Normal and Extensive Form Refinements

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John Hillas

j.hillas@auckland.ac.nz

Department of Economics

University of Auckland

Outline of Lecture

- Normal and Extensive Form Games
  Definitions, Perfect Recall, Strategies, Kuhn’s Theorem

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- Admissible Equilibria and Normal Form Perfect Equilibria

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- The Need For Set Valued Dolutions
- Forward Induction
- Introduction to Strategic Stability

Normal Form Games

Definition 1. A *normal* or *strategic form game* consists of:

1. N, a finite set of players. (We abusively also use N to denote the number of players.)
2. For each player $n \in N$, a finite set of pure strategies S_n, with $S = \times_{n \in N} S_n$.
3. For each player $n \in N$, a payoff function $u_n : S \to \mathbb{R}$.

Normal Form Games

Definition 2. A mixed strategy for Player n in a normal form game is a probability distribution over the player's pure strategies. We denote the set of Player n's mixed strategies Σ_n and call it Player n's mixed strategy space and define $\Sigma = \times_{n \in N} \Sigma_n$ the mixed strategy space. We denote a typical element of Σ_n by σ_n and a typical element of Σ by $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$.

We extend the function u_n to Σ linearly, that is, by taking expectations, and indeed to $\times_n (S_n \cup \Sigma_n)$.

Normal Form Games

Definition 3. Given a game (N,S,u), a mixed strategy profile $(\sigma_1^*,\sigma_2^*,\ldots,\sigma_N^*)$ is a *Nash equilibrium* if for each $n \in N$ and each $s_n \in S_n$,

$$u_n\left(\sigma_1^*,\sigma_2^*,\ldots,\sigma_n^*,\sigma_{n+1}^*,\ldots,\sigma_N^*\right) \geq u_n\left(\sigma_1^*,\sigma_2^*,\ldots,s_n,\sigma_{n+1}^*,\ldots,\sigma_N^*\right).$$

Normal Form Games

Rather than speaking directly of utilities we can define what it means to be a best reply to what the other players are playing.

Definition 4. Given a game (N, S, u), with Σ_n the mixed strategy space of Player n and Σ the space of mixed strategy profiles. Then the *best reply correspondence of Player n*, $BR_n : \Sigma \twoheadrightarrow \Sigma_n$ is defined as

$$BR_n(\sigma_1, \ldots, \sigma_N) = \{\tau_n \in \Sigma_n | u_n(\sigma_1, \ldots, \tau_n, \ldots, \sigma_N) \\ \geq u_n(\sigma_1, \ldots, s_n, \ldots, \sigma_N) \text{ for any } s_n \in S_n\}.$$

Normal Form Games

This allows us to restate the definition of a Nash equilibrium.

Definition 5. Given a game (N,S,u), a mixed strategy profile $(\sigma_1^*,\sigma_2^*,\ldots,\sigma_N^*)$ is a *Nash equilibrium* if for each $n \in N$,

$$\sigma_n^* \in BR_n\left(\sigma_1^*,\sigma_2^*,\ldots,\sigma_N^*\right).$$

Normal Form Games

Theorem 1 **(Nash (1950, 1951)).** *Any finite normal form game* (N, S, u) *has at least one Nash equilibrium.*

Extensive Form Games

The details of the formal definition of an extensive form game are a bit cumbersome and we'll skip it. We shall however briefly list the notation.

- A finite set of players, $N = \{1, 2, \ldots, N\}$. We add an artificial Player 0 or Nature.

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- A finite set of nodes, X, and X is a game tree, where $T \subset X$ is the set of terminal nodes and x_0 is the initial node.

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- A finite set of players, $N = \{1, 2, \ldots, N\}$. We add an artificial Player 0 or Nature.
- A finite set of nodes, X, and X is a game tree, where $T \subset X$ is the set of terminal nodes and x_0 is the initial node.
- A set of actions, A. $\alpha(x) \in A$ is the action at the predecessor of x that leads to x. If x and x' are distinct and have the same predecessor then $\alpha(x) \neq \alpha(x')$.

Extensive Form Games

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- We assume that all information sets in $\mathcal{H}_0$ are singletons and assign a probability $\rho(x)$ to each node that immediately follows such singleton information set of Nature.
- For each terminal node t and each Player n we have $u_n(t)$, the payoff or utility of Player n at terminal node t.

Extensive Form Games

In what follows rather than listing all of the elements of an extensive form game we shall simply refer to the game Γ and understand that all of these elements are specified.

Extensive Form Games

The information partition is said to have *perfect recall* (Kuhn, 1953) if the players remember whatever they knew previously, including their past choices of moves. In other words, all paths leading from the root of the tree to points in a single information set, say Player n's, must intersect the same information sets of Player n and must display the same choices by Player n.

Selten (1975) gave the same definition of perfect recall as Kuhn, but his formal definition is a little more straightforward. We give the definition here in the way that Selten did.

Extensive Form Games

Definition 6. A player is said to have perfect recall if whenever that player has an information set containing nodes x and y and there is a node x' of that player that precedes node x there is also a nodes y' in the same information set as x' that precedes node y and the action of the player at y' on the path to y is the same as the action of the player at x' on the path to x. If all players have perfect recall then we say the game has perfect recall.

Extensive Form Games

One implication of perfect recall is that each path from x_0 to a terminal node cuts each information set at most once. In games without perfect recall we distinguish between linear games, the games defined by Kuhn, and nonlinear games, an extension by Isbell (1957), and later under the name “repetitive games” by Alpern (1988), and more recently discussed under the name “absent-mindedness” by Piccione and Rubinstein (1997), and, following Piccione and Rubinstein, by a number of others. In linear games each play of the game reaches an information set at most once. In a nonlinear game we remove that restriction.

Extensive Form Games

Recall that we introduced the idea of a strategy earlier saying that it was a player's "complete plan of how to play the game." When we discuss normal form games we treat strategies as primitives and so this is an intuitive justification rather than a definition. When we come to deal with extensive form games a strategy is not among the primitive components. Rather it is a derived concept defined in terms of the primitives.

Definition 7. A pure strategy in an extensive form game for Player n is a function that maps each of his information sets to one of the actions available at that information set. We denote the set of Player n's pure strategies by S_n and the set of pure strategy profiles by $S = \times_{n \in N} S_n$

Extensive Form Games

Having now defined pure strategies we can associate to any extensive form game some associated normal form game. The player set is the same; we have just described the strategies, and for each profile of strategies we obtain a probability distribution over the terminal nodes—a probability distribution since there may be moves of nature; if there are no moves of nature then one terminal node will have probability 1—and hence an expected utility.

We define a mixed strategy precisely as we do for the normal form.

Extensive Form Games

Definition 8. A mixed strategy in an extensive form game for Player n is a probability distribution over the player's pure strategies. We denote the set of Player n's mixed strategies by Σ_n and the set of mixed strategy profiles by $\Sigma = \times_{n \in N} \Sigma_n$. We say that the mixed strategy σ_n of Player n is completely mixed if $\sigma_n(s_n) > 0$ for all $s_n \in S_n$. We say that the profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ is completely mixed if σ_n is completely mixed for all $n \in N$.

Extensive Form Games

Rather than having the player randomise over pure strategies we could have them randomise independently at each information set.

Definition 9. A behaviour strategy in an extensive form game for Player n is a function that maps each of his information sets to a probability distribution on the actions available at that information set. We denote the set of Player n's behaviour strategies by B_n and the set of behaviour strategy profiles by $B = \times_{n \in N} B_n$. We say that the behaviour strategy b_n of Player n is completely mixed if at each information set b_n assigns strictly positive probability to each of the actions available at that information set. We say that the profile $b = (b_1, \ldots, b_N)$ is completely mixed if b_n is completely mixed for all n.

Extensive Form Games

In nonlinear games we also need to consider randomisations over behaviour strategies. We *can* consider such strategies for linear games, but we do not need to do so.

Definition 10. A general strategy in an extensive form game for Player n is a probability distribution over the player's behaviour strategies. We denote the set of Player n's general strategies by G_n and the set of general strategy profiles by $G = \times_{n \in N} G_n$.

Extensive Form Games

We now define what it means for two strategies to be equivalent. Simply put two strategies of a player are equivalent if, whatever the other players do, the two strategies induce the same probability distribution over the terminal nodes.

Definition 11. Two strategies of Player n $x_n, y_n \in S_n \cup B_n \cup \Sigma_n \cup G_n$ are said to be Kuhn equivalent if for any strategies of the others x_{-n} the profiles (x_n, x_{-n}) and (y_n, x_{-n}) induce the same probability distributions over the terminal nodes.

Extensive Form Games

Kuhn (1953) showed that in linear games (the only games he considered) for any behaviour strategy there is always an equivalent mixed strategy and that if the player has perfect recall the converse is also true.

Theorem 2 (Kuhn (1953))**.** *In a linear game for any behaviour strategy b_n of Player n there exists a mixed strategy σ_n of Player n that is Kuhn equivalent to b_n. If in some extensive form game Player n has perfect recall then for any mixed strategy σ_n of Player n there exists a behaviour strategy b_n of Player n that is Kuhn equivalent to σ_n.*

Admissible and Normal Form Perfect

Definition 12. In a normal form game (N,S,u) a strategy $s_n \in S_n$ of Player n is *admissible* or *undominated* if there is no mixed strategy $\sigma_n \in \Sigma_n$ such that for all $s_{-n} \in \times_{m|m\neq n} S_m$

$$u_n(\sigma_n, s_{-n}) \geq u_n(s_n, s_{-n})$$

and for at least one $t_{-n} \in \times_{m|m\neq n} S_m$

$$u_n(\sigma_n, t_{-n}) > u_n(s_n, t_{-n}).$$

Admissible and Normal Form Perfect

Definition 13. An admissible equilibrium is a Nash equilibrium σ such that for all n in N if $\sigma_n(s_n) > 0$ then s_n is admissible, that is a Nash equilibrium in which only admissible strategies are played with positive probability.

Admissible and Normal Form Perfect

This seems a mild requirement. A slightly stronger refinement is that of normal form perfection.

Definition 14. A completely mixed strategy profile $\sigma \in \Sigma$ is a ε-perfect equilibrium if $\varepsilon > 0$ and for all $n \in N$ and all $s_n, t_n \in S_n$ if $u_n(s_n, \sigma_{-n}) < u_n(t_n, \sigma_{-n})$ then $\sigma_n(s_n) < \varepsilon$. A strategy profile $\sigma \in \Sigma$ is a normal form perfect equilibrium if there is a sequence of strategy profiles $\sigma^t \to \sigma$ and positive numbers $\varepsilon^t \to 0$ with σ^t a ε^t-perfect equilibrium.

Admissible and Normal Form Perfect

Selten (1975) proved the following theorem.

Theorem 3 (Selten (1975)). *For any finite normal form game (N, S, u) there is at least one normal form perfect equilibrium.*

Admissible and Normal Form Perfect

We give two results relating the two concepts we have just defined.

Theorem 4 (van Damme (1991))**.** *For any finite normal form game* (N, S, u) *if* σ *is a normal form perfect equilibrium then* σ *is admissible.*

Theorem 5 (van Damme (1991))**.** *For any finite two player normal form game* (N, S, u) *if* σ *is admissible then* σ *is a normal form perfect equilibrium.*

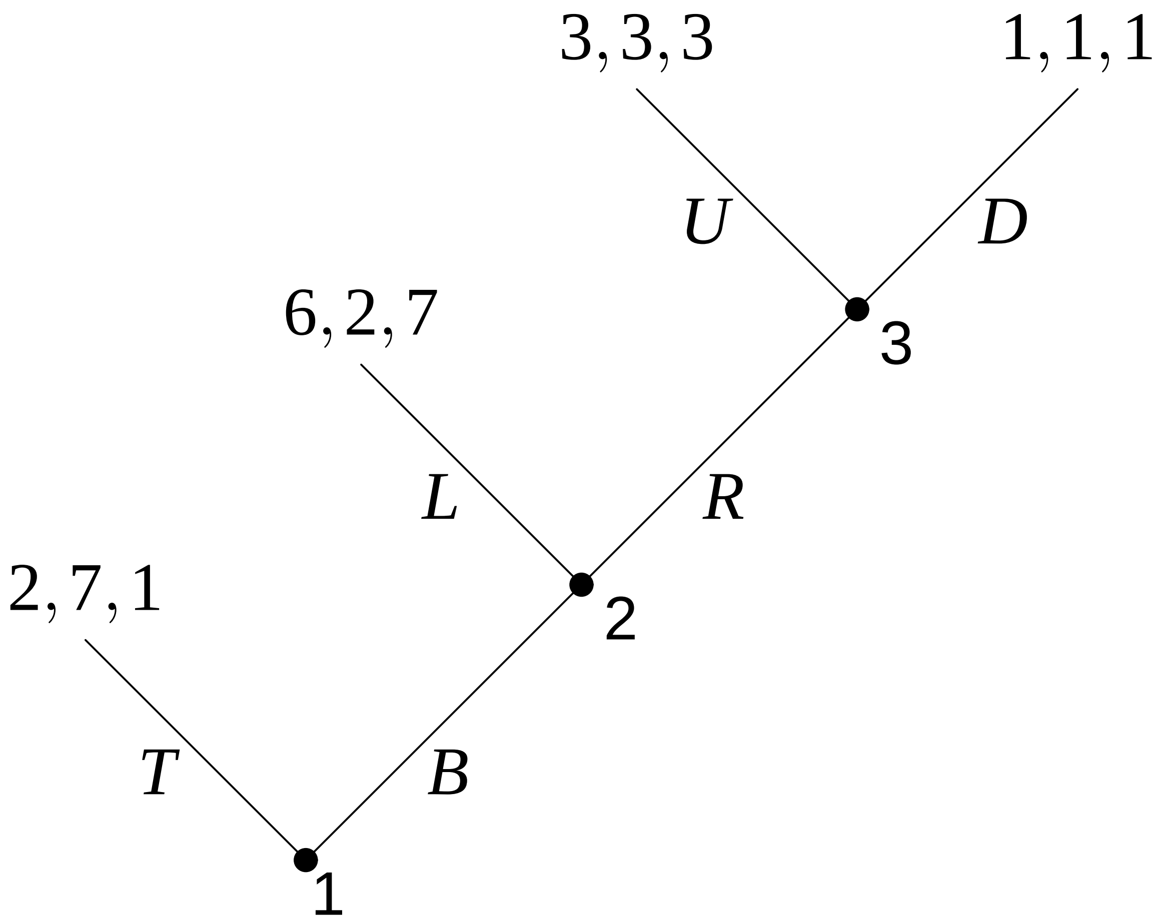
Thus, in two player games the two concepts coincide.

Backward Induction

Definition 15. Given an extensive form game Γ a subgame Γ' of Γ consists of some node of Γ and all nodes following it, together with those structures that pertain to those nodes such that each information set of Γ is either completely in Γ' or completely outside it

Since we may take the starting node of the subgame to be x_0 the initial node of the original game this means that for any extensive form game Γ the whole game Γ is one of the subgames of Γ. It may be the only subgame but often there are others.

Backward Induction



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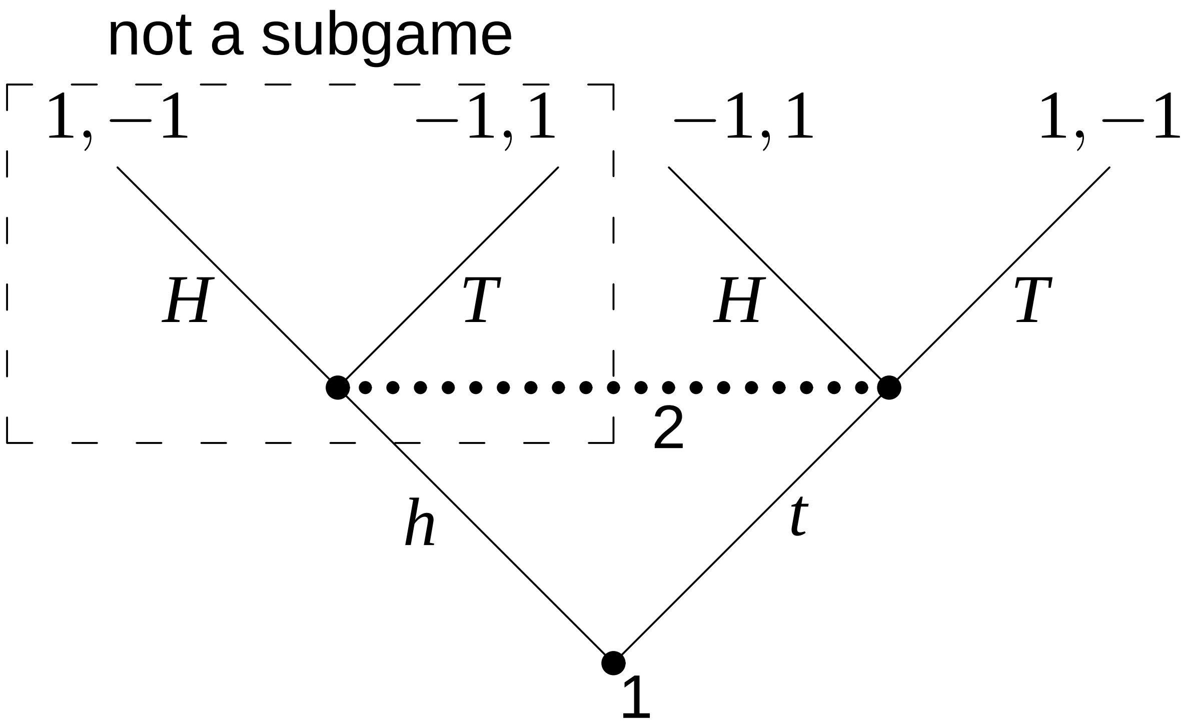
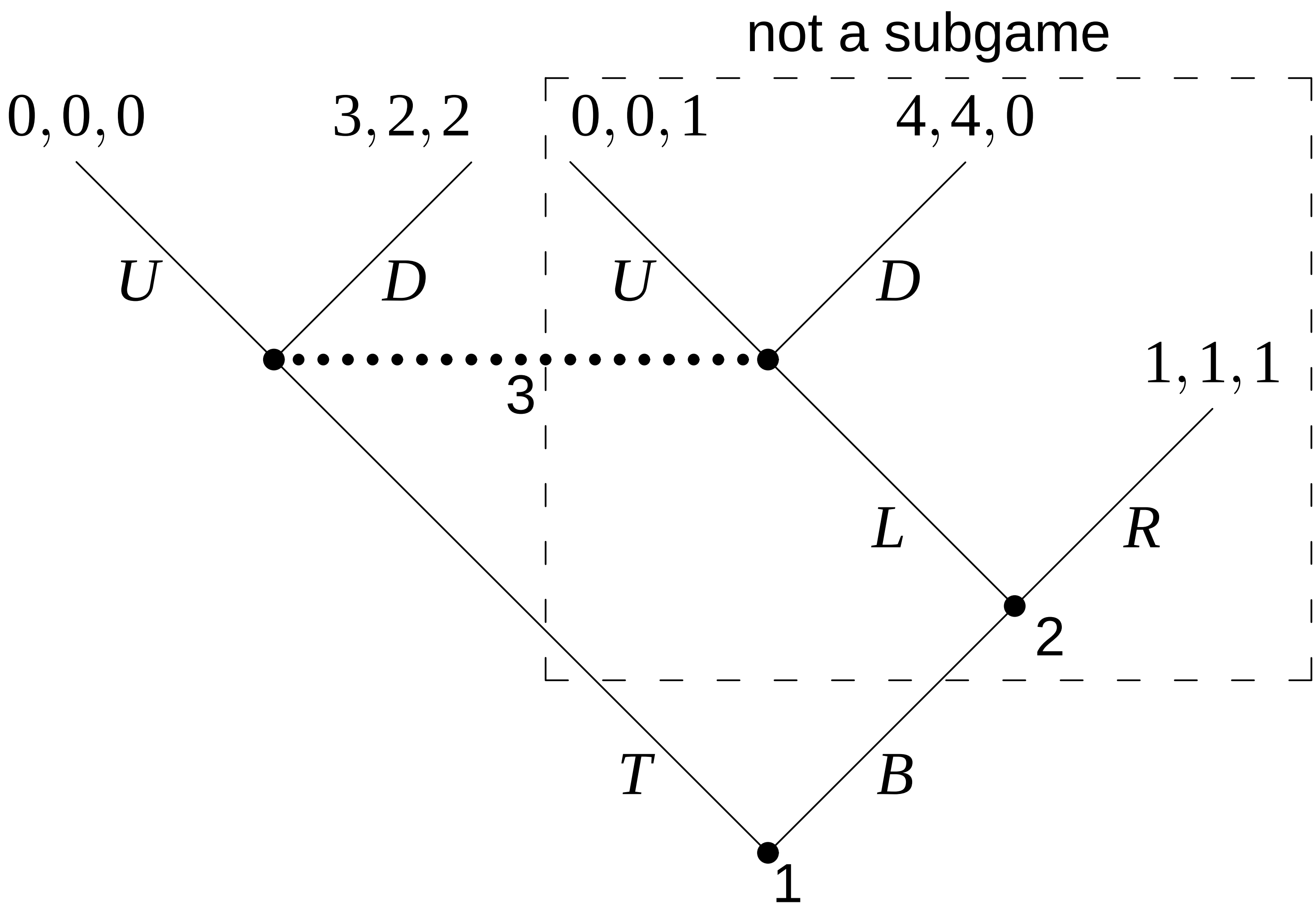


Backward Induction



Backward Induction

Definition 16. In an extensive form game Γ with perfect recall a *subgame perfect equilibrium* is a profile of behaviour strategies b such that for every subgame the parts of b relevant to the subgame constitute an equilibrium of the subgame.

The fact that the whole game is one of the subgames means that a subgame perfect equilibrium is an equilibrium. It is straightforward to see that every finite game with perfect recall has a subgame perfect equilibrium. It is also true that with the appropriate definition every finite extensive form game has a subgame perfect equilibrium.

Backward Induction

The definition we have given is for games with perfect recall. There is nothing about the idea of subgame perfect equilibria that requires perfect recall, though, of course, such a concept cannot be defined in terms of behaviour strategies—in games without perfect recall there may be no equilibria in behaviour strategies. How to formulate such a definition is here left as an exercise for the reader.

Backward Induction

Definition 17. A completely mixed behaviour strategy profile b is an *extensive form perfect equilibrium* of Γ if there is a sequence of completely mixed behaviour strategy profiles b^t converging to b such that for each Player n and each information set of Player n the choices of Player n given by the strategy b_n are optimal given the conditional distribution on the information set implies by b^t and the behaviour of all the players given by b^t at the information sets following that information set.

Backward Induction

Quasi-perfect equilibria were defined by van Damme (1984).

Definition 18. A completely mixed behaviour strategy profile b is a *quasi-perfect equilibrium* of Γ if there is a sequence of completely mixed behaviour strategy profiles b^t converging to b such that for each Player n and each information set of Player n the choices of Player n given by the strategy b_n are optimal given the conditional distribution on the information set implies by b^t and the behaviour of the other players given by b^t and the behaviour of Player n given by b_n at the information sets following that information set.

Backward Induction

If we compare this to the definition of extensive form perfect equilibria above we see that the only difference is that the "mistakes" of the player moving at an information set at later nodes are taken into account in the definition of extensive form perfect equilibria and are not taken into account in the definition of quasi-perfect equilibria.

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Backward Induction

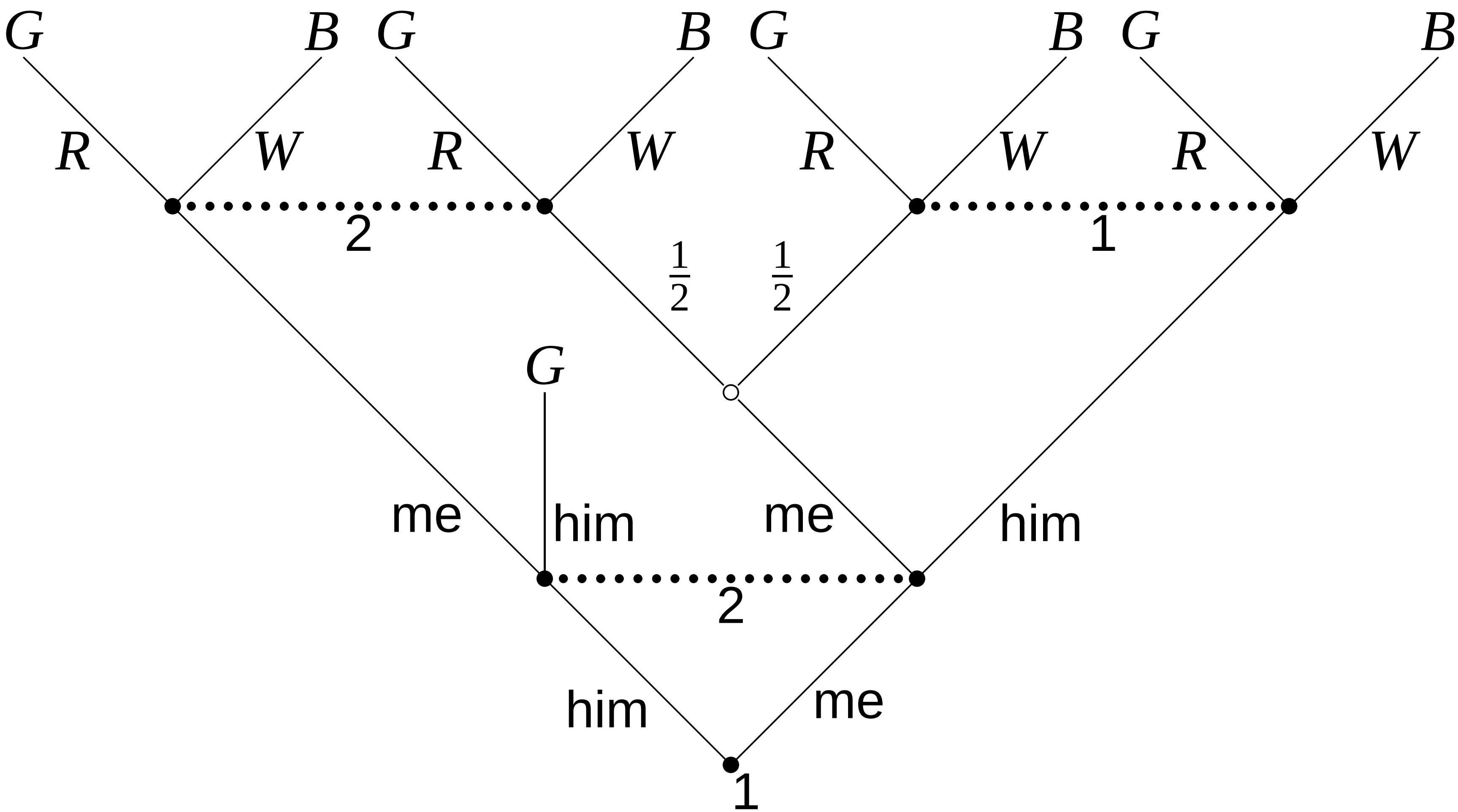
Theorem 6 (van Damme (1984))**.** *A quasi-perfect equilibrium of an extensive form game is Kuhn equivalent to a normal form perfect equilibrium of the associated normal form game, and thus an admissible equilibrium.*

We say Kuhn equivalent since one is a profile of behaviour strategies and the other a profile of mixed strategies. Mertens (1995) argues that quasi-perfect equilibrium is precisely the right mixture of admissibility and backward induction.

Backward Induction

Mertens (1995) offers the following game in which the set of extensive form perfect equilibria and the set of admissible equilibria have an empty intersection and hence also the set of extensive form perfect equilibria and set of quasi-perfect equilibria.

Backward Induction



Backward Induction

Kreps and Wilson (1982) define the concept of sequential equilibrium explicitly incorporating beliefs into the definition of equilibrium.

Definition 19. A *system of beliefs* μ gives, for each information set, a probability distribution over the nodes of that information set. An assessment is a pair (b, μ) where b is a profile of behaviour strategies and μ a system of beliefs.

Backward Induction

Definition 20. Given an assessment (b, μ), the behaviour strategy b_n of Player n is said to be *sequentially rational* with respect to that assessment, if at every information set at which a player moves, it maximizes the conditional payoff of the player, given his beliefs at that information set and the strategies of the other players.

Definition 21. The assessment (b, μ) said to be *consistent* if there is a sequence of completely mixed behaviour strategy profiles b^t converging to b such the the beliefs μ^t obtained from b^t as conditional probabilities converge to μ.

Backward Induction

Definition 22. An assessment (b, μ) is a *sequential equilibrium* if the strategy of each player is sequentially rational with respect to the assessment and the assessment is consistent.

If $(b.\mu)$ is a sequential equilibrium for some beliefs μ then we shall say that b is a sequential equilibrium strategy profile.

Backward Induction

Sequential equilibrium is a weakening of both extensive form perfect equilibrium and of quasi-perfect equilibrium.

Theorem 7 (Kreps and Wilson (1982); van Damme (1984))**.** *Given an extensive form game with perfect recall if b is an extensive form perfect equilibrium then b is a sequential equilibrium strategy profile. Similarly if b is an quasi-perfect equilibrium then b is a sequential equilibrium strategy profile.*

Backward Induction

But not much of a weakening. If we fix the extensive form, that is the game without the payoffs we can think of the space of games with this extensive form as some finite dimensional real space, $\mathbb{R}^K$ for some K.

Theorem 8 (Kreps and Wilson (1982); Blume and Zame (1994))**.** *For any extensive form, except for a closed set of payoffs of lower dimension than the set of all possible payoffs, the sets of sequential equilibrium strategy profiles and extensive form perfect equilibrium strategy profiles coincide.*

Backward Induction

And similarly.

Theorem 9 (Pimienta and Shen (2013); Hillas, Kao, and Schiff (2016))**.** *For any extensive form, except for a closed set of payoffs of lower dimension than the set of all possible payoffs, the sets of sequential equilibrium strategy profiles and quasi-perfect equilibrium strategy profiles coincide.*

Aside: Real Algebraic Geometry

The last two results are proved using techniques from real algebraic geometry, the mathematics of the semi-algebraic sets that Sylvain spoke of yesterday. It is not obvious from the little that Sylvain said that the various sets that we define are semi-algebraic, that is are defined by a finite number of polynomial equalities and inequalities. In fact they don't seem to be, involving statements about the limit of infinite sequences. However the Tarski-Seidenberg Theorem (Tarski, 1951; Seidenberg, 1954) shows that they are.

Aside: Real Algebraic Geometry

A first-order formula is an expression involving variables and constants, the quantifiers $\forall$ and $\exists$, the logical operators $\wedge$, $\vee$, and $\neg$, the operations $+$, $-$, $\cdot$, and $/$, and the relations $=$, $>$, and $<$. Variables in a first-order formula which are quantified are *bound*, while unbound variables are *free*. By definition, $X \subset \mathbb{R}^n$ is semi-algebraic if and only if it is defined by a first-order formula with n free variables and no bound variables. However, the Tarski-Seidenberg Theorem states that every first-order formula with n free variables is equivalent to a first-order formula with n free variables and no bound variables and hence all sets defined by first-order formulas are semi-algebraic.

Aside: Real Algebraic Geometry

An implication of the Tarski-Seidenberg Theorem is that all of the sets, functions, correspondences, and so on, that we have defined are in fact semi-algebraic.

Backward Induction

We turn again to normal for games, though we shall soon see the relevance to the extensive form. Myerson (1978) suggested a strengthening of normal form perfect equilibrium, which he called proper equilibrium.

Definition 23. An ε-*proper equilibrium* is a completely mixed strategy vector such that for each player if, given the strategies of the others, one strategy is strictly worse than another, then the first strategy is played with probability at most ε times the probability with which the second is played. In other words, more costly mistakes are made with lower frequency. A strategy profile is a *proper equilibrium* if it is the limit of a sequence of ε-proper equilibria as ε goes to 0.

Backward Induction

Theorem 10 (Myerson (1978))**.** *Every finite game has at least one proper equilibrium of its normal form. Every proper equilibrium is normal form perfect, and hence admissible.*

Backward Induction

While proper equilibrium is defined in terms of the normal form it does, in fact, have implications for the analysis of extensive form games.

Theorem 11 (van Damme (1984)). *For any normal form game and any extensive form game having that normal form, any proper equilibrium of the normal form game is Kuhn equivalent to a quasi-perfect equilibrium of the extensive form game.*

In the light of Theorem 7, this implies that any proper equilibrium is Kuhn equivalent to a sequential equilibrium strategy, a result proved independently by Kohlberg and Mertens (1982, 1986).

Backward Induction

A partial result in the other direction also holds.

Theorem 12 (Mailath, Samuelson, and Swinkels (1997); Hillas (1997))**.** *An equilibrium* σ *of a normal form game G is proper if and only if there exists a sequence of completely mixed strategies* $\{\sigma^t\}$ *with limit* σ *such that for any extensive form game* Γ *having the normal form G, for any sufficiently small* $\varepsilon > 0$, *for sufficiently large* t, *some behaviour strategy corresponding to* σ^t *is an* ε*-quasi-perfect equilibrium of* Γ.

Backward Induction

The full converse is false; it is not true that an equilibrium that is quasi-perfect in any extensive form game with a given normal form is necessarily proper in that normal form.

	V	W	X	Y	Z
A	$1,1$	$3,1$	$3,1$	$0,0$	$1,1$
B	$1,1$	$2,0$	$2,0$	$0,0$	$1,2$
C	$1,1$	$1,0$	$2,1$	$1,0$	$1,0$
D	$1,1$	$2,1$	$1,0$	$1,0$	$1,0$

Invariance

Right from the start von Neumann and Morgenstern (1944) had argued that the normal form encompassed all the relevant information about a game. Kohlberg and Mertens (1986), and later and more strongly, Mertens (2003, 1989, 1991b, 1992), have argued for even more. Kohlberg and Mertens argue that since the players can, in any case already play mixed strategies it should not matter if we add an existing mixed strategy as a new pure strategy. We would, in fact just be saying twice that the player could play this mixed strategy. Thus the solution should depend only the reduced normal form, the normal form when all strategies equivalent to mixtures of other strategies have been removed. This property is called reduced normal form invariance.

Invariance

Mertens goes further adding that if two games have the same best reply correspondence then their solutions should be the same, and even that if the best reply correspondences of two games were the same on the interior of the strategy space—that is, on the admissible best erply correspondence—then the solutions of the game should be the same. He terms these properties ordinality.

Invariance

I find this argument convincing, but shall not address it here. Rather, I'll just point out two things. First, that Nash equilibrium and normal form perfect equilibrium satisfy reduced normal form invariance and even ordinality define in terms of the best reply correspondence, while normal form perfect equilibria also satisfy ordinality defined with the admissible best reply correspondence. And second that if we also require other properties such as backward induction that adding the requirement of invariance or ordinality can substantially increase the implications of those requirements.

Set-Valued Solutions

We are seeking an answer to the question: What are the self-enforcing behaviours in a game? As we indicated, the answer to this question should satisfy the various invariances we discussed above. We also require that the solution satisfy stronger forms of rationality than Nash equilibrium and normal form perfect equilibrium, the two equilibrium concepts that we have said do satisfy those invariances.

Set-Valued Solutions

In particular, we want our solution concept to satisfy admissibility that we defined earlier, and some form of the iterated dominance condition we shall define later, the backward induction condition we discussed, and the forward induction condition we shall also define in the next section. We also want our solution to give some answer for all games.

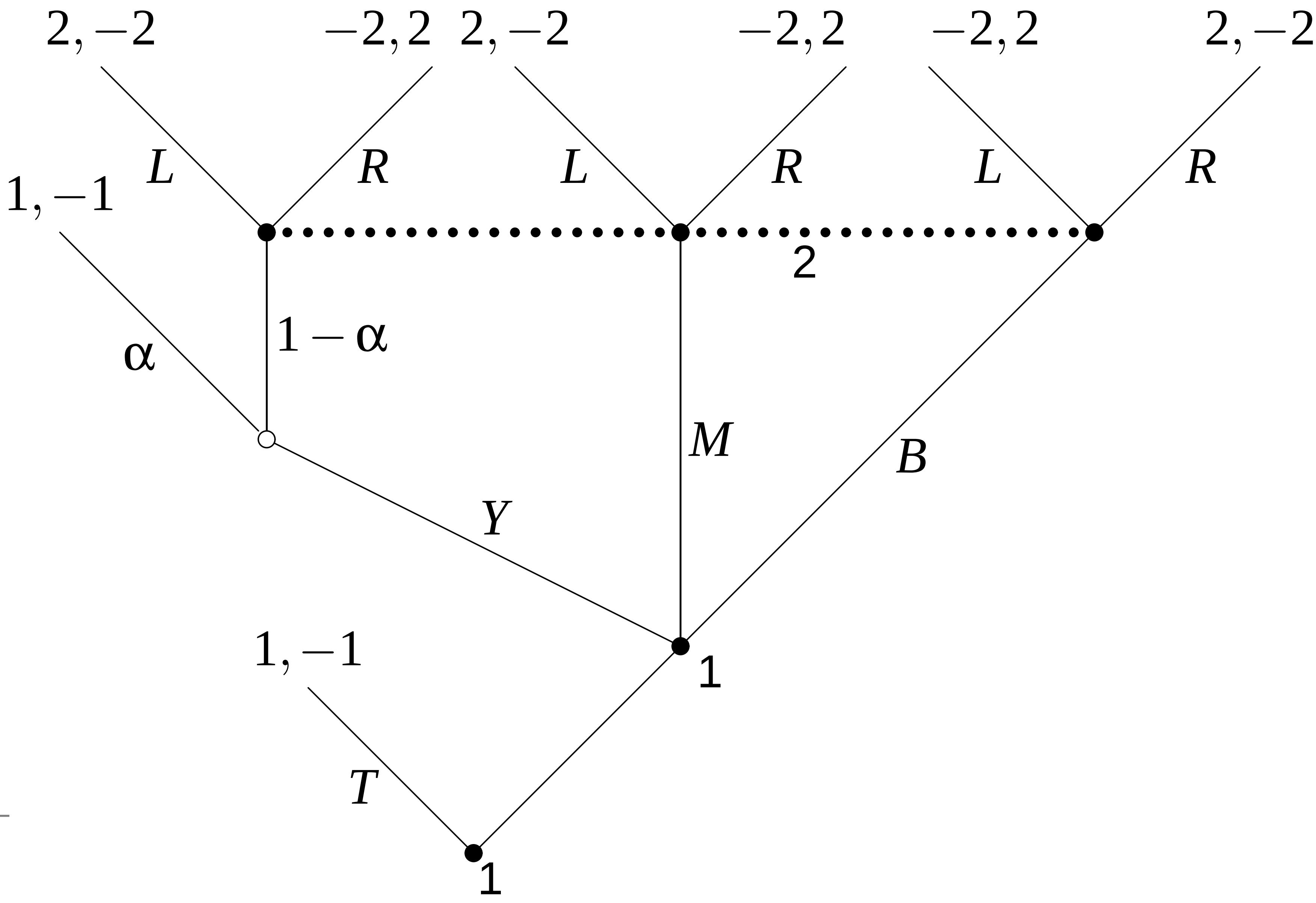
Set-Valued Solutions

It is impossible for a single valued solution concept to satisfy these conditions. In fact, two separate subsets of the conditions are inconsistent for such solutions. Admissibility and iterated dominance are inconsistent, as are backward induction and invariance.

Set-Valued Solutions

	L	R
T	$3, 2$	$2, 2$
M	$1, 1$	$0, 0$
B	$0, 0$	$1, 1$

Set-Valued Solutions



Set-Valued Solutions

	L	R
T	$1, -1$	$1, -1$
M	$2, -2$	$-2, 2$
B	$-2, 2$	$2, -2$

Set-Valued Solutions

Thus it may be that elements of a solution satisfying the requirements we have discussed would be sets. However we would not want these sets to be too large. We are still thinking of each element of the solution as, in some sense, a single pattern of behaviour. In generic extensive form games we might think of a single pattern of behaviour as being associated with a single equilibrium outcome, while not specifying exactly the out of equilibrium behaviour.

Set-Valued Solutions

One way to accomplish this is to consider only connected sets of equilibria. In the definition of Mertens (1989, 1991b) the connectedness requirement is strengthened in a way that corresponds, informally, to the idea that the particular equilibrium should depend continuously on the "beliefs" of the players. Without a better understanding of exactly what it means for a set of equilibria to be the solution we cannot say much more. However some form of connectedness seems to be required.

Forward Induction

The concept of *forward induction* was introduced and discussed by Kohlberg and Mertens (1986). The precise status of this concept is not clear in their paper. They do not list "forward induction" as one of the requirements for a solution but it seems to be important in the motivation and stable sets do, in fact, satisfy their concept of forward induction. The property that Kohlberg and Mertens call forward induction is the following: "A stable set contains a stable set of any game obtained by the deletion of a strategy which is an inferior response in all the equilibria of the set." ((Kohlberg and Mertens, 1986, p. 1029.)

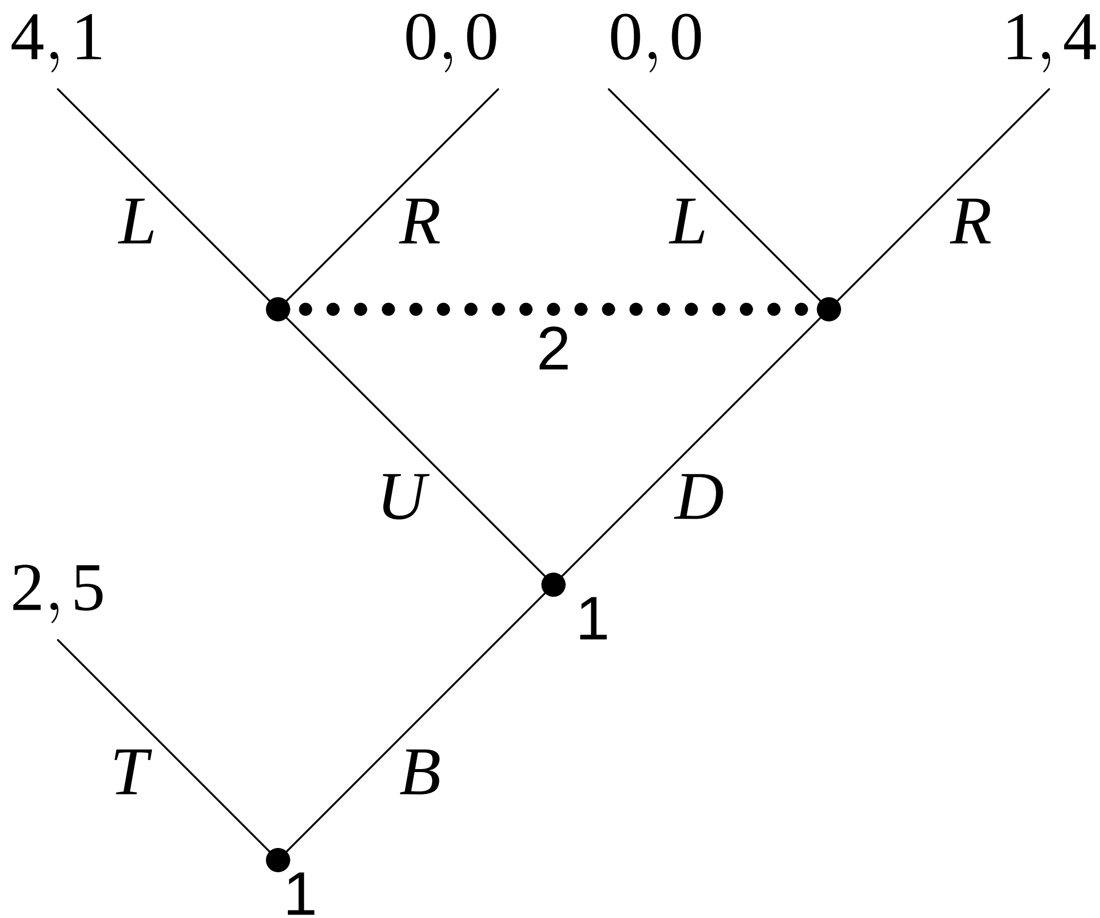
Forward Induction

This is obviously a strong property. For example, it means that if there are two strategies that are both inferior responses then when the more preferred of these strategies is deleted and the less preferred kept the solution should remain stable. Even before this full strength there are those who have argued against this kind of requirement. Cho and Kreps (1987) show how a series of stronger implementations of the forward induction like ideas refine the set of sequential equilibria in signaling games. They argue that the relatively strong implementations are quite unintuitive.

Forward Induction

We next look at an example of a game that had a large role in the motivation of the idea of forward induction and show that a combination of forward induction and reduced normal form invariance is sufficient to eliminate the sets of equilibria that do not satisfy forward induction.

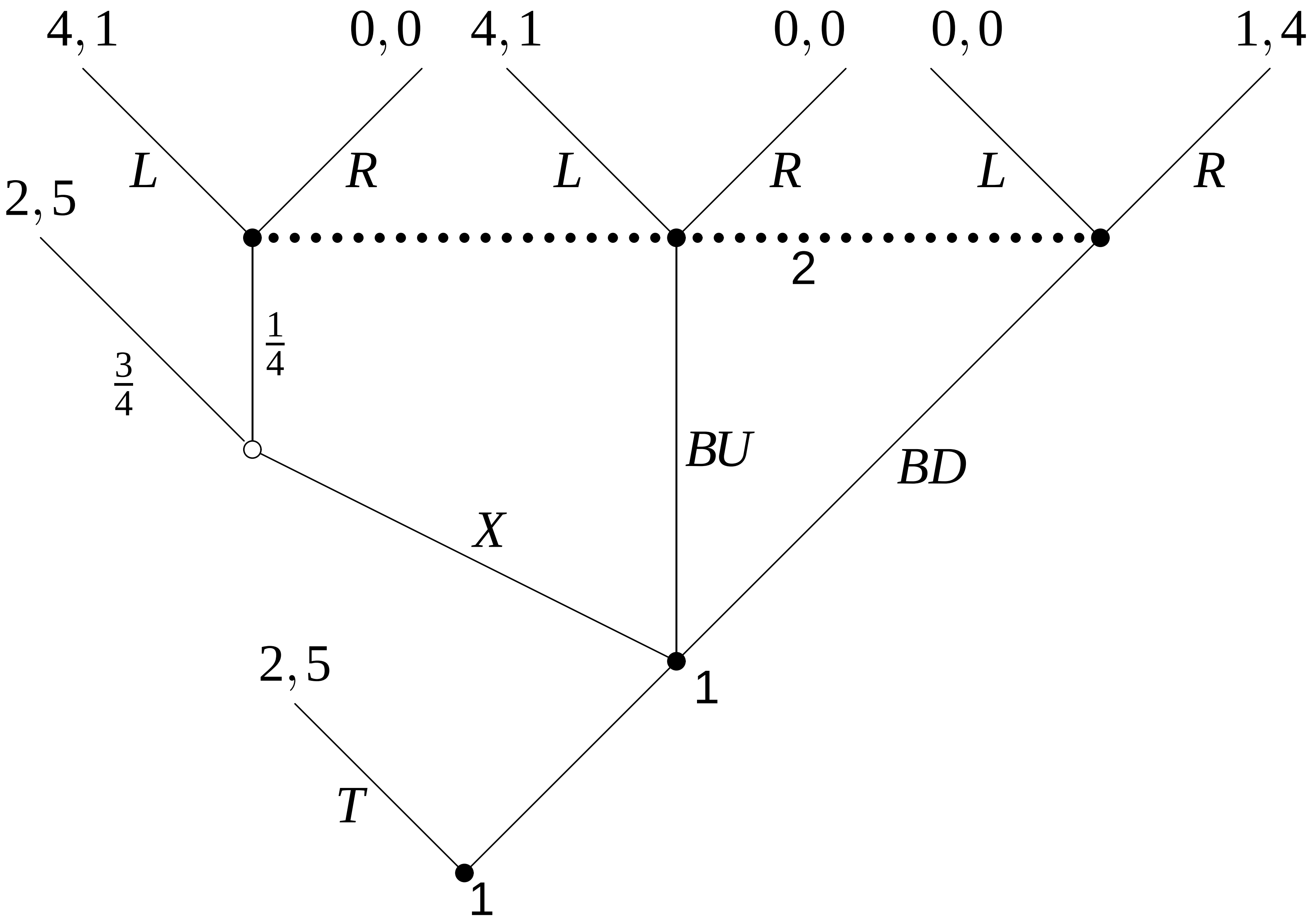
Forward Induction



Forward Induction

	L	R
T	$2, 5$	$2, 5$
BU	$4, 1$	$0, 0$
BD	$0, 0$	$1, 4$

Forward Induction



Forward Induction

	L	R
T	$2, 5$	$2, 5$
BU	$4, 1$	$0, 0$
BD	$0, 0$	$1, 4$
X	$\frac{5}{2}, 4$	$\frac{3}{2}, \frac{15}{4}$

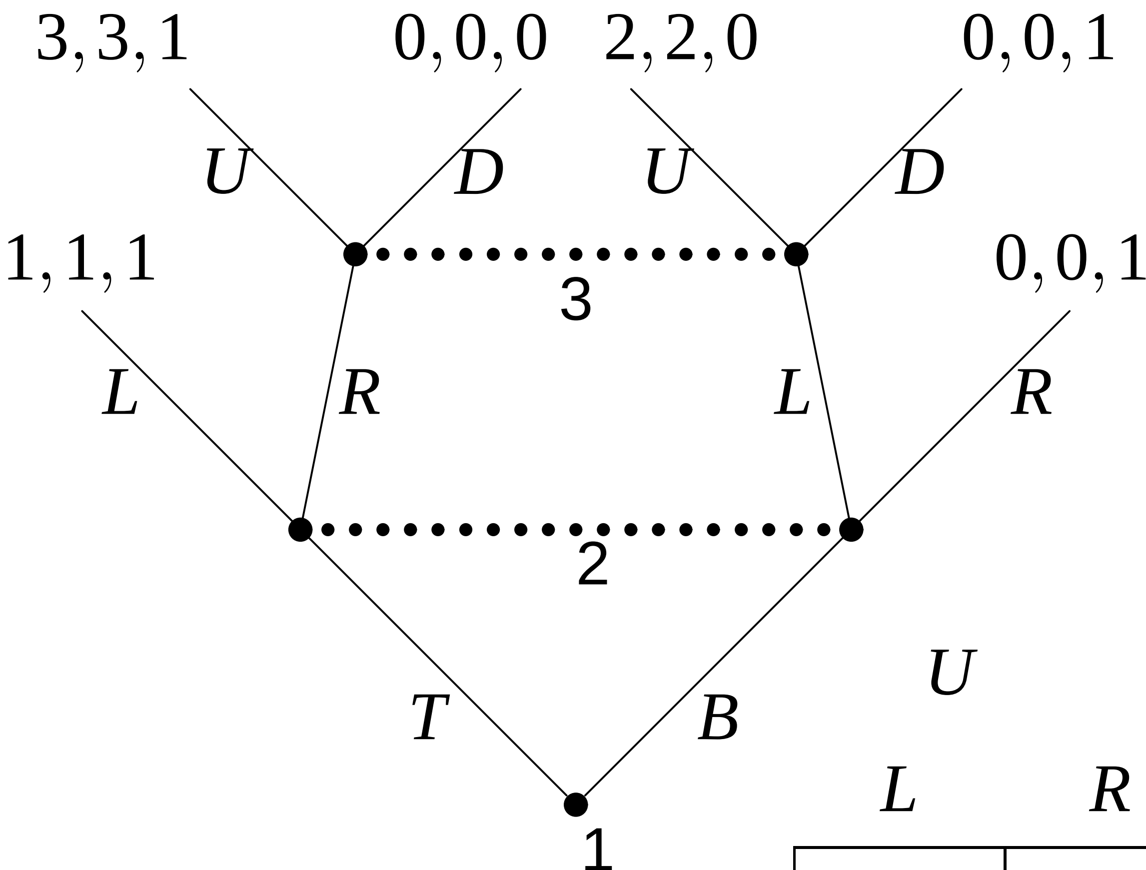
Forward Induction

In a number of games that have been examined in the literature similar thing occur. At one point I had conjectured that some form of forward induction was implied by backward induction and various forms of invariance; I was wrong.

Let us now look at an example of a game in which it will be clear that we cannot get the full strength of forward induction from backward induction arguments.

In some ways this game resembles some of the signaling games that Cho and Kreps (1987) argue show the unreasonable strength of stability, and other strong forms of forward induction. It differs in that the game is unavoidably a three person game, and so the kind of techniques that work in signaling games cannot be applied.

Forward Induction



	U	
	L	*R*
T	1, 1, 1	3, 3, 1
B	2, 2, 0	0, 0, 1

	D	
	L	*R*
T	1, 1, 1	0, 0, 0
B	0, 0, 1	0, 0, 1

Forward Induction

We consider a final example. This is a two person normal form game and the forward induction arguments are in terms of iterated deletion of weakly dominated strategies. I learned this example from Hari Govindan.

In this example we see an equilibrium that is eliminated by forward induction arguments. Nevertheless, the equilibrium is proper, and remains proper no matter what mixtures are added as new strategies.

Forward Induction

	t_1	t_2	t_3
s_1	$1,1$	$0,1$	$1,1$
s_2	$1,0$	$1,1$	$0,-1$
s_3	$1,1$	$-1,0$	$1,1$

Strategic Stability

Kohlberg and Mertens (1986) gave a list of requirements that a concept of strategic stability should satisfy. They showed that even quite weak versions of their requirements implied that the solution concept should assign sets of equilibria as solutions to the game. Thus a stability concept is a rule that assigns to each game in the domain of games under consideration a collection of subsets of the space of (mixed) strategy profiles of the game. Since the paper of Kohlberg and Mertens the list of requirements a concept of strategic stability should satisfy has been modified and expanded, particularly in the work of Mertens (2003, 1989, 1991b,a, 1992, 1995) The list presented here is a somewhat modified and expanded version of the original one.

Strategic Stability

- Existence. *Every game has at least one stable set.*

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- Independence of Inadmissible Strategies. *One form of the forward induction idea.*
- Ordinality. *A stability concept is ordinal.*
- The Small Worlds Axiom *Suppose that the players can be divided into insiders and outsiders, and that the payoffs of the insiders do not depend on the strategies of the outsiders. Then the stable sets of the game between the insiders are precisely the projection of the stable sets of the larger game.*

Strategic Stability

Kohlberg and Mertens (1986) consider a space of perturbations such as that defined by Selten (1975), in which each strategy of a player has attached to it a small probability. Lets assume that all these probabilities are no more than δ and call this set P_δ. Perturbed games are defined in a natural way. (There are a number of alternative methods.)

Strategic Stability

Kohlberg and Mertens then define a *stable set of equilibria* to be a set of Nash equilibria that is minimal with respect to the property that all sufficiently small perturbations of the game have equilibria close to the stable set.

They define a *hyperstable set of equilibria* to se a set of Nash equilibria that is minimal with respect to the property that for any game with the same reduced normal form and for any sufficiently small perturbation of the payoffs of that game the game has equilibria close to the stable set.

Strategic Stability

There are some problems with two aspects of these definitions. The minimality requirement does not achieve exactly what was intended, and may be inconsistent with the ordinality requirement. The attempt to impose invariance in the definition of hyerstability is also, for reasons that we won't go into here, not completely satisfactory.

Thus, we'll define a *KM-stable set of equilibria* to be a connected set of normal form perfect equilibria such that all sufficiently small perturbations of the game have equilibria close to the stable set.

Strategic Stability

Kohlberg and Mertens point out that their original definition does not satisfy the backward induction property and express the hope that some modification of that definition will.)The slightly modified definition we gave above does not satisfy backward induction either.)

The paper of Kohlberg and Mertens gave one model for definitions of strategic stability. One defines a space of perturbations. Defines how each perturbation gives perturbed games, or at least how one can associate a set of “equilibria” to each perturbation and then require that he stable set be such that all small perturbations have nearby equilibria.

Strategic Stability

We give now a definition of stable equilibria that does satisfy backward induction. One defines BR-stability by considering directly perturbations to the best reply correspondence (together with a fairly fine topology on such perturbations). One can show that this definition is equivalent to a definition that looks at continuous functions from Σ to P_δ as the space of perturbations. (This is the main result of Hillas, Jansen, Potters, and Vermeulen (2001))

Strategic Stability

One can also make a definition based on the idea of making the minimal change to the definition of KM-stability to give the desired properties. This was one of the approaches I took in Hillas (1990). It seemed to me at the time that the more radical approach of perturbing the best reply correspondence was more promising. I'm no longer sure this is true. The work of De Stefano Grigis (2014, 2015) seeks to make the minimal modifications to the definition of Kohlberg and Mertans so that it satisfies the requirements.

Strategic Stability

Mertens gave a more fundamentally different way of redefining stability. Rather than keeping the form of the definition and changing the space of perturbations Mertens kept the same space of perturbations, P_δ, and changed the way of defining stability.

Consider a small neighbourhood of 0 in η. Lets call this P_δ and call the boundary of this neighbourhood ∂P_δ. Consider also some part S of the graph of the equilibrium correspondence $E : \eta \twoheadrightarrow \Sigma$ and let the part above P_δ be called S_δ and the part above ∂P_δ be called ∂S_δ.

Strategic Stability

Mertens says that the part of S above zero is a stable set if the projection map from S_δ to P_δ is nontrivial (in some sense) for sufficiently small δ. Mertens gave a number of definitions involving different specifications of “nontrivial”. The “right” definition seems to be the one involving homology theory, which says that the projection should not be homologically trivial.

Since homology theory puts a fairly course structure on things it is relatively “easy” to be homologically trivial, that is, homologous to a map to ∂P_δ. Thus the definition in terms of homology is a strong one.

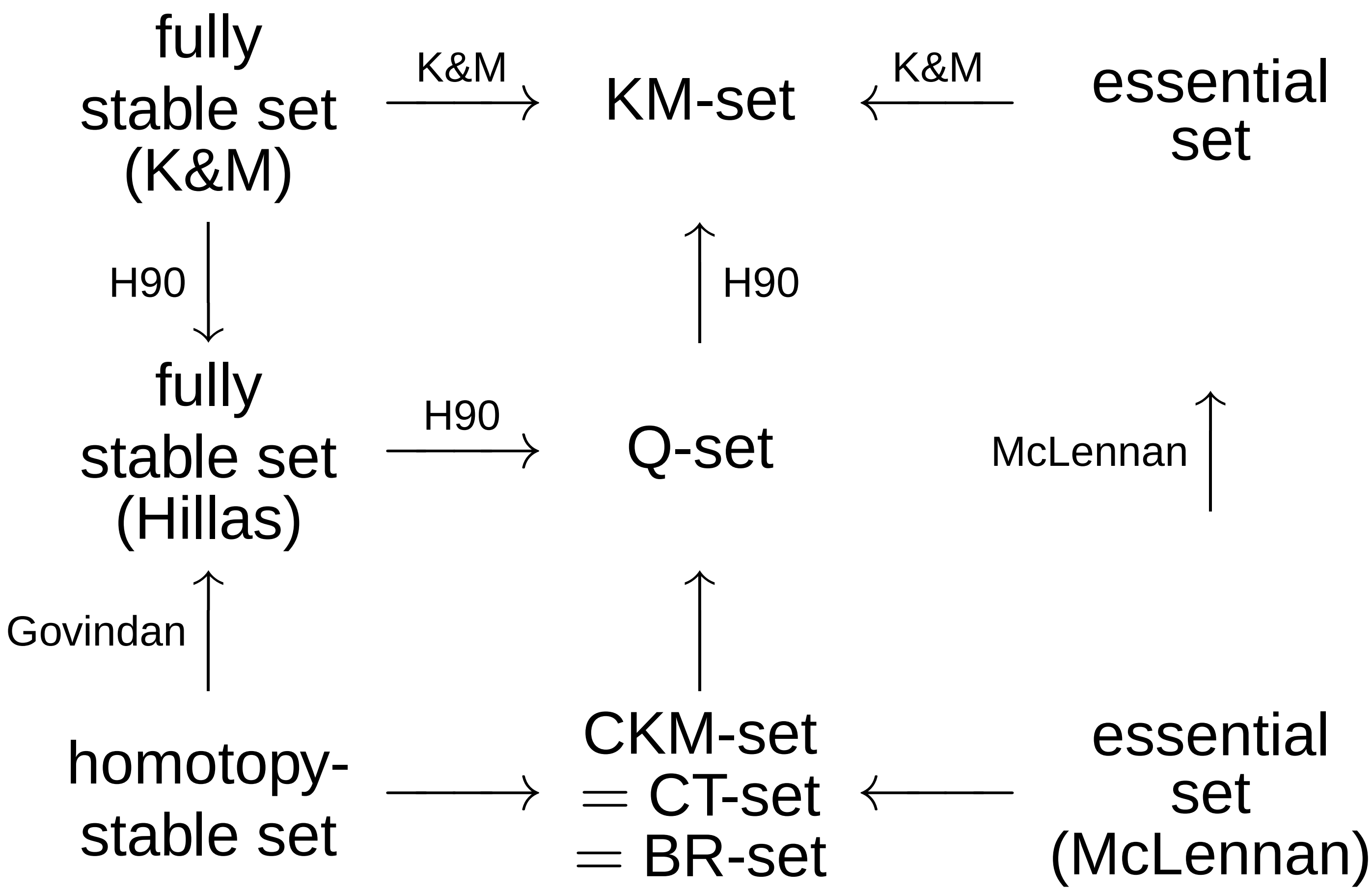
Strategic Stability

The easiest definition to understand simply says that the projection map from S_δ to P_δ should not be homotopic to a map from S_δ to ∂P_δ under a homotopy that leaves the map from ∂S_δ to ∂P_δ unchanged. We'll call sets that satisfy this requirement *homotopy-stable sets*.

Strategic Stability

Hillas, Jansen, Potters, and Vermeulen (2001) show that every stable set in the sense of Mertens is a CKM-stable set. They also show that every CKM-stable set is a BR-stable set. So, every stable set in the sense of Mertens is a BR-stable set. These and some known relations between various stability concepts are displayed below. The relations are shown in the following diagram with those marked K&M were proved in Kohlberg and Mertens (1986); those marked H90 in Hillas (1990)); that marked McLennan in McLennan (1995); that marked Govindan in Govindan (1995); and the unmarked relations proved in Hillas, Jansen, Potters, and Vermeulen (2001).

Strategic Stability



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