Where Strategic- and Evolutionary Stability Depart A Study of Minimal Diversity Games	Dagstuhl
by Dieter Balkenborg and Dries Vermeulen	November 2007

structure of talk:

- objective
- minimal diversity games and their Nash equilibrium components
- behavior of the replicator dynamics
- notions of strategic stability
- application to minimal diversity games
- consequences for evolutionary stability

Objective 1	December 2007

Nash equilibrium: combination of strategies in a game where everybody optimizes given the behavior of the others

page 21 - page 24 of Nash' thesis: Motivation and Interpretation

A) the "mass-action" interpretation
B) the "rational prediction" interpretation

"The basic requirement for non-cooperative game is that there should be no pre-play communication among the players [unless it has no bearing on the game]."

A) the "mass-action" interpretation

- "there is a population [in the sense of statistics] of participants for each position in the game"

- "stable "average play" "

- players have limited information, but enough to judge payoffs from each pure (non-randomizing) strategy

- select pure strategy maximizing (myopically) expected payoff

$\Rightarrow$ population play must be Nash equilibrium

Objective 3	December 2007

Similar model to evolutionary game theory, but

- explicit reference to an adaptive dynamical process instead of stationarity assumption

- not necessarily myopic rationality, instead more successful strategies generate more offspring or are imitated more or a learned more quickly...

- (local) stability requirements

⇒ Nash equilibrium necessary, but not sufficient for a long-run stable outcome
⇒ *here:* asymptotically stable Nash equilibrium components

B) the "rational prediction" interpretation

- Rationality and the game common knowledge of the players

- Prediction unique, can be determined by players

- $\Rightarrow$ prediction must be a Nash equilibrium (because it cannot be a "self-destroying prophecy")

- again, Nash equilibrium *necessary, not sufficient* for Nash equilibrium

⇒ need to refine among Nash equilibria, "...sometimes good heuristic reasons can be found for narrowing down the set of equilibrium points..."

⇒ Harsanyi / Selten theory of equilibrium selection or,

arguably most demanding refinement concept:
Kohlberg /Mertens' notion of *strategically stable set of Nash equilibria*

Objective 6	December 2007

Swinkels, Demichelis / Ritzberger: evolutionary stability + ?
$\Rightarrow$ strategic stability

- concrete examples where evolutionary and strategic stability select different components of Nash equilibria

- evolutionary stable components are not essential fixed point sets

- small perturbations yield examples where generically no trajectory converges

- players $i = 1, \ldots, n$

- each player i has a finite set of pure strategies S_i

- his set of mixed strategies is

$$\Sigma_i = \left\{ \sigma_i : S_i \to \mathbf{R}^{\geq 0} | \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$

- The preferences of the players are described by von Neumann / Morgenstern utility (or "payoff") functions

$$u_i : S = \times_{i=1,\ldots,n} S_i \to \mathbf{R}$$

for each player i.

- The expected utility of a mixed strategy combinations is given by the multilinear function

$$u_i : \Sigma = \times_{i=1,\ldots,n} \Sigma_i \to \mathbf{R}$$

defined by

$$u_i\left(\sigma_1, \ldots, \sigma_n\right) = \sum_{(s_1,\ldots,s_n)\in S} \left(\prod_{i=1}^{n} \sigma_i\left(s_i\right)\right) u_i\left(s_1, \ldots, s_n\right)$$

interpretation:

- simultaneous move game

- simultaneous “planning ahead”;
  (pure) strategy: a plan what to do under all contingencies, as opposed to “crossing the bridge if one gets there”

- For $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ and $\tau_i \in \Sigma_i$, $1 \leq i \leq n$

$$\sigma \backslash \tau_i := (\sigma_1, \ldots, \sigma_{i-1}, \tau_i, \sigma_{i+1}, \ldots, \sigma_n)$$

- A *Nash equilibrium* is a mixed strategy combination $\sigma \in \Sigma$ such that

$$u_i(\sigma \backslash \tau_i) \leq u_i(\sigma)$$

for all $\tau_i \in \Sigma_i$, $1 \leq i \leq n$

A team coordination problem:

- $i = 1, \cdots, N \geq 2$ players must choose simultaneously and independently among
- $k = 1, \cdots, K \geq 2$ pure strategies
  (same for all players)
- identical payoffs for all players:

$$u_i(k_1, \cdots, k_N) = \begin{cases} -1 \text{ if } k_1 = k_2 = \ldots = k_N \\ 0 \quad\quad\quad\quad \text{else} \end{cases}$$

A minimum diversity game has the following Nash equilibrium components:

- $\{\mathfrak{m}\}$ where $\mathfrak{m}$ is the Nash equilibrium in mixed strategies where each player chooses all his strategies with equal probability $1/K$. It is inefficient.

- the set G of efficient strategy combinations yielding expected payoff zero.

- exception $N = K = 2$: G consists of two isolated equilibria.

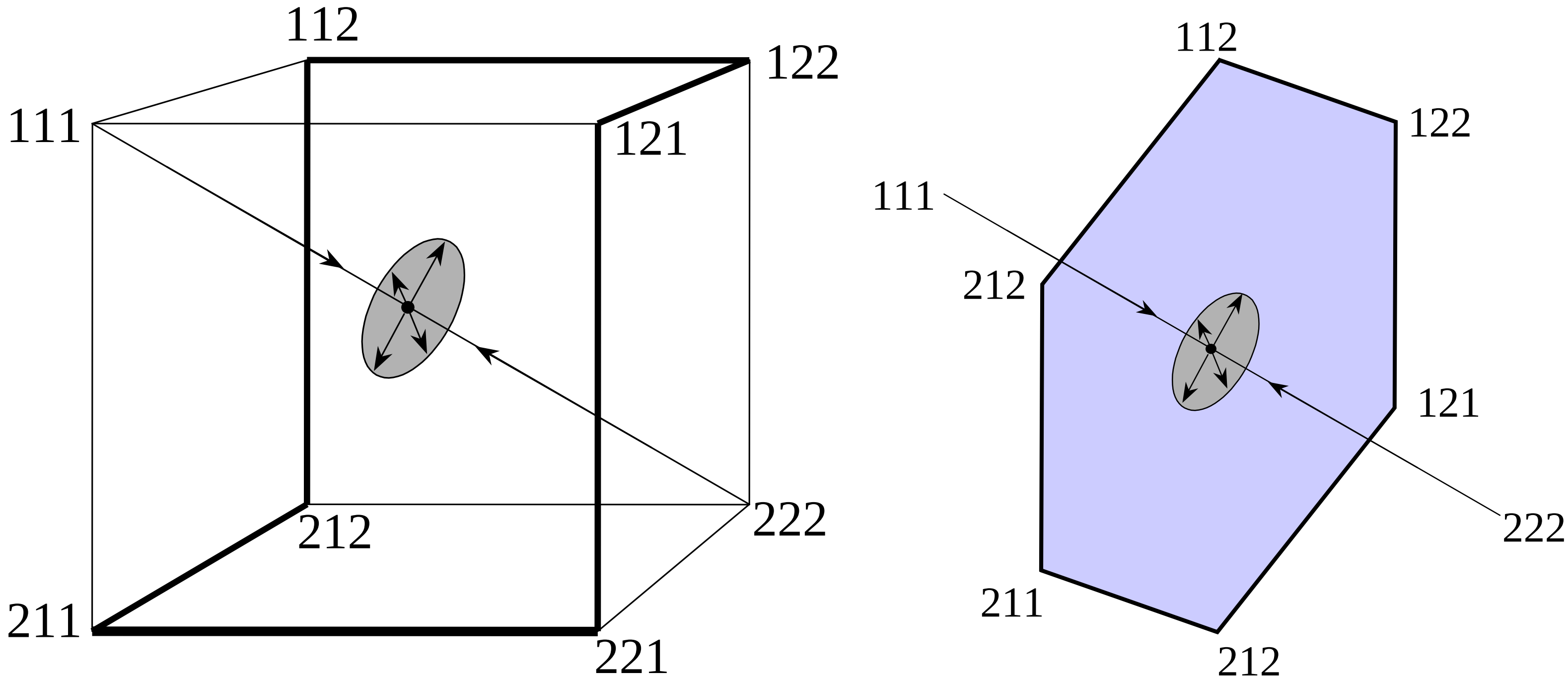
Topology of G	December 2007

Proposition:
G IS A TOPOLOGICAL SPHERE OF DIMENSION

$$(N-1) \times (K-1) - 1$$

Idea: project the $N \times (K-1)$-dimensional polyhedron Σ of all strategy combinations onto the affine subspace through $\mathfrak{m}$ which is orthogonal to the $K-1$ vectors $(e_k, e_k, \cdots, e_k) - \mathfrak{m}$, whereby e_k is the unit vector corresponding to pure strategy $1 \leq k \leq K-1$. The image of Σ is a $(N-1) \times (K-1)$-dimensional compact convex polyhedron. G can be shown to be projected one-to-one onto the boundary of this set. All other strategy combinations are mapped into the interior. The boundary of a compact convex set is always a topological sphere.

Illustration	December 2007



Dimensions	December 2007

$N \vert K$	2	3	4	5
2	0	1	2	3
3	1	3	5	7
4	2	5	8	11
5	3	7	11	15

Euler characteristic zero: two cycles, three 3-dim. spheres etc.

not the dynamic of Taylor + Jonker!!

ODE on Σ:

$$\dot{\sigma}_i(s_i) = \sigma_i(s_i)\left[u_i(\sigma\backslash s_i) - u_i(\sigma)\right] \qquad \text{for all } s_i \in S_i,\ i = 1, \ldots, n$$

- evolution of behavior in n populations
- strategies not played now will not be played in the future (interior of faces invariant)
- growth rate of use of strategy proportional to success of strategy

Notice: Every NE and any pure strategy combination is a rest point.

For any game with identical interests (with identical payoff functions) the payoff is easily shown to increase along a trajectory and is constant only at a rest point.

$$\frac{du}{dt} = \sum_{i=1}^{n} \sum_{s_i \in S_i} \sigma_i\left(s_i\right)\left[u\left(\sigma \backslash s_i\right) - u\left(\sigma\right)\right]^2 \geq 0$$

Therefore any trajectory can only have rest points as ω-limits.

The rest points not in the equilibrium point G of a minimum diversity game are isolated and as follows: Let $\mathfrak{K}$ be a set of $\kappa \geq 1$ strategies. Then $\mathfrak{m}_{\mathfrak{K}}$ is the mixed strategy combination where each player selects one of the strategies in $\mathfrak{K}$ with probability $1/\kappa$.

Proposition: $\mathfrak{m}_{\mathfrak{K}}$ is a hyperbolic rest point with only real Eigenvalues, at least one of them being positive.

Observation: all points in G are stable rest points.

Theorem: There is a subset $X \subseteq \Sigma$ of Lebesque measure 1 such that all trajectories starting in X converge to G. In particular, $\mathfrak{m}$ is unstable.

In brief: Evolution selects G

(Compare Hofbauer/ Swinkels) Consider a dynamic on $\mathbf{R}^S$ that is twice differentiable and forward-invariant on Σ. Suppose

- $\frac{\partial u_i}{\partial t} > 0$ for all players i on all points which are not rest points,
- all Nash equilibria are rest points,
- all rest points are also rest points of the replicator dynamics,
- if a rest point is not hyperbolic, then it is also not hyperbolic for the RD.

Then **Proposition:** Almost all trajectories converge to G. In particular, $\mathfrak{m}$ is unstable.

Various attempts to define the concept exist, search for the right concept guided by a list of criteria (which cannot always be satisfied by a single Nash equilibrium)

A strategically stable set of Nash equilibria should exist and e.g. be

- compatible with the iterated elimination of weakly dominated strategies
- compatible with the never-weakly-best-reply criterion
- compatible with the "small world axiom" (outsider / insider)
- invariant with respect to the duplication of strategies

A set of Nash equilibria G is

- **essential**, if it is minimal with respect to the property that there is a NE close by in every nearby game.

- **hyperstable,** if it is minimal with respect to the property that there is a NE close by in every game nearby an equivalent game.

- **KM-strategically stable,** if it is minimal with respect to the property that there is a NE close by in every nearby trembling-hand perturbed game.

For $0 \leq \delta \leq 1$ let

$$\begin{aligned} P_\delta &= \{\bar{\eta}\sigma | 0 \leq \bar{\eta} \leq \delta, \sigma \in \Sigma\} \\ &\partial P_\delta \text{ relative boundary} \\ &P_\delta^i \text{ relative interior} \end{aligned}$$

For $\eta = \bar{\eta}\tau \in P_1$ the (trembling-hand) *perturbed game* $\Gamma(\eta)$ has the same strategy space Σ and utility functions

$$u_i^\eta(\sigma) = u_i((1-\bar{\eta})\sigma + \eta) = u_i((1-\bar{\eta})\sigma + \bar{\eta}\tau)$$

(Notice that $\bar{\eta} = \sum_{s_i \in S_i} \eta_i(s_i)$ for all $1 \leq i \leq n$ and $\tau = \eta/\bar{\eta}$. Thus P_1 parametrizes the space of all trembling-hand perturbations with equal trembling prob. for all players.)

Strategic Stability 4	December 2007

For closed semialgebraic subsets S^i of

$$\left\{(\eta,\sigma)\in P_1^i\times\Sigma|\sigma \text{ Nash eq. of } \Gamma(\eta)\right\}$$

denote by S their closure in $P_1\times\Sigma$, and let S_δ^i, S_δ, ∂S_δ denote the inverse images in S of P_δ^i, P_δ, ∂P_δ, respectively, under the projection $\text{proj}: S\to P_1$.

(Mertens 1989) The stable sets are the Hausdorff limits of the sets S_0, where S^i is such that, for all sufficiently small δ, S_δ^i is connected and the projection from $(S_\delta,\delta S_\delta)$ to $(P_\delta,\partial P_\delta)$ is homologically non-trivial.

Proposition: $\mathfrak{m}$ is strategically stable under any of these definitions.

THEOREM (DeMichelis, Ritzberger, extensions by DeMichelis, Sorin**)** An asymptotically stable component of Nash equilibria for a dynamic similar to above which has non-zero Euler characteristic contains a Mertens-stable set.

THEOREM (Govindam, Wilson) A Nash equilibrium component is hyperstable iff its index is non-zero.

Since component G for minimum diversity games asymptotically stable, Euler-characteristic =index.

Therefore: If $\dim(G)$ even, G is hyperstable and Mertens-stable. If $\dim(G)$ odd, G is not hyperstable. But is it Mertens-stable or essential?

Proposition: For $N = 3$ and $K = 2$ and for $N = 2$ and K odd G does not contain a Mertens strategically stable set and is not essential.

Here strategic and evolutionary stability depart!

Proof	December 2007

Method: Consider the following perturbed games where the trembles *correlate* with the strategy choices: To each pure strategy combination s select a nearby mixed strategy combination s^{ε} and consider the game

$$u_i^{\varepsilon}(s) = u_i\left(s^{\varepsilon} \backslash s_i\right)$$

If arbitrarily small perturbations exist with no Nash equilibria near to G, then G is not essential, but also, using results by *Hillas, Jansen and Vermeulen*, not Hillas- or Mertens stable.

Construction	December 2007

Set

$$s_i^{k,\varepsilon} = \Sigma_{l=1}^{K} \frac{\varepsilon^{(l-k) \bmod K}}{\Sigma_{l=0}^{K-1} \varepsilon^{k}} s_i^{l}$$

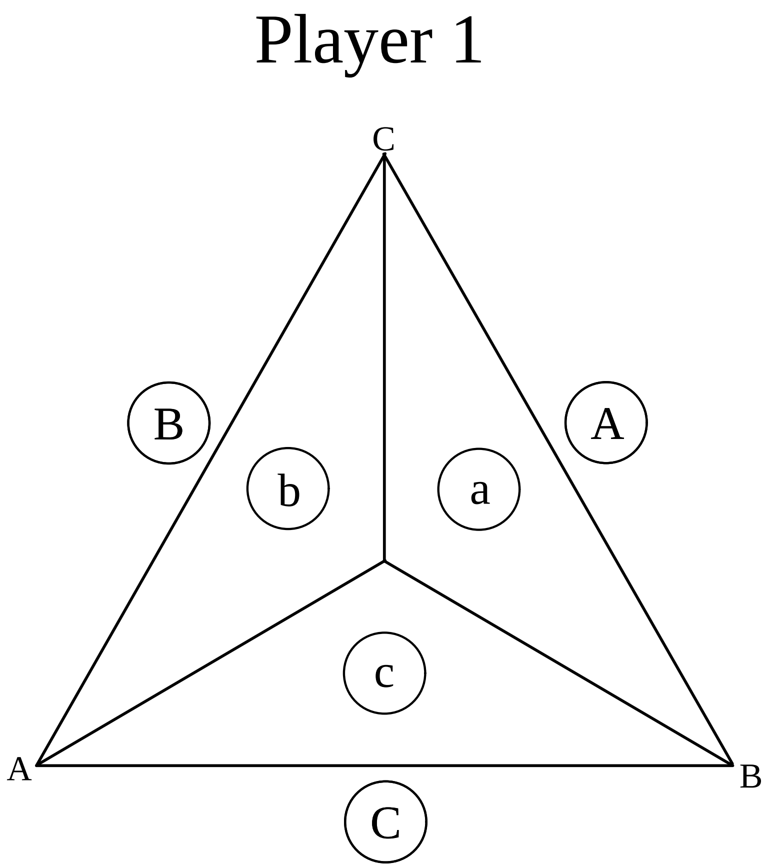
Then:

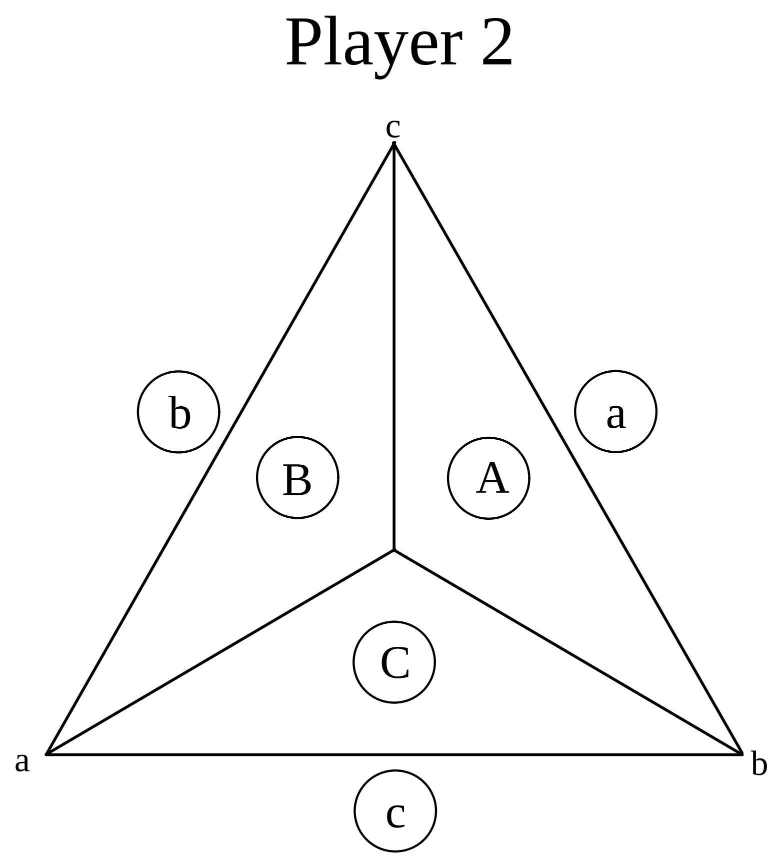
$$u_1^{\varepsilon}(l,k) = u_2^{\varepsilon}(k,l) = -\frac{\varepsilon^{(l-k) \bmod K}}{\Sigma_{l=0}^{K-1} \varepsilon^{k}}$$

Lemma: If, in an equilibrium $\neq \mathfrak{m}$ where player 2 uses strategy k with minimal probability, player 1 is not going to use $k - 1 \bmod^{+} K$.

Illustrations 1	December 2007

$N = 2$: Generalized paper-scissor stone construction.

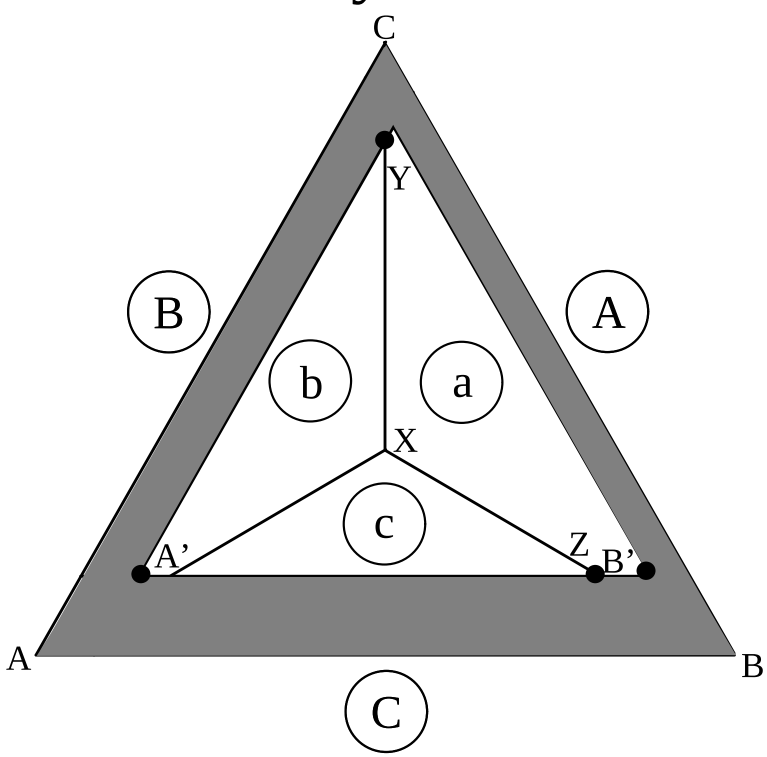




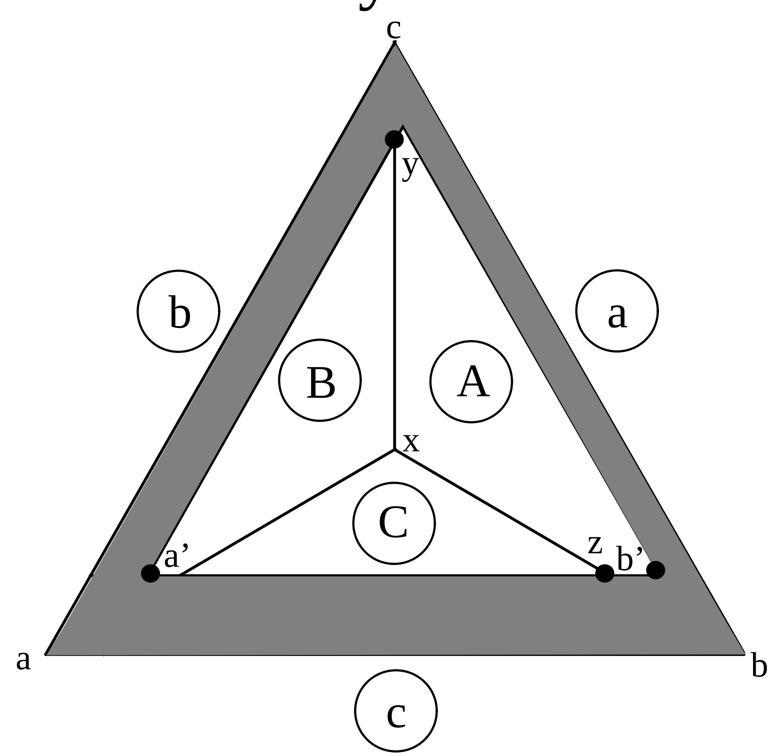
Illustrations 2	December 2007

NE: (A’,b’), (B’,a’), (Y,z), (Z,y), (Z,z)

Player 1



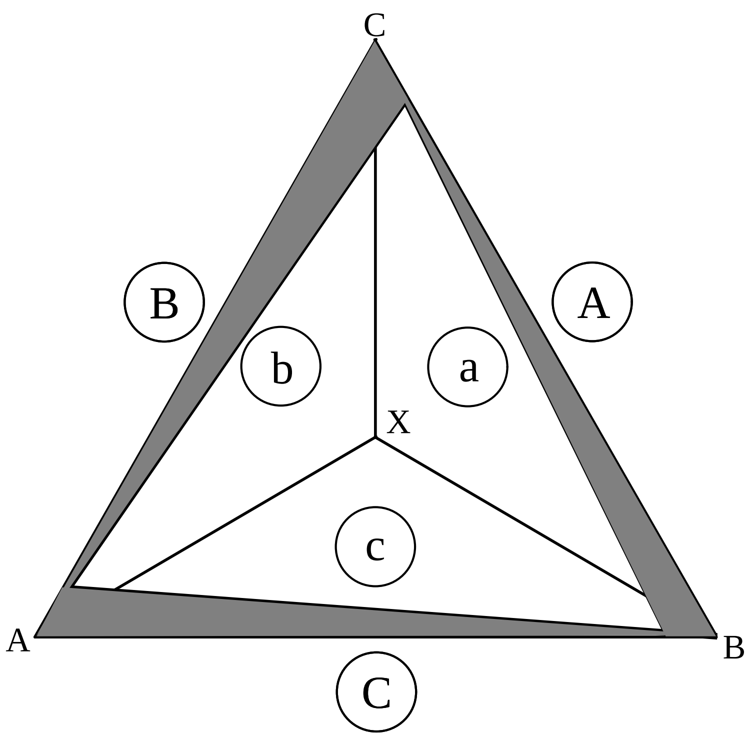
Player 2



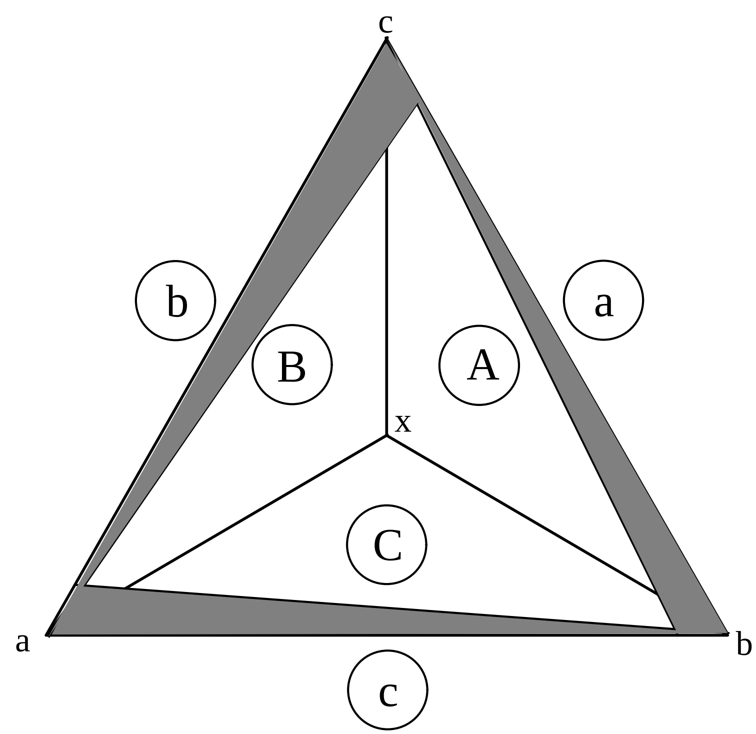
Illustrations 3	December 2007

Only (X,x) NE

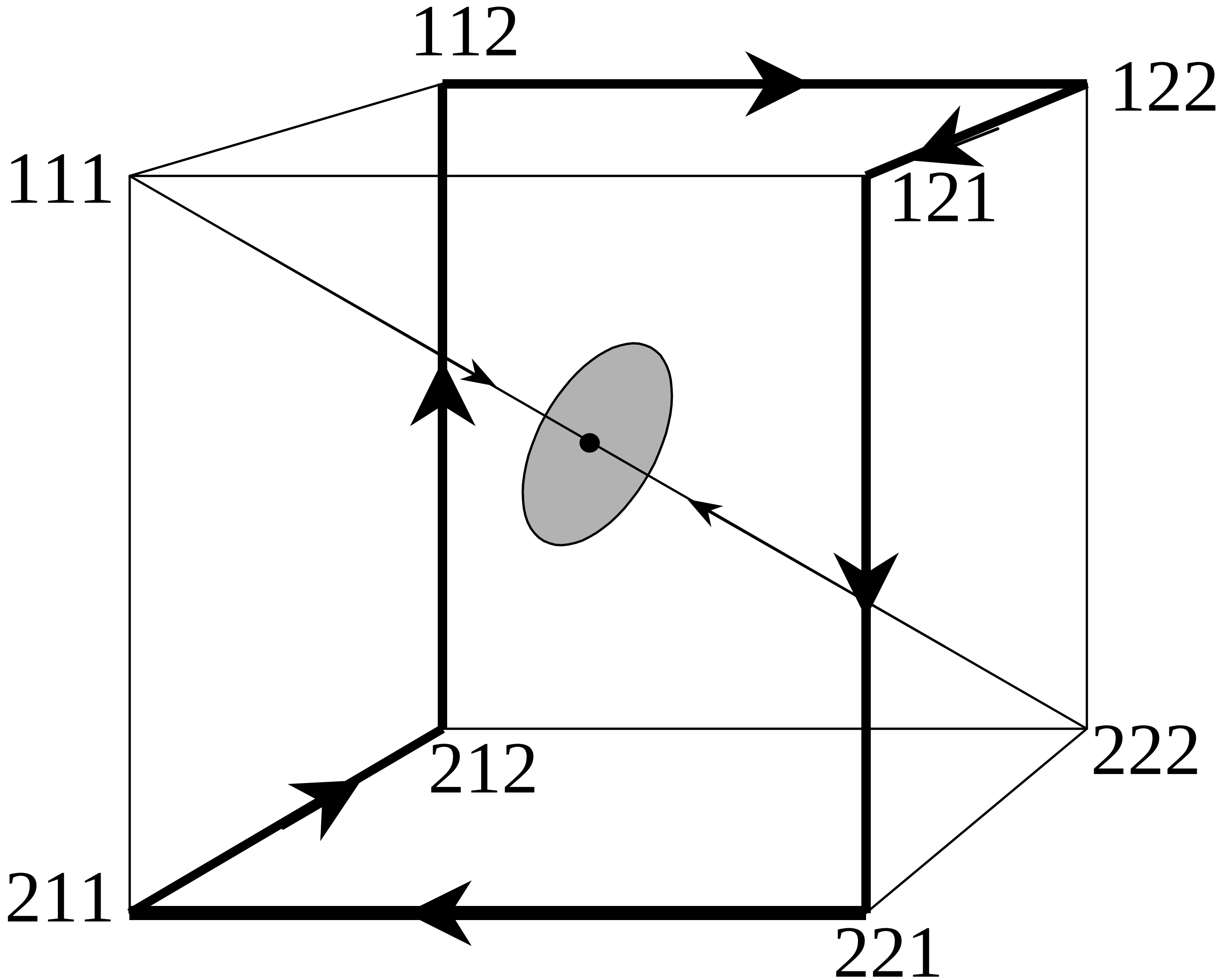
Player 1



Player 2



Illustrations 4	December 2007



Consider dynamic as above which continuously depends on the payoffs. Then in the nearby perturbed games:

$\mathfrak{m}$ dynamically unstable, generically all trajectories converge to G and hence never to Nash equilibria.

$N = 3,\ K = 2:$ Jordan, Hofbauer Swinkels

$N = 2,\ K = 3:$ Shapley

continuous fictituous play: Gaunersdorfer / Hofbauer.

Warning: $\mathfrak{m}$ asymptotically stable for a suitable myopic learning dynamic in the perturbed games (Hofbauer).

Open Questions	December 2007

Do the above examples have cycles as minimal attractors?

What about other minimal diversity games, in particular $N = K = 3$?

game with unique mixed eq.: paper-scissor-stone against both opponents, sum payoffs

What about strict equilibrium sets?