Existence of Nash, Approximate and Sharing Rule Equilibria in Discontinuous Games

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Contents

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- Many classical problems in economics are formulated as discontinuous games.
- Discontinuities may arise when firms choose the same price, location, bid or stopping time.
- Standard existence results, such as Nash-Glicksberg cannot be used directly to prove existence of an"equilibrium".
- Two natural questions arise:
  Q1:) Under which conditions does a Nash equilibrium exist?
  Q2:) When a Nash Eq does not exist, which solution to use?

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- Reny's paper generates a very active literature:
  Bagh and Jofre (2006), Carmona (2005-2009), Barelli-Soza (2009), McLennan-Monteiro-Tourky (2011), Barelli-Menghel (2013), Reny (2010, 2015), Bich-Laraki (2016).

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- Baye, Tian and Zhou (1993) presented (an apparently) quite different approach. We will see (in the conference), that their approach is also linked to Reny.

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- We will also study approximate equilibrium (limit of ϵ-Nash equilibria as $\epsilon \to 0$).

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- $(x,y) = (3,3)$ is a Nash equilibrium of the new game.
- The new sharing rule gives each player its market share (3,1).
- Remark: $(3-\epsilon, 3)$ is an ϵ-equilibrium with payoff $\approx (3,1)$.

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- If the sharing rule gives the good to the agent with the highest valuation, then $(b_1, b_2) = (v_2, v_2)$ is a Nash equilibrium.
- Remark: $(v_2 + \epsilon, v_2)$ is an ϵ-equilibrium where the agent with the highest evaluation wins and pays $\approx v_2$.

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- $\overline{\Gamma}_x = \{v \in \mathbb{R}^N : (x, v) \in \overline{\Gamma}\}$ is the x-section of $\overline{\Gamma}$.

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Where:

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Theorem (Reny 1999)

Any better-reply secure quasiconcave compact game G admits a pure Nash equilibrium.

Reny equilibrium in pure strategies

Let us define the following relaxation of Nash equilibrium.

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Proof: G is better-reply secure <=> Nash and Reny coincide.

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(i) G is payoff secure if $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) = \sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i})$.
(ii) G is reciprocally upper semicontinuous if, whenever $(x, v) \in \overline{\Gamma}$ and $u_i(x) \leq v_i$ for every $i \in N$, then $u(x) = v$.

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Proof.

- Let (x, v) be a Reny equilibrium: for every $i \in N$, $\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \leq v_i$.

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Corollary (Reny 1999)

Every payoff secure and reciprocally upper semicontinuous game is better-reply secure.

Proof.

- Let (x, v) be a Reny equilibrium: for every $i \in N$, $\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \leq v_i$.
- Since the game is payoff secure, $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) \leq v_i$.

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- Thus, $u_i(x) \leq v_i$ for every $i \in N$.
- By reciprocal upper semicontinuity, $v = u(x)$: x is a Nash.

Point security

Lemma

Better-reply secure is equivalent to:
for every $x \in X$ *which is not a Nash equilibrium:*
$\exists \varepsilon > 0$, $\exists U$ *a neighborhood of* x, $\exists d \in X$ *s.t.:*
$\forall y \in U$, $\exists i \in N$ *s.t.* $u_i(d_i, x'_{-i}) > u_i(y) + \varepsilon$ *for all* $x' \in U$.

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- It has been extended, by McLennan et al to multiple security.

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A correspondence is Kakutani if it is closed $\neq \emptyset$ and convex valued.

Theorem (Barelli and Meneghel 2013)

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- It is an extension of McLennan et al 2011 multiple security.
- The result is extendable to games with ordinal preferences (Barelli and Soza 2009, Reny 2015).

Refining Reny equilibrium

Define the following regularization of u_i (introduced by Carmona):

$$\underline{\underline{u_i}}(d_i, x_{-i}) := \sup_{U \in \mathcal{V}(x)} \sup_{\phi_i \in W_U(d_i, x_{-i})} \inf_{x' \in U, d'_i \in \phi_i(x')} u_i(d'_i, x'_{-i}),$$

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Corollary (Extending Bagh and Joffre 2006)

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- For any subset I of the set of the players N, let B_I denote the set of strategies at which every player $j \in N/I$ is best replying:

$$B_I = \{x \in X : u_j(x_j, x_{-j}) \geq u_j(y_j, x_{-j}) : \quad \forall j \in N/I\}$$

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- Note that $B_N = X$ and $B_\emptyset =$ set of Nash equilibria.

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- The theorem is deduced by constructing a surrogate game G^* which is point secure when G is correspondence secure and with the same set of Nash equilibria as G.

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- All above results can be further extended to utilities (ordinal preferences) that are not quasi-concave (convex). The main idea goes back to Bich (2009).

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Theorem (Prokopovych 2011)

Any payoff secure quasiconcave compact game G such that V_i is continuous for every i admits an approximate equilibrium.

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Theorem (Bich Laraki 2016a)

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- Let $(x, v) \in \overline{\Gamma}$ be a Reny equilibrium.
- Let $\underline{\mathcal{S}}(d_i, x_{-i})$ be the space of sequences $(x^n_{-i})_{n\in\mathbb{N}}$ converging to x_{-i} such that $\lim_{n\to+\infty} u_i(d_i, x^n_{-i}) = \underline{u}_i(d_i, x_{-i})$.

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- Define the sharing rule $q : X \to \mathbb{R}^N$ by

$$q(y) = \begin{cases} v & \text{if } y = x, \\ \text{limit point of } (u(d_i, x^n_{-i})) & \text{if } y = (d_i, x_{-i}) \ (x^n_{-i}) \in \underline{\mathcal{S}}(d_i, x_{-i}), \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

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- Show that (x, v) is a sharing rule equilibrium associated to q.

Existence of approximate equilibria

Definition

A game G is *approximately better-reply secure* if for every $(x, v) \in \overline{\Gamma}$:
whenever x is not an approximate equilibrium profile,
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Proof: Approximately better-reply secure $<=>$ Reny and approximated equilibrium profiles coincide.

Proving Prokopovych's theorem

Recall that G is payoff secure if:
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Corollary (Prokopovych 2011)

A payoff-secure compact game G such that V_i is continuous for every i is approximately better-reply secure.

Proof.

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- x' is an ϵ-equilibrium.

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Application to diagonal games

The strategy set of player i is $[0,1]$ and his payoff is:

$$u_i(x_i, x_{-i}) = \begin{cases} f_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) > x_i, \\ g_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) < x_i, \\ h_i(x_i, x_{-i}) & \text{if } \phi(x_{-i}) = x_i, \end{cases}$$

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- $h_i : [0,1]^N \to \mathbf{R}$ is a bounded mapping.
- $\phi : [0,1]^{N-1} \to [0,1]$ is a continuous "aggregation" function, such as $\phi(x_{-i}) = \max_{j \neq i} x_j$, $\min_{j \neq i} x_j$, $\frac{1}{N-1} \sum_{j \neq i} x_j$, or the k-th highest value of $\{x_1, ..., x_{N-1}\}$ for $k = 1, ..., N-1$.

Example: in first price auctions, $\phi(x_{-i}) = \max_{j \neq i} x_j$, $f_i = 0$, $g_i = v_i - \max_{j \neq i} x_j$ and $h_i = \frac{g_i}{k}$ where $k = |\{j : x_j = \max_i x_i\}|$ is the cardinal of the set of players with maximum bid.

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Theorem (Bich Laraki 2016a)

Any quasiconcave diagonal game satisfying condition (C): $\forall x_i = \phi(x_{-i})$: $h_i(x)$ is a strict convex combination of $f_i(x_i, x_i)$ and $g_i(x_i, x_i)$, is approximately better-reply secure.

Sketch of the proof.

Under Assumption (C), the game is payoff secure. Consequently, if $(x, v) \in \overline{\Gamma}$ is a Reny Eq then

$$\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \; i \in N.$$

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- **Case 3:** $\exists i$ s.t. $x_i = \phi(x_{-i}) = 0$. Consider $(x^n, u(x^n)) \to (x, v)$.
  Define (y^n) as: if $v_j \leq f_j(0)$, $y_j^n := 0$ otherwise, $y_j^n := x_j^n$.
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- **Case 4**: $\exists i$ s.t. $x_i = \phi(x_{-i}) = 1$. Similar to case 3.

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(σ, q) is a *sharing rule mixed solution* of G if $\sigma \in M$ is a mixed equilibrium of the auxiliary game $\tilde{G} = ((X_i)_{i\in N}, (q_i)_{i\in N})$ where utilities $q = (q_i)_{i\in N}$ must satisfy: $(SZ)\ \ \forall y \in X,\ \ q(y) \in co\overline{\Gamma}_y$.

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Theorem (Simon and Zame 1990)

Any compact-metric game admits a sharing mixed solution.

Finite deviation equilibrium

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- To every D, one can associate $(m^D, u(m^D))$ where m^D is a Nash mixed equilibrium of the game restricted to D. This defines a mapping from $\mathcal{D}$ to $\overline{\Gamma'}$, called a net.
- Any limit point (m, v) of the net $(m^D, u(m^D))_{D \in \mathcal{D}}$ is a FDE.

Non-empty intersection between Reny and Simon-Zame

Theorem

In compact-metric games, any finite mixed deviation equilibrium is
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and
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Proof of B: adapt Simon and Zame's proof.

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Corollary

If G' is better-reply-secure, G has a Nash mixed equilibrium.

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Strategic approximation

Definition

A game G admits a (**resp. weak**) strategic approximation if:
$\exists$ a sequence of finite sets $D_n \subset M$ such that :
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Proof: use the finite deviation equilibrium.

Contents

Two player diagonal games

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Examples:

Bertrand Duopoly with Discontinuous Costs,
Bertrand-Edgeworth Duopoly with Capacity constraints,
Timing Games (silent or noisy).

Bayesian diagonal games

- At stage 0: a type $t = (t_1, ..., t_N) \in T = T_1 \times ... \times T_N$ is drawn according to some joint probability distribution p.
- At stage 1: each player i is privately informed of his own type t_i (correlations between types are allowed).
- At stage 2: each player i is asked to choose an element $x_i \in [0, 1]$ (interpreted as a bid).

Bayesian diagonal games

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The payoff of player i is assumed of the form:

$$u_i(t, x_i, x_{-i}) = \begin{cases} f_i(t, x_i, \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) > x_i, \\ g_i(t, x_i \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) < x_i, \\ h_i(t, x_i, x_{-i}) & \text{if } \phi_i(x_{-i}) = x_i, \end{cases}$$

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Examples: one unit auctions (first second, all-pay, double), multi-unit first price auctions.

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Counter-example: without condition (a) or (b), we may have non-existence even if the game is zero-sum!

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- When the players stop simultaneously, if the type is A, player 1 gets $h = 3$, if the type is B, he gets $h = -2$.
- The game does not have a value in mixed strategies (and so has no approximated equilibrium).

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This lecture:

- Propose a unifying framework that links Simon-Zame and Reny in pure and mixed strategies,
- Apply it to derive new existence results of approximate equilibria.

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