Catch games:
The impact of modeling decisions

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Outline

Introduction

- Catch games

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- Catch games
- Value

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- Catch games
- Value
  - Measure theoretic assumptions

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- Catch games
- Value
    - Measure theoretic assumptions
    - Set theoretic assumptions

Examples

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- Economic: entry game

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- Political: challenger vs incumbent

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- Economic: entry game
- Political: challenger vs incumbent
- Fashion games

Examples

- Economic: entry game
- Political: challenger vs incumbent
- Fashion games
- Military, security

Outline

Catch game

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- Action set A

Catch game

- Action set A
- Action set B: all finite subsets of A

Catch game

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- Winning set for player 1

Catch game

- Action set A
- Action set B: all finite subsets of A
- Winning set for player 1

$$W^1 = \{(a, b) \in A \times B \mid a \notin b\}$$

A catch game

Player 2

Player 1

1	
2	
3	

A catch game

Player 2

	{1}	{2}	{3}
Player 1 — 1			
2			
3			

A catch game

Player 2

Player 1

	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
1							
2							
3							

A catch game

Player 2

Player 1

u	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
1	0	1	1	0	0	1	0
2	1	0	1	0	1	0	0
3	1	1	0	1	0	0	0

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- if $E_1, E_2, \ldots \in \mathcal{F}_A$, then $\bigcup_{i \in \mathbb{N}} E_i \in \mathcal{F}_A$

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Probability measure

$$p\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} p(A_i) \ ,$$

where $A_i \in \mathcal{F}_A$ and $A_i \cap A_j = \emptyset \ (i \neq j)$

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$$p(W^1) = \qquad W^1 \subseteq \bigcup_{i \in \mathbb{N}} Z^i$$

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$$p(W^1) = \qquad \sum_{i \in \mathbb{N}} p(Z^i) : W^1 \subseteq \bigcup_{i \in \mathbb{N}} Z^i$$

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$$p(W^1) = \inf \left\{ \sum_{i \in \mathbb{N}} p(Z^i) : W^1 \subseteq \bigcup_{i \in \mathbb{N}} Z^i \right\}$$

Probability of winning

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$$p(W^1) + p(W^2) \geq 1$$

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$$p(W^1) \geq 1 - p(W^2)$$

Values for player 1

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v_1^{o}

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Values for player 2

Values for player 2

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Outline

Conditions

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Condition [A]

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- For every $a \in A$:

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- For every $a \in A$: $\{a\} \in \mathcal{F}_A$

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Condition [B]

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Condition [B]

- For every $a \in A$, $m \in \mathbb{N}$:

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Condition [B]

- For every $a \in A$, $m \in \mathbb{N}$:

$$\{b \in B_m | a \in b\} \in \mathcal{F}_B$$

Values for player 2

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Conditions [A] + [B]

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$$v_2^p = v_2^o = 1$$

Values for player 1: Case 1

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Example

$A = \mathbb{N}$
$\mathcal{F}_A = \mathcal{P}(\mathbb{N})$

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$$\sum_{n \in \mathbb{N}} p(\{n\})$$

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Example

$A = [0, 1]$

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Condition [A] , Condition [B']

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$$\{b \in B | D \cap b \text{ has exactly } k \text{ elements}\} \in \mathcal{F}_B$$

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$A = [0, 1]$
$\mathcal{F}_A = \mathcal{M}$

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Example

$A = [0, 1]$
$\mathcal{F}_A = \mathcal{M}$
$\mathcal{M}$ σ-algebra of Lebesgue-measurable sets in $[0, 1]$

Values for player 1: Case 3

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Non-trivial measure on $\mathcal{F}_A$, but each such measure has an atom

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$\mathcal{F}_A$: E countable, E^c countable

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$$p(E) = \begin{cases} 0\ , & \text{if } \ E \text{ countable} \\ 1\ , & \text{if } \ E^c \text{ countable} \end{cases}$$

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$$v_1^p = 0$$

Values

	v_1^p	v_1^o
no non-trivial measure on $\mathcal{F}_A$:	0	0
atomless measure on $\mathcal{F}_A$:	1	1
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no non-trivial measure on $\mathcal{F}_A$:	0	0
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$$v_2^p = v_2^o = 1$$

Example

$A = [0,1]$, $\mathcal{F}_A = \mathcal{P}([0,1])$

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Case 1: ZF+(V=L)

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No non-trivial measure on $\mathcal{P}([0, 1])$: $v_1^p = v_1^o = 0$

Example

$A = [0,1]$, $\mathcal{F}_A = \mathcal{P}([0,1])$

Case 1: ZF+(V=L)

No non-trivial measure on $\mathcal{P}([0,1])$: $v_1^p = v_1^o = 0$
AC $\Rightarrow \mathcal{M} \neq \mathcal{P}([0,1])$

Example

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Case 2: ZF+AD

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Case 2: ZF+AD

$\mathcal{M} = \mathcal{P}([0, 1])$

Lebesgue-measure: atomless, $v_1^p = v_1^o = 1$

Outline

Probability measure

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) ,$$

where $A_i \cap A_j = \emptyset \ (i \neq j)$

Finitely-additive probability

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \ ,$$

where $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $n < \infty$

Values (AC)

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$$w_1^p = w_2^o = 0$$

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$$w_1^p = w_2^o = 0$$
$$w_1^o = w_2^p = 1$$

Outline

Results

	v_1^p	v_1^o
no non-trivial m. on $\mathcal{F}_A$:	0	0
atomless m. on $\mathcal{F}_A$:	1	1
non-trivial m. on $\mathcal{F}_A$, but has an atom:	?	1

Results

	v_1^p	v_1^o	w_1^p	w_1^o
no non-trivial m. on $\mathcal{F}_A$:	0	0		
atomless m. on $\mathcal{F}_A$:	1	1	0	1
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	v_1^p	v_1^o	w_1^p	w_1^o
no non-trivial m. on $\mathcal{F}_A$:	0	0		
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$$v_2^p = v_2^o = 1$$

Results

	v_1^p	v_1^o	w_1^p	w_1^o
no non-trivial m. on $\mathcal{F}_A$:	0	0		
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$$v_2^p = v_2^o = 1 = w_2^p$$

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	v_1^p	v_1^o	w_1^p	w_1^o
no non-trivial m. on $\mathcal{F}_A$:	0	0		
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$$v_2^p = v_2^o = 1 = w_2^p$$

$$w_2^o = 0$$

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    - Sigma-algebra, probability measure, finitely-additive probability

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  - Measure theoretic assumptions:
    - Sigma-algebra, probability measure, finitely-additive probability
  - Set theoretic assumptions:
    - Cases for probability measures, finitely-additive probability