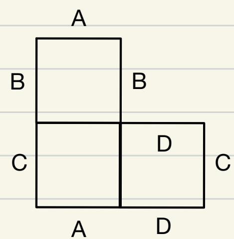
Geometry and Dynamics on Flat Surfaces

Vaibhav Gadre (Glasgow)

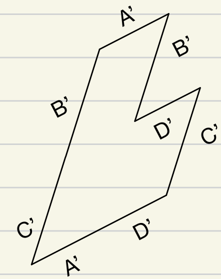
Lecture 3

I Recap

$SL(2,\mathbb{R})$ action on moduli spaces of flat surfaces.



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow$$



Sub actions

g_t : diagonal action also known as Teich. geodesic flow

h_s : horocycle flow

r_θ : rotations.

I Recap

$$SL(X,q) = \{ A \in SL(2,\mathbb{R}) \text{ such that } A(X,q) = (X,q) \}$$

$\downarrow$ as unmarked flat surfaces.

Theorem (Smillie):

$SL(2,\mathbb{R})$ orbit of q closed $\Longleftrightarrow$ $SL(X,q)$ is a lattice.

II Veech dichotomy

A flat surface (X,q) has optimal (straight line) dynamics if for any $\theta \in [0,2\pi)$, the foliation F_θ on (X,q) in the direction θ is either

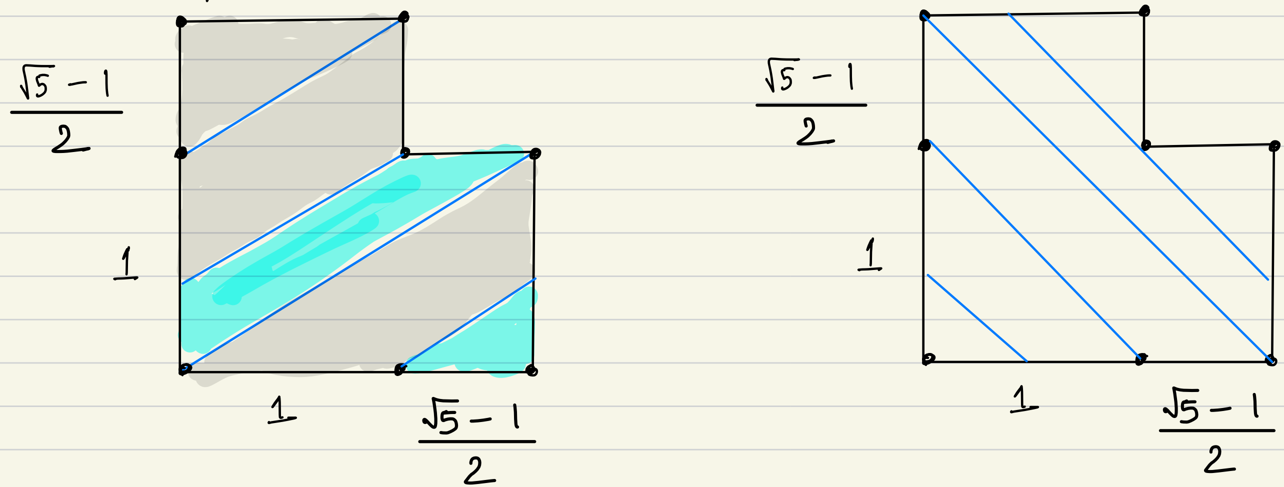
1) uniquely ergodic
2) completely periodic, namely $(X,q) = \bigcup_{\substack{\text{finite}\\ \text{union}}} C_i$

where C_i are metric cylinders with slope θ.

II

Veech dichotomy

For example



Theorem : $SL(X,q)$ lattice $\Rightarrow$ (X,q) has optimal dynamics.

Caution : Converse is not true !

II

Lattice $\Rightarrow$ optimal dynamics

Recall the setup

$$\begin{tikzcd} SL(2,\mathbb{R})\, q \arrow[r, hook] \arrow[d] & Q\,\mathrm{Teich}(\underline{k}) \arrow[d] \\ SO(2,\mathbb{R})\backslash SL(2,\mathbb{R}) = \mathbb{D}_q \arrow[r, hook] & \mathrm{Teich}(S_{g,n}) \end{tikzcd}$$

isometrically embedded hyperbolic disc

called Teich. disc of q

$$\mathbb{D}_q / SL(X,q) \hookrightarrow \mathrm{Teich}(S_{g,n}) / \mathrm{Mod}$$

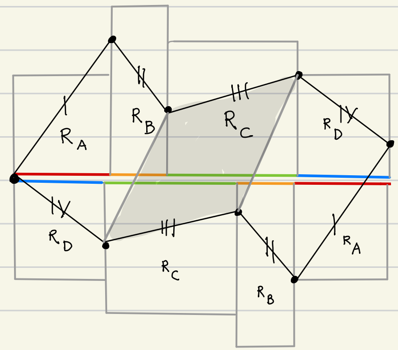
$$\shortparallel$$

$M_{g,n} \longrightarrow$ moduli of Riem. surfaces.

II

Lattice $\Rightarrow$ optimal dynamics

Step 1: (X, q) contains a cylinder.



Step 2: $\mathbb{D}_q / SL(X, q)$ is not compact

Suppose the said cylinder is in direction θ with respect to vertical.

Then the core curve gets arbitrarily short (cylinder modulus gets arbitrarily large) in $g_t r_{-\theta}(X, q)$ as $t \to \infty$.

II

Lattice $\Rightarrow$ optimal dynamics

Step 3 : Conclude $SL(X,q)$ is a non-uniform lattice, that is $\mathbb{D}_q / SL(X,q)$ a finite area hyperbolic surface with cusps.

Step 4: Consider any direction ϕ on (X,q).
Consider Teich. ray $\gamma_t = g_t \, r_{-\phi}(X,q)$ on $\mathbb{D}_q / SL(X,q)$.

Rays on $\mathbb{D}_q / SL(X,q)$ either recur to a compact part or head straight out a cusp.

If γ_t recurs then foliation F_ϕ is uniquely ergodic by a theorem of Masur.

II

Lattice $\Rightarrow$ optimal dynamics

Step 5 : Suppose γ_t heads out a cusp.

Then $SL(X,q)$ contains a corresponding parabolic element.

Suppose the parabolic is $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

Step 6 : Deduce every vertical leaf that does not run into singularities is closed.

Substeps

1 : Argue some power of the parabolic fixes each vertical separatrix pointwise

2 : Conclude every vertical sep. is a saddle connection

3 : Further conclude cylinder decomposition

QED

III Dynamics of the Teichmüller flow

Inhomogeneity : A flat surface (X, q) is ϵ-thin if there exists a saddle connection α s.t $|\mathrm{per}_q(\alpha)| < \epsilon$.

Thick part influences rank 1 behaviour ; thin parts higher rank.

Dynamics of g_t analogous to rank $\underline{1}$.

III Dynamics of the Teichmüller flow

For any $SL(2,\mathbb{R})$-invariant orbit closure

- g_t is ergodic with respect to the $SL(2,\mathbb{R})$-invariant measure

- g_t is exponentially mixing satisfying the same decay of correlations as proved by Moore-Ratner.

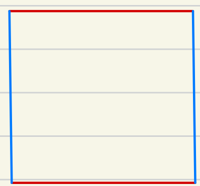
  Avila-Gouëzel-Yoccoz

  Avila-Gouëzel.

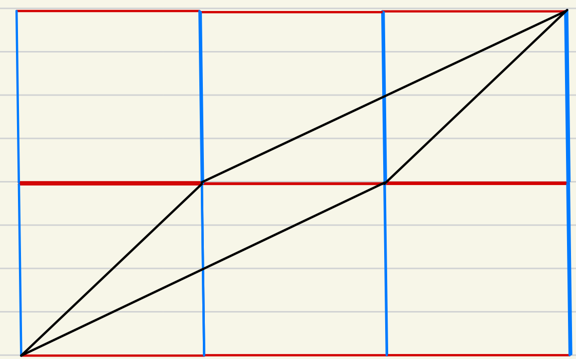
spectral gap for g_t action on appropriate function space.

IV Kontsevich-Zorich co-cyle

The co-cyle that records the action on the absolute homology of the surface.



$$\xrightarrow{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}$$

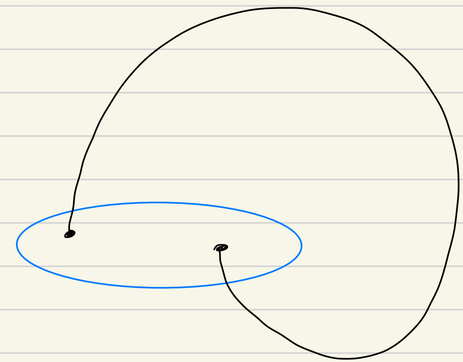


Red and Blue curve a $\mathbb{Z}$-basis for homology.

Action of g_t in the eigendirections give a change of basis which in this case is the same matrix.

IV Kontsevich-Zorich co-cyle

Cocycle along the flow



trivialise absolute homology over contractible period chart

Record action on the absolute homology when g_t returns us to the period chart.

Cocycle preserves algebraic intersection on H_1 and hence takes values in $Sp(2g, \mathbb{Z})$.

IV Kontsevich-Zorich co-cyle

Overall picture: Suppose $\mathcal{C}$ is a stratum component.

$$\pi_1(\mathcal{C}) \xrightarrow[\text{monodromy}]{\text{modular}} \mathrm{Mod} \xrightarrow[\text{symplectic}]{\text{standard}} \mathrm{Aut}(H_1(S;\mathbb{Z})) = Sp(2g, \mathbb{Z})$$

Co-cycle over the flow is the part of the symplectic monodromy that the flow detects.

IV Kontsevich-Zorich co-cyle

A map $B : \mathcal{C} \times \mathbb{R}_{\geq 0} \longrightarrow Sp(2g, \mathbb{Z})$ is a cocycle if

1) $B(q, 0) = Id$; and

2) $B(q, t+s) = B(g_t q, s)\, B(q, t)$.

Oseledets :

If the maps $q \longrightarrow \log \|B(q,t)\|$ and $q \longrightarrow \log \|B^{-1}(q,t)\|$ are L^1 with respect to an ergodic g^t- invariant measure μ then for μ-almost every q and every $v \in \mathbb{R}^{2g}$

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\|B(q,t)\, v\|}{\|v\|}$$ exists and assumes

up to $2g$ different values $\boxed{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2g}.}$ Lyapunov Spectrum.

IV Kontsevich-Zorich co-cyle

Zorich: Oseledets theorem applies to the Kontsevich-Zorich co-cycle.

Kontsevich-Zorich conjecture

The Lyapunov spectrum of the Kontsevich-Zorich co-cycle over the full stratum component is simple, that is the exponents λ_i are distinct.

V Why the interest

1 Interval exchange maps:

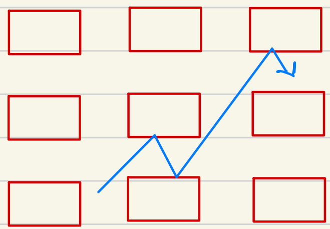
Zorich relates counting visits to subintervals under iteration to the co-cycle getting deviations of ergodic averages in terms of the exponents.

2 Wind-free models

V

Why the interest

Ehrenfest wind-tree model: infinite billiards in the plane.



$\mathbb{Z} \oplus \mathbb{Z}$ periodic identical rectangular obstacles with sides aligned along the lattice.

Fraczek - Ulcigrai : Non-ergodicity; almost every direction does not equidist.

Avila - Hubert : Recurrence properties; a.e direction

Delecroix - Hubert - Lelievre : Abnormal diffusion rate $\sim T^{\boxed{2/3}}$

independent of obstacle shape.

second Lyapunov exponent for assoc. flat surface.

V

Why the interest

Counting periodic trajectories of length $\leq L$ / $\mathbb{Z}^2$-translations

Pardo: For almost-every wind-tree

$$N(L) = \frac{1}{2\pi^2}\,\pi L^2 + \underbrace{o(L^2)}_{\text{bad trajectories}}.$$

counting of bad trajectories related to Lyapunov exponents of an associated co-cycle.

VI Lyapunov simplicity

Approach hinges on two ingredients

1) g_t dynamics has a coding with approximate product structure.

2) Large co-cycle in terms of image in $Sp(2g, \mathbb{Z})$.

VI Lyapunov simplicity

Forni 2002 : simplicity in genus 2

Avila-Viana 2007 : simplicity for all abelian strata.

Avila-Matheus-Yoccoz 2017 : Zariski density of co-cycle for hyper-elliptic strata.

Gutierrez-Romo 2018 : Zariski density of cocycle for all abelian strata + many quadratic strata.

Bell-Delecroix-G-Gutierrez Romo-Schleimer : Zariski density in +/- pieces for all quadratic $\Longrightarrow$ Lyapunov simplicity.