The Index of Nash Equilibria

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Finite Normal-Form Games

- We consider in these lectures, the set of finite games with fixed strategy sets and parametrized by the payoff functions.
- Set of players is $\mathcal{N} = \{ 1, \ldots, N \}$.
- For each n, a finite set S_n of pure strategies. $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- Σ_n is player n's set of mixed strategies. $\Sigma \equiv \Sigma_n$.
- For each n, let $S_{-n} = \prod_{m \neq n} S_m$ and $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$.
- A game is now specified by a payoff function $G_n : S \to \mathbb{R}$ for each n.

The Space of Games

- The space of games with the strategy space S is $\Gamma \equiv \mathbb{R}^{\mathcal{N} \times S}$. A typical game is a payoff vector $G \in \Gamma$.
- The Nash equilibrium correspondence from Γ to Σ assigns to each G its set $E(G)$ of Nash equilibria.
- Let E be the graph of the equilibrium correspondence, i.e., E is the set of $(G, \sigma) \in \Gamma \times \Sigma$ such that σ is a Nash equilibrium of G.
- Let $p : E \to \Gamma$ be the natural projection, $p(G, \sigma) = G$.

The Structure Theorem for Finite Games

- The structure theorem shows that the graph of equilibria, E, is a manifold of the same dimension as Γ. As we shall see below, it shows a bit more than that, actually.
- First, given G, for each $n \in \mathcal{N}$ and $s_n \in S_n$, let $g_{n,s_n} \equiv G(s_n, \sigma^*_{-n})$, where σ^* is the strategy profile consisting of uniform mixtures for each player.
- Then let $\tilde{G}(s) = G(s) - g_{n,s_n}$.
- Each G can be decomposed uniquely into the pair $(\tilde{G}, g) \in \mathbb{R}^{\mathcal{N}\times S} \times_n \mathbb{R}^{S_n} \equiv \tilde{\Gamma} \times Z$ and we will frequently represent G as such a pair.

The Structure Thm—-contd.

Let $(\tilde{G}, g, \sigma) \in E$. Then for each n, σ_n solves the maximization problem:

$$\max_{\tau_n} \tilde{G}(\sigma_{-n}, \tau_n) + \sum_{s_n} g_{n,s_n} \tau_{n,s_n}$$

$$\text{s.t.} \sum_{n,s_n} \tau_{n,s_n} - 1 = 0$$

and

$$\tau_{n,s_n} \geqslant 0.$$

The Structure Thm—contd.

From the Kuhn-Tucker conditions for Player n:

$$\begin{aligned} \tilde{G}_n(\sigma_{-n}, s_n) + g_{n,s_n} - v_n + \mu_{n,s_n} &= 0 \\ \sigma_{n,s_n}\mu_{n,s_n} &= 0. \end{aligned}$$

where v_n is player n's equilibrium payoff.

- From $(\tilde{G}, g, \sigma) \in E$, we can uniquely get $(\tilde{G}, \sigma, v, \mu)$.
- Likewise, from any $(\tilde{G}, \sigma, v, \mu)$ such that $(\sigma, \mu) \in \Sigma \times_n \mathbb{R}_+^{S_n}$ satisfies complementary slackness, we can obtain a unique g such that $(\tilde{G}, g, \sigma)$ is in E.

The Structure Thm—contd.

- If we let $z_n = \sigma_n + v_n - \mu_n$, then $\sigma_n = r_n(z_n)$, where r_n is a retraction, mapping z_n to the point in Σ_n that is closes to it in ℓ_2-distance and
- $v_n - \mu_n = z_n - \sigma_n$ so that: $g_n = z_n - \sigma_n - \tilde{G}(s_n, \sigma_{-n})$.
- If we define $h : E \to \tilde{G} \times Z$ by $h(\tilde{G}, g, \sigma) = (\tilde{G}, z)$ where $z_{n,s_n} = \sigma_n + \tilde{G}(s_n, \sigma_{-n}) + g_{n,s_n}$, then h is a homeomorphism.
- $h^{-1}(\tilde{G}, z) = (\tilde{G}, g, r(z))$ with $g_{n,s_n} = z_{n,s_n} - \sigma_{n,s_n} - \tilde{G}(s_n, \sigma_{-n})$.
- $h \circ h^{-1}$ and $h^{-1} \circ h$ are easily verified to be identities.

The Structure Thm—contd

We are now ready to state the structure theorem.

Theorem (Kohlberg-Mertens, 1986)

h is a homeomorphism between E and Γ. The homeomorphism extends to the one-point compactifications $\bar{E}$ and $\bar{\Gamma}$ of E and Γ. Moreover, $p \circ h^{-1} : \bar{\Gamma} \to \bar{\Gamma}$ is homotopic to the identity.

- $p \circ h^{-1}$ is homotopic to the identity on the domain of Γ using the linear homotopy.
- The linear homotopy extends to a continuous map at infinity because $\|z - g\|_\infty$ is bounded for each fixed $\tilde{G}$.

Some Implications of the Structure Theorem

- Since the retraction r is piecewise-linear, h^{-1} is piecewise polynomial and hence E is a piecewise-analytic manifold of the same dimension as Γ.
- Since E is a semi-algebraic set (i.e., it is defined by finitely many polynomial inequalities) there exists a lower-dimensional set Γ_0 such that for each $G \in \Gamma \setminus \Gamma_0$: $p^{-1}(G)$ is a finite set; p is locally a differentiable at $(G, \sigma) \in p^{-1}(G)$ and has a differentiable inverse; h^{-1} is differentiable at $h(G, \sigma)$.
- We say that G is a regular game if it does not belong to Γ_0 and we say that an equilibrium is regular if it is an equilibrium of a regular game.

Degree of Equilibria

For simplicity, let $\psi = p \circ h^{-1}$.

Definition

Let G be a regular game and σ an equilibrium of G. The degree of σ, denoted $\deg(\sigma; G)$ is $+1$ (resp. -1) if the determinant of $D\psi$ computed at $h(G, \sigma)$ is positive (resp. negative).

The definition extends to an arbitrary game G and a component C of the set $E(G)$ of equilibria of G as follows.

Definition

Take an open neighborhood V of $\{ G \} \times C$ in E whose closure is disjoint from the other components of G. Let U be an open neighborhood of $G \in \Gamma$ s.t. $p^{-1}(U)$ does not contain points on the boundary of V. The degree of C, denoted $\deg(C; G)$, is the sum of the degrees of all equilibria of a regular game G' that are contained in V.

Remarks on the degree

- $\sum_C \deg(C; G) = 1$ for each G, which is an implication of the fact that ψ is homotopic to the identity.
- The degree of a strict equilibrium is $+1$, since $D\psi$ is the identity map.
- There exist pure strategy equilibria with degree zero.
- For each integer d there exists a game and a component of its equilibria with degree d. (Govindan, von Schemde, von Stengel, 2003).

Index of Equilibria

- Since Nash equilibria are obtained as solutions to fixed-point problems or rest points of dynamical systems, there is a natural notion of an index for a component that derives from fixed-point theory.
- Multiple maps and dynamical systems have been studied in the literature. Here are some examples.
- Nash gave two existence proofs, one applying the Kakutani fixed point theorem to the best-reply correspondence; the other applying Brower to a function.
- Gul Pearce and Stacchetti defined a map that exploits a variational inequality argument in the definition of Nash equilibria.
- Evolutionary game theory considers a lot of dynamical systems whose rest points are, or at least include, Nash equilibria.

Questions about the index

- Does the index of a component depend on the fixed-point map we consider?
- Is the index the same as the notion of degree we have seen?
- Without additional restrictions on the fixed-point maps, the answers to these questions would be no.
- When the fixed points are continuous in the game, then we get a full equivalence.

Invariance of the Index

Definition

A Nash map is a continuous function $f : \Gamma \times \Sigma \to \Sigma$ such that for each G, the fixed points of $f(G, \cdot)$ are the Nash equilibria of G.

For each Nash map f, each game G and each component C of $E(G)$, we can define its index, which is denoted $\text{ind}_f(C; G)$.

Theorem (Demichelis-Germano, 2002)

For any two Nash maps, f_1 and f_2, $ind_{f_1}(C; G) = ind_{f_2}(C; G)$ for all games G and components C of $E(G)$.

Sketch of the proof

- Let Δ be a ball around Σ. Define $H : \Gamma \times \Delta \to \tilde{\Gamma} \times Z \times \Delta$ to $H(\tilde{G}, g, \tau) = (\tilde{G}, z, \tau)$ where $z_{n,s_n} = \tau_{n,s_n} + G(\sigma_n, \tau_{-n})$. H is a homeomorphism.
- Let $D = (z, \tau)$ such that $r(z) = \tau$. Then $H(E) = \tilde{\Gamma} \times D$.
- $\tilde{\Gamma} \times Z \times \partial\Delta$ is a deformation retract of $\tilde{\Gamma} \times (Z \setminus D)$: retract (z, τ) to (z, τ_0) where τ_0 is the unique point on the line from $r(z)$ to τ that lies on $\partial\Delta$.
- $\Gamma \times \partial\Delta$ is a deformation retract of $(\Gamma \times \Delta) \setminus E$. Let φ be the retraction.

Sketch—-contd.

- for $i = 1, 2$ if we let d_i be the displacement map, $d_i(G, \sigma) = \sigma - f_i(G, \sigma)$, then the restriction of d_i to $\Gamma \times \Delta \setminus E$ is homotopic to $d_i \circ i \circ \varphi$, where i is the inclusion $i : \Gamma \times \partial\Delta \to (\Gamma \times \Delta) \setminus E$.
- the restriction of the d_i's to $\Gamma \times \partial\Delta$ are homotopic, therefore the restrictions of d_i to $\Gamma \times \Delta \setminus E$ are.
- extend the homotopy to E as well.

Remarks

- The map H extends the homeomorphism of the K-M structure theorem from E to the whole of $\Gamma \times \Sigma$ mapping now to $\tilde{\Gamma} \times Z \times \Sigma$ where E is mapped to $\tilde{\Gamma} \times D$.
- Thus H "unknots" the manifold of equilibria.
- The result extends to vector fields, since what is used here is the displacement map.
- Systems like the replicator dynamics are somewhat different, since they admit spurious rest points. As long as their indices are zero, the results can be extended.

Equivalence of Index and Degree

- To prove the equivalence of the index (defined using a Nash map) and the degree, we use the map defined by Gul-Pearce-Stacchetti (1993).
- The GPS map is defined as follows. First for each $G \in \Gamma$ and $\sigma \in \Sigma$, and n, s_n, let $\pi_{n,s_n}(G, \sigma) = \sigma_{n,s_n} + G_n(s_n, \sigma_{-n})$.
- The GPS map $f : \Gamma \times \Sigma \to \Sigma$ is given by $f(G, \sigma) = r \circ \pi$, where r is the retraction we defined before.
- σ is a Nash equilbrium of G iff $\pi(G, \sigma)$ retracts to σ.

Equivalence of Index and Degree

Theorem (See Govindan and Wilson, 2005)

$ind_f(C; G) = deg(C; G)$ for each game G and each component C of its equilibria.

Sketch of Proof:

- Fix G and consider π as a function defined on Σ. σ is an equillbrium of G iff it is a fixed point of $r \circ \pi$.
- $\psi(\tilde{G}, z) = (\tilde{G}, g)$ iff z is a fixed point of $\pi \circ r$.
- By the commutativity, the index of a component of fixed points of the first map is the same as that of its image under the second.
- the displacement map for the second map is precisely the map that is used to compute the degree of equilibria.

Index of the BR-correspondence

- Again, we the GPS map to show that the index defined using the BR-correspondence coincides with that of any Nash map.
- Fix a game G. For each $\lambda > 0$, the game λG is ordinally equivalent to G and has the same BR correspondence; also the index of the equilibria computed under a Nash map is invariant under λ.
- For large λ, the GPS map of λG is a good approximation of the BR correspondence.
- Specifically, For $\varepsilon > 0$ the graph of the GPS map of λG is within ε of the graph of BR if λ is large.
- It now follows that the index of a component computed using BR coincides with that computed using λG for large λ.

Some Properties of Index

- The fact that the sum of the indices of equilibria is $+1$ gives rise to the "generic oddness" result, namely that regular equilibria have an odd number of equilibria.
- Since pure strategy equilibria have index $+1$, if a game has $n \geqslant 1$ pure equilibria, it has at least $n - 1$ mixed equilibria.
- If the index of a component is non-zero, then every game close by has a close-by equilibrium. The converse is not true. (see the example by Hauk and Hurkens, 2002).

Some Remarks on dynamical systems

- The index of equilibria has implications for dynamical stability.
- Equilibria with index -1 typically have an unstable manifold and are therefore not stable.
- It is possible that even equilibria with index $+1$ have an unstable manifold.
- Demichelis and Ritzberger (2002) show that a necessary condition for a component of equilibria to be stable under a dynamic is that its index agree with its Euler characteristic.
- They show furthermore that if the index is zero, and if it asymptotically stable under some dynamic, then there exists a perturbation of the dynamic for which there are no zeros in a neighborhood of the component.

Harsanyi's Purification Theorem

- The interpretation of a mixed strategy is straightforward in a zero-sum game: typically, randomization is optimal if a player assumes that his opponent will discover the exact mixture he is employing.
- In a non-zero-sum game, the precise distribution a player uses matters to his opponents and not to himself.
- One interpretation of a mixed strategy is that it represents the beliefs of a player's opponents about what he is likely to play. Thus the equilibrium is in beliefs.
- Harsanyi (1973) showed that if the payoffs of players are perturbed and the realized perturbation is private knowledge to the players, then in the resulting game with incomplete information, players employ a pure strategy, though the ex ante distribution approximates a mixed equilibrium, thus "purifying" it.

Harsanyi's model

- Given a game G, and a product distribution μ over Γ, consider the following game with incomplete information where the strategy sets of the players are as in G.
- Nature draws a vector $\eta \in \Gamma$ according to the distribution μ and each player is informed of his component of η.
- Given the realization η_n, player n's payoff function is $G_n + \eta_n$.
- A strategy for player n is a measurable map from his perturbation to S_n. Given the structure of payoffs, the only relevant aspect of his opponents' strategies is the induced distribution over their actions.

Harsanyi's model—contd.

- Given a distribution σ_{-n} of actions, a strategy s_n is better than t_n for n for type η_n iff $\eta_{n,s_n} - \eta_{n,t_n} \geqslant G(s_n, \sigma_{-n}) - G_n(t_n, \sigma_{-n})$.
- The set of types that are indifferent between s_n and t_n lie on a hyperplane.
- If we assume that each μ_n is absolutely continuous w.r.t. to the Lebesgue measure, then for a.e. type, there is a unique best reply.
- From the incomplete-information game, we now have a fixed point problem on Σ. Start with a distribution σ; compute the unique best reply to it in the incomplete-information game and recompute the implied distribution.

The Purification Theorem

- This best-reply function is continuous and has a fixed point.
- If μ_n is close to putting all its mass on 0, then the graph of this best-reply function is close to the best-reply correspondence of the original game.
- If the game is regular, then every regular equilibrium is approximated by an equilibrium of this game. Hence the theorem:

Theorem

Let G be a regular game and σ a regular equilibrium. Let μ^k be a sequence of distributions converging to point mass at zero that are absolutely continuous w.r.t. the Lebesgue measure. There exists a sequence of pure-strategy equilibria of these games such that the corresponding sequence of distributions σ^k converges to σ.

Addition or Deletion of Duplicate Strategies

- Kohlberg-Mertens argued that solutions to games should be invariant to the addition or deletion of duplicate strategies. More specifically:
- Suppose we have for each n, a set S_n^* that contains S_n and, letting Σ_n^* be the set of mixtures over S_n, an affine map $\rho_n : \Sigma_n^* \to \Sigma_n$ whose restriction to Σ_n is the identity such that in the game G^* with strategies S^*, $G(\sigma^*) = G(\rho(\sigma^*))$.
- Then we say that G^* is obtained from G by the addition of duplicate strategies. And K-M require that G and G^* have the same solutions.
- Clearly the set of Nash equilibiria of G are the images of the Nash equiilbria of G^* under ρ.
- Here we ask: what is the relation between $\text{ind}(C; G)$ and $\text{ind}(\rho^{-1}(C); G^*)$?

Invariance under the addition of duplicate strategies

Theorem (See Govindan and Wilson, 2005)

$ind(C; G) = ind(\rho^{-1}(C); G^*)$

Sketch of Proof.

- Take a neighborhood U of C that whose closure is disjoint from the other components of G.
- Let $U^* = \rho^{-1}(U)$ and $C^* = \rho^{-1}(C)$. Take a neighborhood W^* of the graph of BR^* such that the index of C^* can be computed by sum of the indices of the fixed points in U^* for any function h^* whose graph is in W^*.
- Let W be the neighborhood of BR consisting of (σ, τ) such that $\rho^{-1}(\sigma) \times \rho^{-1}(\tau)$ is contained in W^*.

Proof—contd.

- Take a function h whose graph is contained in W and that can be used to compute the index of C.
- Define h^* by $h^* = i \circ h \circ \rho$ where $i : \Sigma \to \Sigma^*$ is the inclusion.
- Letting $h^0 = i \circ h$, we have $h^* = h^0 \circ \rho$ and $h = \rho \circ h^0$.
- Use now the commutativity property of index to conclude the result.

Essential Equilibria

- An equilibrium σ of a game G is essential if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each δ-perturbation of G there is an equilibrium that is within ε of σ.
- Every regular equilibrium is essential, but in non-generic games there might not be an essential equilibrium.
- To get existence, we could say that a component C is is essential if: for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each δ-perturbation of G there is an equilibrium that is within ε of C.

Hyperstability

- Every component with a non-zero index is essential
- However, if a component has index zero, it could still contain an essential set. (Hauk and Hurkens, 2002).
- A component is hyperstable if it is equivalent to an essential component of every equivalent game, i.e. a game obtained by the addition or deletion of duplicate strategies.
- From our previous result, if a component has a non-zero index, it is hyperstable.
- The questions is: does the converse hold?

Uniform Hyperstability

We do not know the answer to this question (except when the game is a generic extensive-form game). But the converse holds if we strengthen the requirement of hyperstability as follows.

Definition

A component C is uniformly hyperstable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each equivalent game G^* and each δ-perturbation of G^*, there exists an equilibrium whose image under ρ is within ε of C.

We now have the following theorem.

Theorem (Govindan and Wilson, 2005)

A component C is uniformly hyperstable iff its index is nonzero.

Sketch of the proof

- Suppose the index of C is non-zero. Fix $\varepsilon > 0$. We can assume, if necessary by replacing ε with a smaller number, that the closed ε-neighborhood U of C does not intersect the other components of equilibria.
- There exists $\delta > 0$ such that for each profile σ that is on the boundary of the neighborhood U some player gets at least $\delta/2$ more by choosing an optimal reply against σ than playing σ_n.
- Take now an equivalent game G^* and a δ-perturbation of G^* (in the ℓ_∞ norm).
- The δ-perturbation of G^* does not contain an equilibrium in the boundary of the inverse image of U under the map ρ.
- The sum of the degrees of the components of this perturbation is the same as that of C. Hence, it is nonzero and thus it has an equilibrium that projects to a point in U.

Sketch of the proof—contd.

- Suppose that a component C has index zero. Take a neighborhood U of C such that its closure is disjoint from the other components of G. Fix $\delta > 0$.
- There exists a function $h : \Sigma \to \Sigma$ whose graph is within δ of the graph of BR and which does not have a fixed point in U.
- By adding duplicate strategies we can construct a game G^* and a $K\delta$-perturbation of G^* (for some constant K) of G^* such that for this perturbation to have an equilibrium in $\rho^{-1}(U)$ requires h to have a fixed point in U.

An Application of the Structure Theorem

- Index theory is quite a useful tool in computing fixed points, especially in path-following or homotopy algorithms.
- The homotopy principle works as follows. Suppose we want to compute a fixed point of a function f_0; we first transform f_0 to a function f_1 with a unique, easily-computed solution.
- Then we trace a homotopy from f_1 back to f_0, tracing fixed points along the way, i.e., we track a path in the graph of the fixed-point correspondence over the homotopy.
- In the case of games, f_0 and f_1 correspond to two games and we are tracing a path of equilibria over the interval between the games.

A Global Newton Method

- Let $G = (\tilde{G}, g)$ be a game whose equilibria we want to compute.
- For each $h \in Z$ with norm 1, let $L(h)$ be the set of games $(\tilde{G}, g + \lambda h)$ for $\lambda > 0$ and let $E(h)$ be the cross-section of E over $L(h)$.
- The structure theorem implies that for generic h, the closure of $E(h)$ is a 1-dimensional manifold with finitely many connected components.
- Moreover for generic h, $(\tilde{G}, g + \lambda h)$ has a unique equilibrium in dominant strategies for λ large.
- We can now trace a path in the graph starting at this unique equilibrium for some large λ.

Global Newton Method—contd.

- We now define a dynamical system in the graph over a generic ray or equivalently under its image in Z
- This path has an image under the map ψ in the space of games that is merely tracing the ray.
- Let the unique equilibrium of the game $g + \lambda_0 h_0$ be represented as the point $z_0 \in Z$.
- $\dot{h} = -(\text{sign}|D\psi|)h$ and $\dot{z} = D\psi^{-1}h$; whether the movement is positive or negative depends on whether the current equilibrium has index ± 1.
- The limit point of this system computes a $+1$ equilibrium if the game is generic.

Extensive-Form Games

- For a fixed finite game tree $\mathcal{T}$, consider the space of games obtained by varying the payoffs at the terminal nodes.
- What can we say about the structure of equilibria over this space of games?
- Since even for generic payoff assignments, there exists a continuum of equilibria, the restriction of the equilibrium correspondence over all normal form games (with the same pure strategy set as in $\mathcal{T}$) to extensive form games does not give us a structure theorem.
- However, we know from Kreps-Wilson that for generic extensive form games, there exist finitely many equilibrium distributions (over the terminal nodes).
- Could one obtain a structure theorem for outcomes?

A Counterexample

Consider the following parametrized game $G_{x,y}$:

$$\begin{pmatrix} 2,2 & 2,2 \\ x,3 & y,1 \end{pmatrix}$$

- The game $G_{3,3}$ has a unique outcome; $G_{3,1}$ has two outcomes; both are robust; hence, there is no hope for a degree theory.
- Actually, the graph is not even a boundaryless manifold (even though it has the same dimension as the space of games) because:
- The game $G_{3,2}$ has a neighborhood that is a manifold with boundary.
- The game $G_{2,1}$ represents a bifurcation point.

Enabling Strategies

- The misbehavior of the graph arises from the fact that some of the outcomes are on the boundary of the set of distributions over the terminal nodes.
- We will now show that if we restrict players to play in a polyhedron in the interior of the strategy space, so that we are in the interior of the space of outcomes, there is a structure theorem for games.
- To do that, we will introduce a new class of strategies, we call enabling strategies
- The structure theorem will be be like that for normal-form games using completely-mixed enabling strategies

Enabling Strategies—contd.

- Enabling strategies are equivalent to mixed strategies and behavioral strategies (and strategies in sequence form, introduced by von Stengel, 1996).
- The set of enabling strategies is a polyhedron of the same dimension as that of behavioral strategies (which is typically much smaller than that of mixed strategies).
- Payoffs are multilinear in enabling strategies (unlike with behavioral strategies).
- Thus with enabling strategies, we have a game in strategic form, where the strategy sets are polyhedra and the payoffs are multilinear.

Definition of Enabling Strategies

- Say that an action i of player n is a "last action" of player n if there exists a terminal node x such that i is the last action along the unique path from the root of the tree to x.
- Let L_n be the set of last actions of player n.
- We now construct a map $f_n : \Sigma_n \to [0,1]^{L_n}$ as follows: for each σ_n and $i \in L_n$, let $f_{n,i}(\sigma)$ be the total probability of the pure strategies that play all actions leading up to (and including) i.
- f_n is linear and so its image is a polyhedron, which we call the set of enabling strategies.

A Structure Theorem in Extensive-Form Games

Let Γ be the space of all games with the tree $\mathcal{T}$. Fix now a polyhedron P_n contained in the interior of the set of enabling strategies. Let E be the graph of the equilibrium correspondence over Γ.

Theorem

E is homeomorphic to Γ and the projection map has degree 1.

Remark: The equilibria of a game G is obtained as the solution to a variational inequality involving the polyhedron P. Like in the K-M theorem the dual to this map gives us the structure theorem. We need to restrict strategies to the interior in order to avoid problems of division by zero. (See Govindan and Wilson 2002 for details.)

Finite Stochastic Games

- The structure theorem extends in a straightforward way to stationary equilibria of finite stochastic games.
- Suppose that there is a finite state space Ω. The set of actions of each player n in each state is S_n. There is a probability transition function $P : \Omega \times S \to \Omega$.
- The payoffs in state ω are $G(\omega, \cdot)$. The space of games is $\mathbb{R}^{\mathcal{N} \times \Omega \times S}$. The discount factor is $0 < \beta < 1$.
- A stationary equilibrium is a pair $(\sigma, v) \in \Sigma^{\Omega} \times \mathbb{R}^{\mathcal{N} \times \Omega}$ such that for each state ω, $\sigma(\omega)$ is an equilibrium of the finite normal-form game where the payoffs of each player n are given by : $G_n(\omega, \cdot) + \beta \sum_{\omega'} P(\omega, \sigma(\omega)) v_n(\omega')$.

Stochastic Games

- Decompose a game, state-by-state $(\tilde{G}(\omega), g(\omega))_{\omega \in \Omega}$.
- Given an equilibrium (G, σ, v), let $z_{n,s_n}(\omega) = \sigma_{n,s_n}(\omega)) + G_n(s_n, \sigma_{-n}(\omega)) + \beta \sum_{\omega'} P(\omega, s_n, \sigma_{-n}(\omega)) v_n(\omega')$
- With this modification, the structure theorem works as in the case of normal-form games.
- All the implications of the structure theorem go through as well.

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