An introduction to “equilibria in games”

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1 Basic results

11 Strategic game

A strategic game G is defined by:
- a set I of N players
- a set of strategies S^i for each $i \in I$
- a mapping g from $S = \Pi_{i=1}^N S^i$ into $\mathbb{R}^N$.

$g^i(s^1, \cdots, s^N)$ is the payoff of player i when the profile $s = (s^1, \cdots, s^N)$ is played.

Denote $s = (s^i, s^{-i})$ where s^{-i} is the vector $\{s^j\}$ for $j \neq i$ and $S^{-i} = \Pi_{j \neq i} S^j$.

More generally a game form is a map F from S to an outcome set R. Each player has preferences $\succ_i$ on R. If one uses a representation by a real utility function u^i on R, the composition $u^i \circ F$ gives g^i which is the evaluation by player i of the result of the interaction.

When S^i has a measurable structure, one introduces the mixed extension of G as the game with strategy sets $\Sigma^i = \Delta(S^i)$ (probabilities on S^i) and payoff γ defined by (assuming that Fubini's theorem holds):

$$\gamma^i(\sigma) = \int_S g^i(s) \quad \sigma^1(ds^1) \otimes \ldots \otimes \sigma^N(ds^N).$$

σ^i is a mixed strategy.
Explicitly in the finite case:

$$\gamma^i(\sigma) = \sum_{s=(s^1,\ldots,s^N)\in S} \prod_{j=1}^{N} \sigma^j(s^j)\, g^i(s).$$

is the multilinear extension of g.
This allows to keep the topological properties (compactness, continuity) on S and g and to add geometrical properties (convexity).

For $\varepsilon \geq 0$, the ($\varepsilon-$) best reply correspondence BR^i_ε from S^{-i} to S^i, is defined by:

$$BR^i_\varepsilon(s^{-i}) = \{s^i \in S^i : g^i(s^i, s^{-i}) \geq g^i(t^i, s^{-i}) - \varepsilon, \forall t^i \in S^i\}.$$

Write $BR : S \rightrightarrows S$, for the global best reply correspondence that maps $s \in S$ to $\prod_{i \in I} BR^i(s^{-i})$.

12 Equilibrium: definition

A Nash equilibrium is a profile of strategies $s \in S$ where no player can gain by changing his strategy.

More generally, for $\varepsilon \geq 0$, an $\varepsilon-$equilibrium is a profile $s \in S$, such that for all i, $s^i \in BR^i_\varepsilon(s^{-i})$, which is:

$$g^i(t^i, s^{-i}) \leq g^i(s) + \varepsilon, \quad \forall t^i \in S^i, \quad \forall i.$$

s is a Nash equilibrium iff:

$$s \in BR(s).$$

An equilibrium s is strict if $\{s\} = BR(s)$.

13 Equilibrium: existence

Theorem (Nash, 1951, Glicksberg, 1952)

1) If S^i is compact convex, g^i continuous and quasi concave w.r.t. s^i, for all $i \in I$, the set of equilibrium is compact and non empty.
2) If S^i is compact , g^i continuous, for all $i \in I$, the mixed extension has an equilibrium.

Proof:
1) g quasi-concave implies that for all s, $BR(s)$ is convex. By continuity and compactness, $BR(s)$ is compact and non-empty for each s. The joint continuity hypothesis implies that the graph of the correspondence BR is closed.
Then use Ky Fan's fixed point theorem for the correspondence BR on S.
2) If S^i is compact, $\Delta(S^i)$ is convex and compact (for the weak$*$ topology). Similarly if g^i is continuous, γ^i is continuous (again for the weak$*$ topology using for example Stone -Weierstrass theorem) and multilinear. Then use part 1. ∎

The game G has a symmetry ϕ if :
(1) ϕ is a permutation of I and if $j = \phi(i)$, ϕ is a bijection between S^i and S^j with :
(2) for all $i \in N$ and all $s \in S$, $g^{\phi(i)}(\phi(s)) = g^i(s)$.

Theorem

(Under the previous hypotheses)
A game with symmetry ϕ has a symmetric equilibrium ϕ ($\sigma = \phi(\sigma)$).

14 Equilibrium: finite case

A mixed equilibrium of G is an equilibrium of the mixed extension of G.

Lemma

σ is a mixed equilibrium iff for all i and all $s^i \in S^i$:

$$g^i(s^i, \sigma^{-i}) < \max_{t^i \in S^i} g^i(t^i, \sigma^{-i}) \Rightarrow \sigma^i(s^i) = 0.$$

Proof: Follows from $g^i(\sigma^i, \sigma^{-i}) = \max_{t^i \in S^i} g^i(t^i, \sigma^{-i})$ and $g^i(\sigma^i, \sigma^{-i}) = \sum_{t^i \in S^i} \sigma^i(t^i) g^i(t^i, \sigma^{-i})$. ■

Theorem (Nash, 1950)

Every finite game G has a mixed equilibrium.

Proof: Let $\Sigma = \prod \Delta(S^i)$. Define the Nash map f from Σ to Σ by:

$$f(\sigma)^i(s^i) = \frac{\sigma^i(s^i) + (g^i(s^i, \sigma^{-i}) - g^i(\sigma))^+}{1 + \Sigma_{t^i \in S^i}(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+}$$

where $a^+ = max(a, 0)$.
f is well defined and with values in Σ: $f(\sigma)^i(s^i) \geq 0$ and $\sum_{s^i \in S^i} f(\sigma)^i(s^i) = 1$.
Since f is continuous and Σ convex, compact, Brouwer's fixed point theorem implies the existence of $\sigma \in \Sigma$ with $f(\sigma) = \sigma$.
Such σ is an equilibrium.
Otherwise there exists $i \in I$ and $u^i \in S^i$ with $g^i(u^i, \sigma^{-i}) - g^i(\sigma) > 0$. Since there exists s^i with $\sigma^i(s^i) > 0$ and $g^i(s^i, \sigma^{-i}) \leq g^i(\sigma)$ one get:

$$\sigma^i(s^i) = f(\sigma)^i(s^i) = \frac{\sigma^i(s^i)}{1 + \Sigma_{t^i \in S^i}(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+} < \sigma^i(s^i)$$

hence a contradiction.

Note that reciprocally any equilibrium is a fixed point of f since all quantities $(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+$ are vanishing. ∎

Semi-algebraicity

Recall that each S^i is finite, with cardinal m^i. Let $m = \Pi_i m^i$.
A game can be identified with a point in $\mathbb{R}^{Nm}$. For example, with 2 players having each 2 strategies one obtains $g \in \mathbb{R}^8$:

	L	R
T	(a_1, a_2)	(a_3, a_4)
B	(a_5, a_6)	(a_7, a_8)

Proposition

The set of equilibria is defined by a finite family of large polynomial inequalities.

Proof: σ is an equilibrium iff:

$$\sum_{s^i \in S^i} \sigma^i(s^i) - 1 = 0, \sigma^i(s^i) \geq 0, \quad \forall s^i \in S^i, \forall i \in I,$$

$$g^i(\sigma) = \sum_{s=(s^1,\dots,s^N) \in S} \prod_i \sigma^i(s^i) g^i(s) \geq g^i(t^i, \sigma^{-i}), \forall t^i \in S^i, \forall i \in I,$$

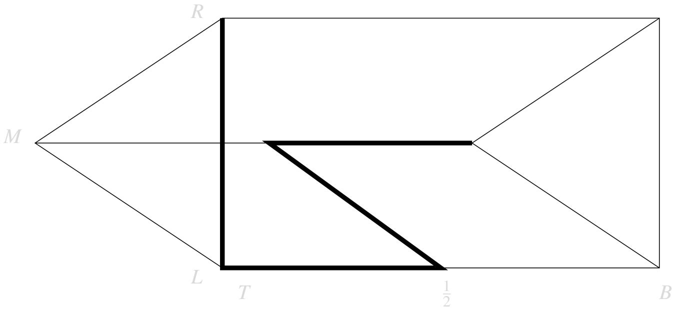
the unknown being the $\sigma^i(s^i)$.
(By linearity it is enough to compare to extreme points). ■

Corollary

The set of equilibria is semi-algebraic.
It is a finite union of closed connected components.

Example 1

	L	M	R
T	$(2,1)$	$(1,0)$	$(1,1)$
B	$(2,0)$	$(1,1)$	$(0,0)$



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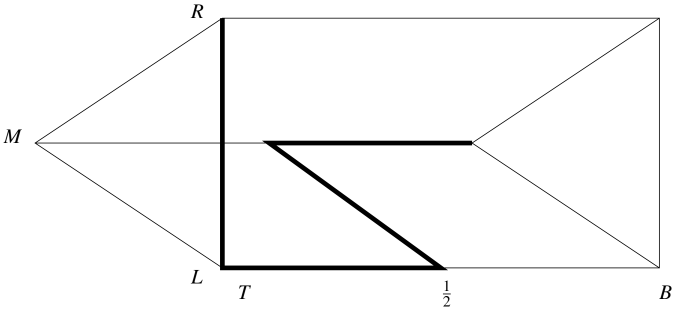
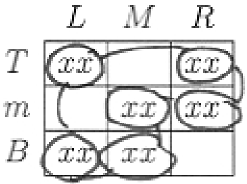


Example 2 (Kohlberg and Mertens, 1986)

	L	M	R
T	$(1,1)$	$(0,-1)$	$(-1,1)$
m	$(-1,0)$	$(0,0)$	$(-1,0)$
B	$(1,-1)$	$(0,-1)$	$(-2,-2)$

There is only one connected component of equilibria which is of the form:



hence homeomorphic to a circle in Σ.

In addition each point is limit of an equilibrium of a close-by game, like with $\varepsilon > 0$:

	L	M	R
T	$(1, 1-\varepsilon)$	$(\varepsilon, -1)$	$(-1-\varepsilon, 1)$
m	$(-1, -\varepsilon)$	$(-\varepsilon, \varepsilon)$	$(-1+\varepsilon, -\varepsilon)$
B	$(1-\varepsilon, -1)$	$(0, -1)$	$(-2, -2)$

with equilibrium $[(\varepsilon/(1+\varepsilon), 1/(1+\varepsilon), 0); (0, 1/2, 1/2)]$ close to $[(0, 1, 0); (0, 1/2, 1/2)]$.

15 Supermodular games

Consider the euclidean space $\mathbb{R}^n$, with the product (partial) order $x \geq y$ iff $x_i \geq y_i$ for all i.
$S \subset \mathbb{R}^n$ is a lattice if for all $x, y \in S$: $\sup\{x, y\} \in S$ and $\inf\{x, y\} \in S$.

Theorem (Tarski, 1955)

Let S be a non empty compact lattice and f an increasing function from S to itself.
Then f has a fixed point.

Consider a strategic game G, where for each $i \in I$, S^i is a non-empty compact subset of $\mathbb{R}^{m_i}$ and g^i is u.s.c. in s^i for each fixed s^{-i}. Assume moreover the game supermodular:

(i) For all i, S^i is a lattice in $\mathbb{R}^{m_i}$.

(ii) g^i has increasing differences in (s^i, s^{-i}):

$$g^i(s^i, s^{-i}) - g^i(s'^i, s^{-i}) \geq g^i(s^i, s'^{-i}) - g^i(s'^i, s'^{-i})$$

as soon as $s^i \geq s'^i$ and $s^{-i} \geq s'^{-i}$.

(iii) g^i is supermodular w.r.t. s^i: $\forall s^{-i} \in S^{-i}$,

$$g^i(s^i, s^{-i}) + g^i(s'^i, s^{-i}) \leq g^i(s^i \vee s'^i, s^{-i}) + g^i(s^i \wedge s'^i, s^{-i}).$$

Proposition (Topkis, 1979)

The game G has an equilibrium.

Proof: For each i and s^{-i}, $BR^i(s^{-i})$ is a non-empty compact lattice of $\mathbb{R}^{m_i}$. If $s^{-i} \geq s'^{-i}$, $\forall t'^i \in BR^i(s'^{-i})$, $\exists t^i \in BR^i(s^{-i})$ such that $t^i \geq t'^i$. Apply Tarski's theorem to the maximal element of the best reply map. ∎

16 Examples

Zero-sum games

Let $f : S \times T \to \mathbb{R}$.

$$\underline{v} = \sup_S \inf_T f(s,t) \qquad \overline{v} = \inf_T \sup_S f(s,t);$$

s is $\varepsilon(\geq 0)$-optimal if:

$$f(s,t) \geq \underline{v} - \varepsilon, \quad \forall t \in T.$$

Proposition

Assume that the game has a value:

$$\underline{v} = \overline{v}$$

and that s,t are ε optimal. Then they form a $2\varepsilon-$ equilibrium:

$$f(s,t') + 2\varepsilon \geq f(s,t) \geq f(s',t) - 2\varepsilon, \quad \forall s',t' \in S \times T.$$

Minmax principle (Aumann and Maschler, 1968)

	L	R
T	$(2,0)$	$(0,1)$
B	$(0,1)$	$(1,0)$

Values = $[2/3, 1/2]$,
optimal strategies $x = (1/3, 2/3), y = (1/2, 1/2)$

Equilibrium payoff = $[2/3, 1/2]$,
equilibrium strategies $x = (1/2, 1/2), y = (1/3, 2/3)$

Dominant strategy and Pareto optimality

	L	R
T	$(0,4)$	$(3,3)$
B	$(1,1)$	$(4,0)$

Adding a constant payoff to player 1 when L is played does not change BR:

	L	R
T	$(3,4)$	$(3,3)$
B	$(4,1)$	$(4,0)$

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	L	R
T	$(3,1)$	$(0,0)$
B	$(4,4)$	$(1,3)$

Battle of the sexes

	L	R
T	$(3,1)$	$(0,0)$
B	$(0,0)$	$(1,3)$

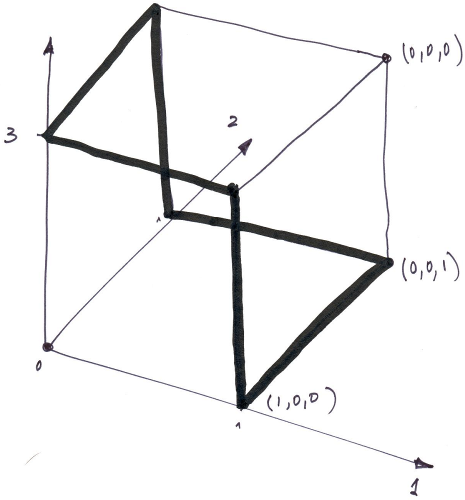
the symmetric equilibrium is dominated by the two non-symmetric ones.

Adding a dominant strategy

	L	R
T	$(1,1)$	$(10,0)$
B	$(0,0)$	$(5,5)$

Minority game

3 players with two actions (rooms) A, B.
The payoff is 1 if the player is alone and 0 otherwise.



Cournot duopoly

2 firms, $i = 1,2$ produce each a quantity $q^i \in [0,100]$. The cost function is $C^i(q^i) = 10q_i$, and the price p is a function of the total production $q = q^1 + q^2$: $p = 100 - q$. The payoff of firm i is thus:

$$g^i(q^1,q^2) = p \times q^i - C^i(q^i) = \left(90 - (q^1 + q^2)\right) q^i.$$

The strategy sets are convex compact and the payoff functions are continuous and concavet, hence an equilibrium exists.
Note that $q^i = 0 \Rightarrow q^j = 45 \Rightarrow q^i = \frac{45}{2}$ and
$q^i = 100 \Rightarrow q^j = 0 \Rightarrow q^i = 45.$
Hence the equilibrium is interior and the first order optimality conditions are necessary and sufficient:
$q_1 = \frac{90-q_2}{2}$ and $q_2 = \frac{90-q_1}{2}$.
The equilibrium is thus $q_1 = q_2 = 30$.

Population game

Population games : each participant $i \in I$ corresponds to a nonatomic set of agents (with a given mass m^i) having all the same characteristics. x^{ip} is the proportion of agents of choosing $p \in S^i$ (finite) in population i. X^i is the corresponding set.
A typical example is congestion games.

The payoffs are defined by a family of continuous functions $\{F^{ip}, i \in I, p \in S^i\}$, all from $X = \prod_i X^i$ to $\mathbb{R}$, where $F^{ip}(x)$ is the outcome of a member in population i choosing p, when the environment is given by x.
An equilibrium is a point $x \in X$ satisfying:

$$x^{ip} > 0 \Rightarrow F^{ip}(x) \geq F^{iq}(x), \quad \forall p, q \in S^i, \forall i \in I. \qquad (1)$$

This corresponds to a Wardrop equilibrium.

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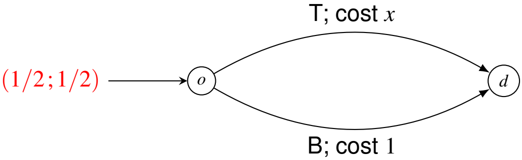
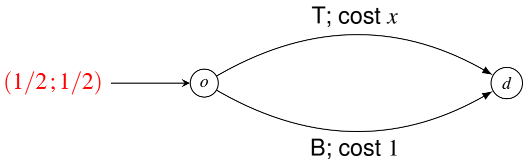
2 participants, size $1/2$ each



Set of equilibria, with probability of T:

2 participants, size $1/2$ each



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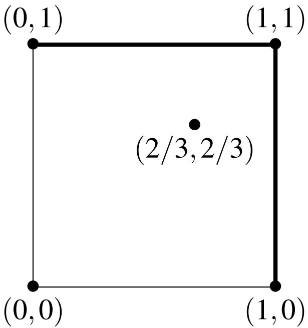
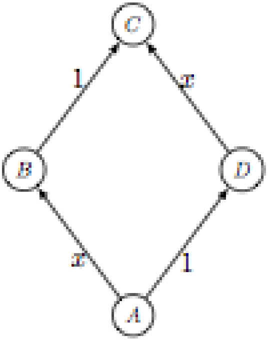
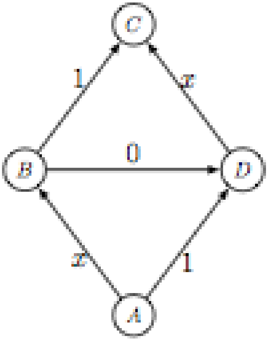


Braes paradox

2 participants: size 1/2 each





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2 Complements

21 Equilibrium and Variational Inequalities

Finite games

The game is defined by a family of functions $\{g^i, i \in I\}$ from (the finite set) $S = \prod_{i \in I} S^i$ to $\mathbb{R}$ with mixed extension to Σ.
Let Vg denote the vector payoff associated to g.
Explicitly, $Vg^i : \Sigma^{-i} \to \mathbb{R}^{m^i}$ is defined by :

$$Vg^{ip}(\sigma^{-i}) = g^i(p, \sigma^{-i}), \forall p \in S^i.$$

Hence $g^i(\sigma) = \langle \sigma^i, Vg^i(\sigma^{-i}) \rangle = \sum_{p \in S^i} \sigma^i(p) Vg^{ip}(\sigma^{-i})$.

An equilibrium σ is thus a solution of :

$$\langle Vg(\sigma), \sigma - \tau \rangle = \sum_{i \in I} \langle Vg^i(\sigma^{-i}), \sigma^i - \tau^i \rangle \geq 0, \quad \forall \tau \in \Sigma. \qquad (2)$$

$\mathscr{C}^1$games

Assume S^i convex in some euclidean space, g^i $\mathscr{C}^1$ and concave w.r.t. s^i for each $i \in I$. Then:

$$g^i(s) \geq g^i(t^i, s^{-i}), \quad \forall t^i \in S^i, \forall i \in I \tag{3}$$

is equivalent to :

$$\langle \nabla g(s), s - t \rangle = \sum_{i \in I} \langle \nabla^i g^i(s), s^i - t^i \rangle \geq 0, \qquad \forall\, t \in S. \tag{4}$$

where ∇^i is the gradient w.r.t. s^i.

Population games

Recall that the payoffs are defined by a family of continuous functions $\{F^{ip}, i \in I, p \in S^i\}$, all from $X = \prod_i X^i$ to $\mathbb{R}$, where $F^{ip}(x)$ is the outcome of a member in population i choosing p, when the environment is given by x.

An equilibrium is a point $x \in X$ satisfying:

$$x^{ip} > 0 \Rightarrow F^{ip}(x) \geq F^{iq}(x), \quad \forall p, q \in S^i, \forall i \in I. \tag{5}$$

An equivalent characterization is through the variational inequality:

$$\langle F(x), x - y \rangle = \sum_{i \in I} \langle F^i(x), x^i - y^i \rangle \geq 0, \quad \forall y \in X. \tag{6}$$

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Note that F, ∇g and Vg play similar roles in the three frameworks.
Call them evaluation functions and denote them by Φ with for each (i,p), $\Phi^{ip} : X \longrightarrow \mathbb{R}$.

Definition

$NE(\Phi)$ is the set of $x \in X$ satisfying:

$$\langle \Phi(x), x - y \rangle \geq 0, \qquad \forall y \in X. \tag{7}$$

More generally let $C \subset \mathbb{R}^d$ be a closed convex set and Ψ a map from C to $\mathbb{R}^d$.
Consider the variational inequality, for $x \in C$:

$$\langle \Psi(x), x - y \rangle \geq 0, \qquad \forall y \in C. \tag{8}$$

Proposition

The solutions to (8) are the solutions in C of:

$$\Pi_C[x + \Psi(x)] = x. \tag{9}$$

where Π_C is the projection operator on the closed convex subset C.

Proof : The projection operator is defined on $\mathbb{R}^d$ by :

$$\langle z - \Pi_C[z], c - \Pi_C[z] \rangle \leq 0, \quad \forall c \in C$$

thus $\Pi_C[x + \Psi(x)] = x$ is equivalent to:

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$$\langle \Psi(x), c - x \rangle \leq 0, \qquad \forall c \in C.$$

Corollary

Assume Ψ *continuous on* C *convex, compact. Then* $NE(\Psi) \neq \emptyset$.

Proof.

The map $x \mapsto \Pi_C[x + \Psi(x)]$ is continuous from the convex compact set C to itself, hence a fixed point exists. □

22 Manifold of equilibria

We deal here with finite games. Each action set S^i is finite, with cardinal m^i and $m = \Pi_i m^i$. A game g is thus identified with a point in $\mathbb{R}^{Nm}$.
We consider the manifold of equilibria obtained by taking payoffs as parameters. The equilibrium equations correspond to a finite family of polynomial inequalities with the variables (g, σ):

$$F_k(\sigma, g) \leq 0, \qquad k \in K$$

where K is finite, $g \in R^{Nm}$ is the game and σ the strategy profile.
Let $\mathscr{G}$ be the family of games and $\mathscr{E}$ the graph of the equilibrium correspondence:

$$\mathscr{E} = \{(g, \sigma); g \in \mathscr{G},\ \sigma \text{ equilibria of } \ g\}.$$

Theorem (Kohlberg and Mertens, 1986)

$\mathscr{E}$ is homotopic to the graph of a function .

Corollary

The projection π from $\mathscr{E}$ to $\mathscr{G}$ is homotopic to an homeomorphism.

Proof:
Fix a game g and consider, for $h \in H = \mathbb{R}^{\Sigma m^i}$, the affine space defined by $[g;h] = \{Vg^i(.) + h^i\}$, $h^i \in \mathbb{R}^{m^i}$.
The homotopy if defined for $(\sigma;[g;h]) \in \mathscr{E}$ by :

$$\Phi_t(\sigma;[g;h]) = (\sigma,[g,h+t(\sigma+Vg(\sigma))])$$

where $t \in [0,1]$.
Let $\mathscr{F}$ be the graph of the map $[g;h] \mapsto \Pi_\Sigma(h), h \in H$. Note that $\Phi_1(\sigma;[g;h]) \in \mathscr{F}$ since

$$\Pi_\Sigma(\sigma+Vg(\sigma)+h) = \sigma$$

or

$$\langle Vg(\sigma)+h, \sigma-\tau\rangle \geq 0, \quad \forall \tau \in \Sigma.$$

Conversely, starting from $h \in H$ let $\tau = \Pi_\Sigma(h)$. Then $(\tau, [g : h']) \in \mathscr{E}$ with $h' = h - \tau - Vg(\tau)$ since

$$\langle h - \tau, z - \tau \rangle \leq 0, \quad \forall z \in \Sigma$$

writes

$$\langle h' + Vg(\tau), \tau - z \rangle \geq 0, \quad \forall z \in \Sigma.$$

■

Let g be a game and $NE(g)$ its set of equilibria which consists of a finite set of connected components $C_k, k \in K$.

Definition

C_k is essential if for any neighborhood V of C_k in Σ, there exists a neighborhood W of g in $\mathscr{G}$ such that for any $g' \in W$ there exists $\sigma \in NE(g') \cap V$.

Proposition

i) Generically, the set of equilibria is finite and odd.
ii) Every game has an essential component.

Proof

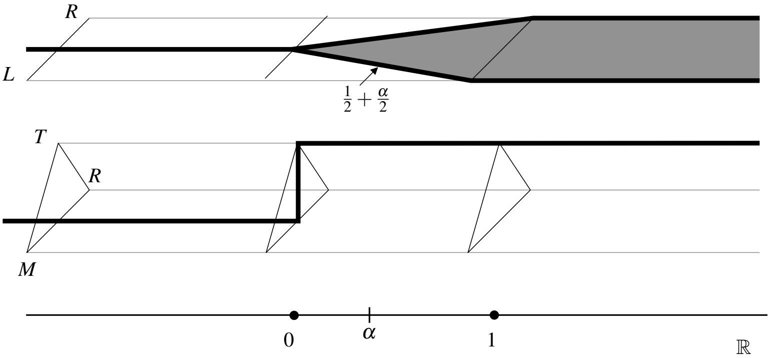
$i)$ Generically an equilibrium is isolated and transverse to the projection from $\mathscr{E}$ to $\mathscr{G}$. If it is "in" for the projection map the degree is 1 and -1 if it is "out". The global degree of the projection (which is the sum over all components) is invariant via homotopy thus equal to 1: thus there are $p+1$ equilibria with degree + 1 and p with degree -1.

$ii)$ By induction it is enough to show that if $NE(g)$ is contained in $U \cup V$, where U et V are two open sets with disjoint closures, there exists a neighborhood W of g such for any $g' \in W$, $NE(g') \cap U \neq \emptyset$ or for any $g' \in W$, $NE(g') \cap V \neq \emptyset$.
Let $\Psi(g)$ be the graph of the best reply correspondence at g (in $\Sigma \times \Sigma$). There exists a neighborhood C of $\Psi(g)$ (with convex sections) such that the intersection with the diagonal belongs to $U \cup V \times U \cup V$.
By contradiction assume the existence of g_1 near g with $NE(g_1) \cap U = \emptyset$ and similarly for g_2 and V. On the other hand one can assume $\Psi(g_i) \subset C$. Let α be a continuous function from Σ to $[0,1]$, with value 1 on U and 0 on V. The correspondence defined by $T(\sigma) = \alpha(\sigma)\Psi(g_1)(\sigma) + (1 - \alpha(\sigma))\Psi(g_2)(\sigma)$ is u.s.c. with convex values. Its graph is included in C hence its non-empty set of fixed points is included in $U \cup V$. Consider such a point, $\sigma = T(\sigma)$. If $\sigma \in U$, $\alpha(\sigma) = 1$ and σ is a fixed point of $\Psi(g_1)$ hence in V. We obtain a similar contradiction if $\sigma \in V$. ■

In the next game, with parameter $\alpha \in \mathbb{R}$:

	L	R
T	$(\alpha, 0)$	$(\alpha, 0)$
M	$(1, -1)$	$(-1, 1)$
B	$(-1, 1)$	$(1, -1)$

the equilibrium correspondence is given by:



23 Nash maps and dynamics

Definition

A Nash field is a continuous map (or a u.s.c. correspondence) Ψ from $\mathcal{G} \times \Sigma$ to Σ such that:

$$NE(g) = \{\sigma \in \Sigma; \Psi(g, \sigma) = \sigma\}, \forall g \in \mathcal{G}.$$

The regularity w.r.t. g is crucial.

Proposition

The following maps are Nash fields:
1. (Nash, 1950)

$$\Psi(g, \sigma)^i(s^i) = \frac{\sigma^i(s^i) + (g^i(s^i, \sigma^{-i}) - g^i(\sigma))^+}{1 + \Sigma_{t^i}(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+}$$

2. (Gul, Pearce and Stacchetti, 1993)
Recall that Π_Σ is the projection on the convex compact set Σ.

$$\Psi(g, \sigma) = \Pi_\Sigma(\{\sigma^i + Vg^i(\sigma^{-i})\}).$$

Each Nash field Ψ induces, on each game g, a dynamical system:

$$\dot{\sigma} = \Psi(g, \sigma) - \sigma,$$

on Σ, whose rest points are $NE(g)$.
(An alternative definition of Nash field would be a continuous map from $\mathscr{G} \times \Sigma$ to T_Σ (tangent vector space at Σ), vanishing on $\mathscr{E}$ and inwards pointing at the boundary of Σ.)

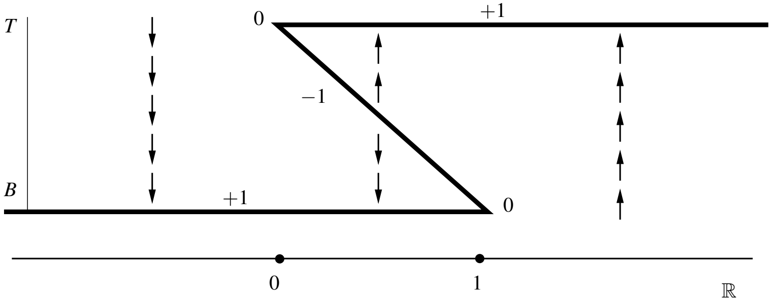
Every component of the set of fixed points has an index and the sum of indices is 1 which is the Euler characteristic of the simpex Σ (Poincaré-Hopf Theorem, see Milnor (1965), p. 35).

Moreover the index of a component C is independent of the Nash field and is equal to the local degree at C of the projection from $\mathscr{E}$ to $\mathscr{G}$ (Demichelis and Germano (2000), Govidan and Wilson (1997)).

For the next game, with parameter $t \in \mathbb{R}$:

	L	R
T	(t,t)	$(0,0)$
B	$(0,0)$	$(1-t,1-t)$

one obtains for the manifold and the dynamics the following configuration:



For $t \notin [0, 1]$ there is only one equilibrium, hence there is only one configuration for the Nash field (inwards at the boundary and vanishing only on the manifold of equilibria).
By continuity we obtain the configuration for $t \in (0, 1)$, thus the mixed equilibria has index -1: the projection reverses the orientation of the manifold - or the vector field has index -1 there.

24 Potential games and dynamics

Finite games

Definition

A real function P defined on Σ is a potential (Monderer and Shapley, 1996) for the game (g, Σ) if:

$$g^i(s^i, u^{-i}) - g^i(t^i, u^{-i}) = P(s^i, u^{-i}) - P(t^i, u^{-i}), \forall s^i, t^i \in S^i, u^{-i} \in S^{-i}, \forall i \in I. \quad (10)$$

This means that the impact due to a change of action of player i is the same on g^i than on P, for all $i \in I$.

Population games

Definition

A real function W, of class $\mathscr{C}^1$ on a neighborhood Ω of X, is a *potential* for Φ if for each $i \in I$, there is a strictly positive function $\mu^i(x)$ defined on X such that

$$\left\langle \nabla_i W(x) - \mu^i(x)\Phi^i(x), y^i \right\rangle = 0, \quad \forall x \in X, \forall y^i \in TX^i, \forall i \in I, \tag{11}$$

where $TX^i = \{y \in \mathbb{R}^{|S^i|}, \sum_{p \in S^i} y_p = 0\}$ is the tangent space to X^i and $\nabla_i W(x)$ is the gradient w.r.t. x^i.

Theorem

Let $\Gamma(\Phi)$ be a game with potential W.
1. Every local maximum of W is an equilibrium of $\Gamma(\Phi)$.
2. If W is concave on X, then any equilibrium of $\Gamma(\Phi)$ is a global maximum of W on X.

Dissipative games

Definition

The game $\Gamma(\Phi)$ is dissipative if Φ satisfies:

$$\langle \Phi(x) - \Phi(y), x - y \rangle \leq 0, \qquad \forall\, (x,y) \in X \times X.$$

(Hofbauer and Sandholm, “stable games”)

Let $SNE(\Phi)$ be the set of $x \in X$ satisfying:

$$\langle \Phi(y), x - y \rangle \geq 0, \qquad \forall y \in X.$$

Proposition

If $\Gamma(\Phi)$ is dissipative

$$SNE(\Phi) = NE(\Phi).$$

in particular $NE(\Phi)$ is convex.

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Proposition

If $\Gamma(\Phi)$ is dissipative

$$SNE(\Phi) = NE(\Phi).$$

in particular $NE(\Phi)$ is convex.

Proof:
One direction is clear. If Φ is dissipative and x is an equilibrium, then:

$$\langle \Phi(y), x-y \rangle \geq \langle \Phi(x), x-y \rangle \geq 0, \quad \forall y \in X.$$

On the other hand, given $z \in X$ and $t \in (0,1]$, let $y = x + t(z-x)$, hence:

$$\langle \Phi(x + t(z-x)), t(x-z) \rangle \geq 0.$$

Dividing by t and then letting t go to 0 gives, by continuity of Φ, the result.

■

Dynamics

The general form of a dynamics describing the evolution of the strategic interaction in game $\Gamma(\Phi)$ is

$$\dot{x}_t = \mathscr{B}_{\Phi}(x_t), \quad x \in X,$$

where for each $i \in I$, $\mathscr{B}^i_{\Phi}(x) \in TX^i$ and X is invariant.

Replicator dynamics (RD) (Taylor and Jonker)

$$\dot{x}^{ip}_t = x^{ip}_t[\Phi^{ip}_t(x_t) - \overline{\Phi}^i(x_t)], \quad p \in S^i, i \in I,$$

where

$$\overline{\Phi}^i(x) = \langle x^i, \Phi^i(x) \rangle = \sum_{p \in S^i} x^{ip}\Phi^{ip}(x)$$

Brown-von-Neumann-Nash dynamics (BNN) (Brown and von Neumann, Smith, Hofbauer)

$$\dot{x}^{ip}_t = \hat{\Phi}^{ip}(x_t) - x^{ip}_t \sum_{q \in S^i} \hat{\Phi}^{iq}(x_t), \quad p \in S^i, i \in I,$$

where $\hat{\Phi}^{iq} = [\Phi^{iq}(x) - \overline{\Phi}^i(x)]^+$ is the “excess evaluation” of q.

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Smith dynamics (Smith)

$$\dot{x}_t^{ip} = \sum_{q\in S^i} x_t^{iq}[\Phi^{ip}(x_t) - \Phi^{iq}(x_t)]^+ - x_t^{ip}\sum_{q\in S^i}[\Phi^{iq}(x_t) - \Phi^{ip}(x_t)]^+, \quad p\in S^i, i\in I,$$

where $[\Phi^{ip}(x) - \Phi^{iq}(x)]^+$ corresponds to pairwise comparison.

Local/direct projection dynamics (LP) (Dupuis and Nagurney, Lahkar and Sandholm)

$$\dot{x}_t^i = \Pi_{T_{X^i}(x_t^i)}[\Phi^i(x_t)], \quad i\in I,$$

where $T_{X^i}(x^i)$ denotes the tangent cône to X^i at x^i.

Global/target projection dynamics (GP) (Friesz, Bernstein, Mehta, Tobin and Ganjalizadeh, Tsakas and Voorneveld)

$$\dot{x}_t^i = \Pi_{X^i}[x_t^i + \Phi^i(x_t)] - x_t^i, \quad i\in I.$$

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$$\dot{x}_t^i \in BR^i(x_t) - x_t^i, \quad i\in I,$$

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The dynamics $\mathscr{B}_\Phi$ satisfies:

i) positive correlation (PC)(Sandholm) if:

$$\langle \mathscr{B}^i_\Phi(x), \Phi^i(x)\rangle > 0, \quad \forall i \in I, \forall x \in X \text{ s.t. } \mathscr{B}^i_\Phi(x) \neq 0.$$

This corresponds to MAD (myopic adjustment dynamics, Swinkels)

ii) Nash stationarity if:
for $x \in X$, $\mathscr{B}_\Phi(x) = 0$ if and only if $x \in NE(\Phi)$.

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Proposition

All previous dynamics (RD), (BNN), (Smith), (LP), (GP) and (BR) satisfy (PC).

Proposition

(BNN), (Smith), (LP), (GP) and (BR) satisfy Nash stationarity on X.
(RD) satisfy Nash stationarity on $\mathrm{int}X$.

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Dynamics in potential games

Proposition

Consider a potential game $\Gamma(\Phi)$ with potential function W. If the dynamics $\dot{x} = \mathscr{B}_\Phi(x)$ satisfies (PC), then W is a strict Lyapunov function for $\mathscr{B}_\Phi$.
Moreover, all ω-limit points are rest points of $\mathscr{B}_\Phi$.

Proof:

$$\frac{d}{dt}W(x_t) = \sum_i \langle \nabla^i W(x_t), \dot{x}_t^i \rangle = \sum_i h^i(x_t) \langle \Phi^i(x_t), \dot{x}_t^i \rangle > 0$$

■

Table of contents

3 Extensions

31 Correlated equilibria

This section deals with "correlated equilibrium" (Aumann, 1974) which is an extension of Nash equilibrium and has good properties from strategic, analytic and dynamic viewpoints.

Example 1

$3,1$	$0,0$
$0,0$	$1,3$

2 pure equilibria, not symmetric
one mixed, symmetric $(1/4,3/4)$ and dominated
Use a public coin: $(3,1)$ if Head and $(1,3)$ if Tail.
Induced distributions on moves:

$1/2$	0
0	$1/2$

Example 2

	g	d
H	$2,7$	$6,6$
B	$0,0$	$7,2$

Let a signal space (B,G,N), with uniform probability $(1/3,1/3,1/3)$. Assume that the players get private messages:
1 knows $a = \{B,G\}$ or $b = \{N\}$
2 knows $\alpha = \{B\}$ or $\beta = \{G,N\}$
Consider the strategies:
H if a, B if b for player1;
g if α, d if β for player 2.
They induce on S the correlation matrix:

$1/3$	$1/3$
0	$1/3$

and no deviation is profitable.

Definition

An information structure $\mathscr{I}$ is given by:
- a probability space $(\Omega, \mathscr{A}, P)$
- a measurable map θ^i from $(\Omega, \mathscr{A})$ to A^i (signals of i), for each $i \in I$.

Let G defined by $g : S = \Pi_i S^i \to \mathbb{R}^n$.

Definition

The game G extended by $\mathscr{I}$, denoted $[G, \mathscr{I}]$, is the game played in 2 stages:
stage 0 : the random variable ω is selected according to P and the signal $\theta^i(\omega)$ is sent to player i.
stage 1 : the players play in G.

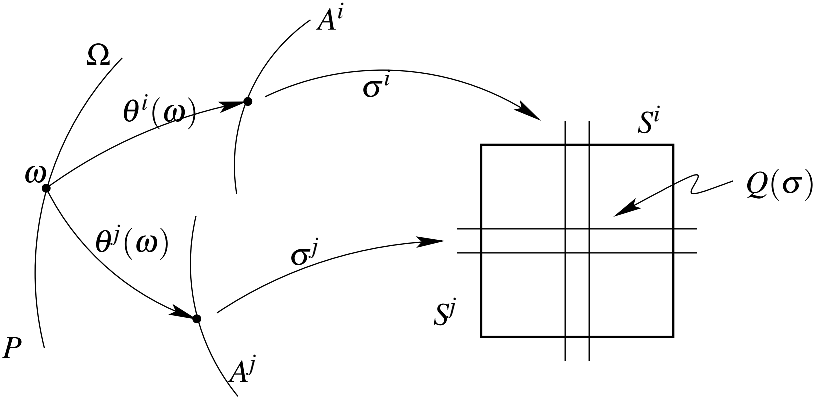
A strategy μ^i of player i in the game $[G, \mathscr{I}]$ is a measurable map from A^i to S^i.
A profile μ of such elements is called a correlated strategy.

Correlated equilibrium

Definition

A correlated equilibrium of G is a Nash equilibrium of some extended game $[G, \mathscr{I}]$.

A profile μ of strategies in $[G, \mathscr{I}]$ maps the probability P on Ω to probability $Q(\mu)$ on S: random variable $\rightarrow$ signal $\rightarrow$ move .



Explicitly, for each ω, $Q(\omega,\mu)$ is the probability on S given by $\prod_i \mu^i(\theta^i(\omega))$ and $Q(\mu)$ is the expectation w.r.t. P.

Definition

$CED(G)$ is the set of distributions of correlated equilibria in G:

$$CED(G) = \cup_{\mathscr{I},\mu}\{Q(\mu); \mu \text{ equilibrium in } [G,\mathscr{I}]\}$$

$CED(G)$ is a convex set: consider the convex combination of information structures.

A canonical information structure for G is given by:
$\Omega = S$; $\theta^i : S \to S^i, \theta^i(s) = s^i$.
P is a probability on S and each player is informed upon his component.
A canonical correlated equilibrium is an equilibrium of $[G,\mathscr{I}]$ where $\mathscr{I}$ is a canonical information structure and where equilibrium strategies are given by:

$$\mu^i(\omega) = \mu^i(s) = \mu^i(s^i) = s^i$$

“each player follows his signal".

The induced distribution of canonical correlated equilibrium ($\in CCED$) is obviously P.

Theorem (Aumann, 1974)

$$CCED(G) = CED(G)$$

Proof: Let μ be an equilibrium profile in an extension $[G, \mathscr{I}]$ and $Q = Q(\sigma)$ the induced distribution.
Then Q belongs to $CCED(G)$.
In fact, each player i get less information: his move s^i rather than the signal a^i such that $\mu^i(a^i) = s^i$. But s^i is a best reply to the correlated strategy of $-i$, conditional to a^i. It is then enough to use the convexity of BR^i on $\Delta(S^{-i})$. ∎

Characterization

Theorem

$Q \in CED(G)$ iff:

$$\forall s^i, t^i \in S^i, \forall i = 1, \ldots, n \qquad \sum_{s^{-i} \in S^{-i}} [g^i(s^i, s^{-i}) - g^i(t^i, s^{-i})] Q(s^i, s^{-i}) \geq 0.$$

Proof: Assume $Q \in CCED(G)$.
If s^i is announced (i.e. its marginal $Q^i(s^i) = \sum_{s^{-i}} Q(s^i, s^{-i}) > 0$) we consider the conditional distribution on S^{-i}, $Q(.|s^i)$, and the equilibrium condition writes:

$$s^i \in BR^i(Q(.|s^i).$$

s^i is a best reply of player i to the distribution of the moves of $-i$, conditional to s^i. ∎

The approach in terms of Nash equilibrium of an extended game is “ex-ante”.
The previous characterization corresponds to an “ex-post” criteria.

Corollary

The set of CED is the convex hull of finitely many points.

Proof: It is a subset of $\Delta(S)$ defined by a finite set of linear inequalities. ■

Comments

1) There is an elementary proof of existence of correlated equilibria via the minmax theorem: CED corresponds to the set of optimal strategies in a finite 0-sum game, Hart and Schmeidler (1989).

2) There exists correlated equilibrium distributions outside the convex hull of Nash equilibria.
In the game:

$0,0$	$5,4$	$4,5$
$4,5$	$0,0$	$5,4$
$5,4$	$4,5$	$0,0$

the only equilibrium is symmetric: $(1/3, 1/3, 1/3)$ with payoff 3.
The following is a CED

0	$1/6$	$1/6$
$1/6$	0	$1/6$
$1/6$	$1/6$	0

inducing the payoff 9/2.

32 Incomplete information games

Like in the previous section, there is an information structure $\mathscr{I} = (\Omega, \mathscr{A}, P)$, but the game itself may depend upon ω.
One call type space the set of signals A^i of each player i (each player knows his type).
Assume A^i finite.
A strategy σ^i of player i is still a map from A^i to $\Delta(S^i)$.
The payoff corresponding to a profile σ is given by :

$$\gamma(\sigma) = \int_{\omega} g(\{\sigma^i(\theta^i(\omega))\}_{i \in I}; \omega) P(d\omega)$$

It is enough to work with Q, the induced probability on $A = \Pi_i A^i$ and $g(., a)$, the expectation of $g(., \omega)$ on $\theta^{-1}(a)$.

The payoff can be written as:

$$\gamma(\sigma) = \sum_a g(\{\sigma^i(a^i)\}; a) Q(a)$$

hence for player i:

$$\gamma^i(\sigma) = \sum_{a^i} Q^i(a^i) B^i(a^i)$$

where

$$B^i(a^i) = \sum_{a^{-i}} g^i(\sigma^i(a^i), \{\sigma^j(a^j)\}_{j\neq i}; a) Q(a^{-i}|a^i).$$

Thus if σ is an equilibrium profile, for each i and each signal a^i, $\sigma^i(a^i)$ maximizes the "Bayesian" payoff facing σ^{-i}:

$$\sum_{a^{-i}} g^i(., \{\sigma^j(a^j)\}_{j\neq i}; (a^i, a^{-i}) Q(a^{-i}|a^i).$$

The first maximization (in γ) is "ex-ante", the second one "ex-post".

Complements

A pure strategy of player i is a map from A^i to S^i.
A behavioral strategy sends A^i to $\Delta(S^i)$.
A mixed strategy is a distribution on the set of pure strategies.

A distributional strategy (Milgrom and Weber, 1985), μ^i is an element of $\Delta(A^i \times S^i)$ that respects the data: the marginal on A^i is Q^i .
The conditional probability $\mu^i(.|a^i)$, as a function of a^i, corresponds to a behavioral strategy.

Example: war of attrition

Consider a symmetric 2 player game: each player $i = 1,2$ has a type $a^i \in A^i = \mathbb{R}^+$ and choses an action $s^i \in S^i = \mathbb{R}^+$. The payoff of each player is a function of his own type and of both actions:

$$f_i(a^i; s^1, s^2) = \begin{cases} a^i - s^j & \text{if} \quad s^j < s^i \\ -s^i & \text{otherwise} \end{cases}$$

The distribution of the types are independent and given by a repartition function G, known to the players. In addition each player knows his type.

1. If G is the Dirac mass at some point v. The only symmetric equilibrium is to play s according to the law Q_v where:

$$Prob\,(s \le t) = 1 - \exp(-\frac{t}{v}).$$

2. If G has a density g, the only symmetric equilibrium is to play, given the type a^i, the (pure) action:

$$s(a^i) = \int_0^{a^i} \frac{t\,g(t)}{1 - G(t)} dt$$

3. Let G_n with density g_n on $[v-\frac{1}{n}, v+\frac{1}{n}]$ converge weak* to the Dirac mass at v. At equilibrium the distribution of actions of i is given by :

$$\Phi_n(t) = Prob(s \leq t) = P_{G_n}(a^i \ ; s(a^i) \leq t).$$

Then Φ_n converges to Q_v.

Link with non-atomic games (Schmeidler,1973; Mas Colell, 1984).
Framework :
A, set of agents, endowed with measure μ
action space S
payoff F, as a fonction of type, action, distribution of actions of the other players.
Equilibrium:
measure λ on $A \times S$
with marginal μ on A and
ν on S with

$$\lambda\{(a,s); s \in argmax F(a,s,\nu)\} = 1.$$

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