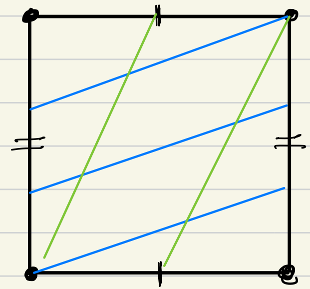
- the interior of γ embeds in $S - Z$;
- the endpoints of γ are contained in Z; and
- γ is a geodesic in the flat metric.

Choosing $\sqrt{q}$, the period of a saddle connection is defined as

$$\mathrm{per}(\gamma) = \int_\gamma \sqrt{q}$$

II Periods

Square torus



period (blue) = $3+i$

period (green) = $1+2i$

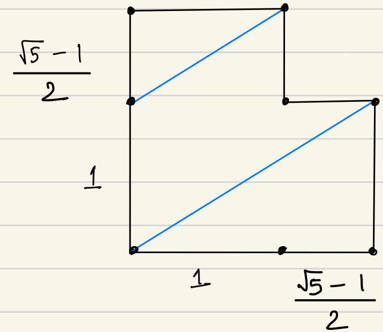
Saddle connection periods (without multiplicities) are given by primitive lattice points in $\mathbb{Z} \oplus i\mathbb{Z} \subset \mathbb{C} = \mathbb{R}^2$

Counting asymptotics very interesting

$\#$ of primitive points in $B(x_0, R) \sim \frac{6}{\pi^2} R^2$

II Periods

Golden L



Saddle connection periods thought of as vectors in $\mathbb{R}^2$

$$\Lambda_\omega = SL(X,\omega)\cdot 1 \text{ union } SL(X,\omega)\cdot \frac{\sqrt{5}-1}{2}$$

where $SL(X,\omega)$ is the $(2,5,\infty)$ Hecke triangle group.

Counting asymptotics $\sim \frac{3\pi}{10} R^2$

Siegel-Veech constant

II Period Co-ordinates

Fix a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ for $H_1(S, \mathbb{Z})$.

The period map

$$q \longrightarrow \{ per_{\sqrt{q}}(\alpha_1), \ldots, per_{\sqrt{q}}(\alpha_n) \}$$

define local co-ordinates in a stratum component.

III $SL(2,\mathbb{R})$ action

Recall : A meromorphic quadratic differential is equivalent to charts from $S - Z$ to $\mathbb{C}$ with translation or half translation transition functions.

$SL(2,\mathbb{R})$ acts on $\mathbb{R}^2 = \mathbb{C}$ by linear transformations

The action takes (half)-translations to (half)-translations.

So it descends to components of strata.

III SL(2,ℝ) action

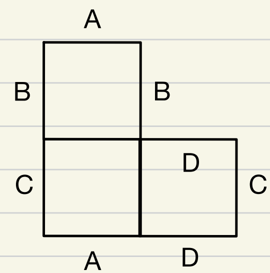
Overall Picture :

$$\begin{tikzcd} SL(2,\mathbb{R}) \cdot q \arrow[r, hook] \arrow[d] & QTeich(\underline{k}) \arrow[d] \\ SO(2,\mathbb{R}) \backslash SL(2,\mathbb{R}) \cdot q = \mathbb{D}_q \arrow[r, hook] & Teich(S_{g,n}) \end{tikzcd}$$

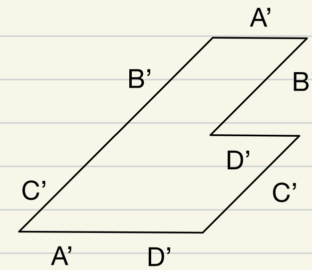
called a Teichmüller disc ; it is an isometrically embedded hyperbolic disc in $Teich(S_{g,n})$.

III $SL(2,\mathbb{R})$ action

Example



$$\xrightarrow{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}$$

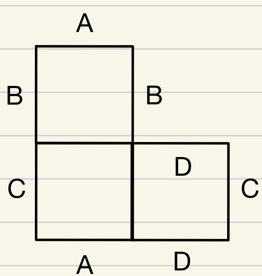


Define $SL(X,q) = \left\{ A \in SL(2,\mathbb{R}) \text{ such that } \underbrace{A(X,q) = (X,q)}_{\text{as unmarked flat surfaces}} \right\}$.

III $SL(2,\mathbb{R})$ action

For any q, $SL(X,q)$ is a discrete subgroup of $SL(2,\mathbb{R})$.

Example : square-tiled L-shaped surface

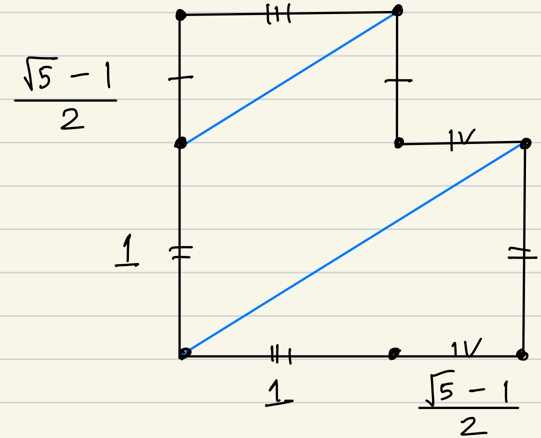


$$SL(X,\omega) = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle$$

$$= \Gamma_2 \longrightarrow$$ congruence-2 subgroup of $SL(2,\mathbb{Z})$

III $SL(2,\mathbb{R})$ action

For the golden L



$SL(X,\omega) = (2,5,\infty)$ - Hecke triangle group.

III $SL(2,\mathbb{R})$ action

A subgroup $\Gamma \subset SL(2,\mathbb{R})$ is called a lattice if

$$\mathrm{vol}\left(SL(2,\mathbb{R})/\Gamma \right) < \infty$$

Theorem (Smillie):

$SL(2,\mathbb{R})$ orbit of q closed $\Longleftrightarrow$ $SL(X,q)$ is a lattice. //

Such a flat surface is called a lattice (Veech) surface; the closed curve of moduli space it generates a Teichmüller curve.

III $SL(2,\mathbb{R})$ action

Eskin - Mirzakhani - Mohammadi : $SL(2,\mathbb{R})$ orbit closures are cut out in period co-ordinates by linear (homogeneous) equations with real coefficients

Eskin - Mirzakhani : Every $SL(2,\mathbb{R})$-invariant ergodic measure is supported on some orbit closure and is in the Lebesgue class.

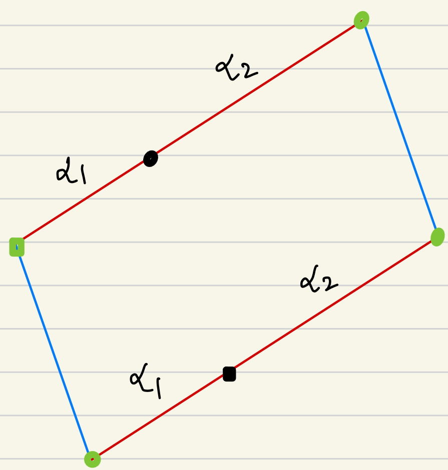
Non-homogeneous analogues of Margulis-Ratner theory.

III SL(2, ℝ) action

A simple SL(2, ℝ) orbit closure

The 2 : 1 locus in $\mathcal{H}(0,0)$

$$\mathrm{per}(\alpha_2) = 2\,\mathrm{per}(\alpha_1)$$



IV

Masur - Veech measure

By periods, differentials in a stratum component can be thought of as elements in relative co-homology.

Normalize the Lebesgue measure in relative cohomology so that the co-volume of the integral lattice is 1.

Different choices of period co-ordinates differ by unimodular action on relative cohomology.

So the measure is well defined and is $SL(2,\mathbb{R})$ invariant.

V Sub-actions of the $SL(2,\mathbb{R})$-action

g_t : diagonal action $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$

Teichmüller geodesic flow

h_s : horocycle flow $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$

r_θ : rotations $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

V Sub-actions of the $SL(2,\mathbb{R})$-action

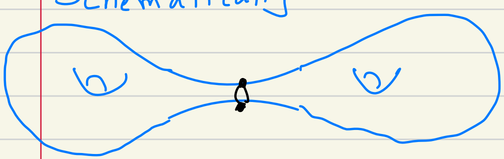
Lot is known for g_t, h_s in the homogeneous setting

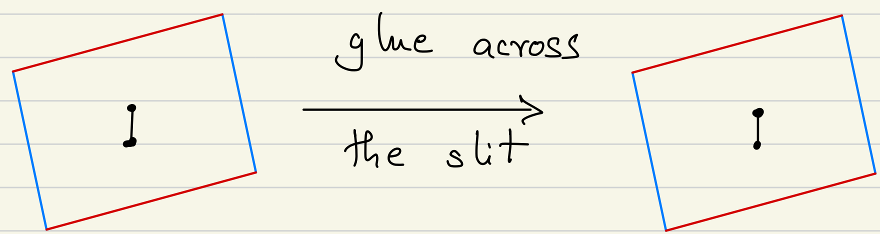
In the Teichmüller setting, there is in-homogeneity.

A flat surface q is ϵ-thin if it contains a saddle connection α s.t $|\mathrm{per}_q(\alpha)| < \epsilon$.

Geometry around thin q is different from geometry around thick q.

Schematically





V Sub-actions of the $SL(2,\mathbb{R})$-action

Think $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$

Rank 1 versus higher rank

Thick part is "negatively curved"

Thin parts have product metric features and complicated intersections. Think $SL(d,\mathbb{R})/SL(d,\mathbb{Z})$

$d>2$

By and large, rank 1 behaviour prevails.

V Sub-actions of the $SL(2,\mathbb{R})$-action

g_t : Teichmüller geodesic flow behaves analogous to the homogeneous setting.

g_t is ergodic w.r.t the Masur-Veech measure

g_t is exponentially mixing satisfying Moore-Ratner type decay of correlations

— Avila-Gouezel-Yoccoz

Avila-Gouezel

in fact, has spectral gap for its action on the appropriate class of functions.

V Sub-actions of the $SL(2,\mathbb{R})$-action

h_s : horocycle flow differs significantly from the homogeneous setting

: examples of exotic h_s invariant measures

— Chaika - Smillie - Weiss .

VI Sample theorem demonstrating analogies

Discussion related Sullivan, Masur log-laws, Khintchine theorems

Continued fractions : Given $x \in [0,1]$, write it as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Gauss: For Leb-almost every x

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \dots + a_n}{n} = \infty$$

VI Sample theorem with trimmed sums

Khintchine : for any $\epsilon > 0$

$$\lim_{n\to\infty} \text{Leb}\left\{ x \in [0,1] \text{ such that } \left| \frac{a_1 + \dots + a_n}{n \log n} - \frac{1}{\log 2} \right| > \epsilon \right\} = 0$$

Borel-Bernstein : For Leb-almost every $x \in [0,1]$

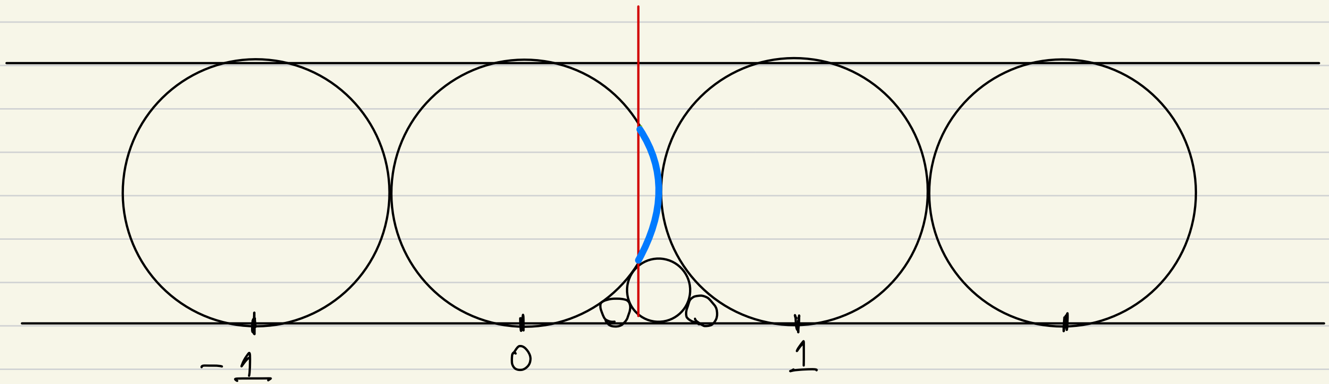
$a_n > n(\log n)(\log \log n)$ for ∞ many n.

Diamond - Vaaler : For Leb-almost every $x \in [0,1]$

$$\lim_{n\to\infty} \frac{a_1 + \dots + a_n - \max_{k \le n} a_k}{n \log n} = \frac{1}{\log 2}$$

VI Sample theorem with trimmed sums

Connection to hyperbolic geometry of the modular surface



Excursions in horoballs : $E_1(x) = \ell_{\partial H}($ blue segment $)$

Similarly $E_2, E_3, \ldots$

Turns out $E_k(x) = a_k(x)$

up to fixed additive error.

VI Sample theorem with trimmed sums

Continuous time versions of along typical hyperbolic geodesics of all results on previous slides.

— [G]

Exact results along typical Teichmüller geodesics on $SL(2,\mathbb{R})$ orbit closures

— [G]

Limits related to Siegel-Veech constants.