Does Evolution Eliminate Irrational Behaviours?

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Santiago, Chile, January 12-13, 2017

Evolution and irrational behaviours

In economics and game theory, agents often assumed “rational”

Possible justification: ‘as-if rationality”

Does evolution really eliminate irrational behaviours?

Do evolutionary game dynamics eliminate dominated strategies?

This talk: pure strategies strictly dominated by other pure strategies

Framework: Single population dynamics

Interactions within a single, large population

- finite set of pure strategies $I := \{1, ..., N\}$.
- $x_i(t)$: frequency of strategy i at time t
- $\mathbf{x}(t) := (x_i(t))_{i \in I}$: state of the population
- evolves in $\Delta(I) = \left\{\mathbf{x} \in \mathbb{R}^N_+, \sum_{i \in I} x_i = 1\right\}$
- Payoff for i-strategists : $u_i(\mathbf{x}(t))$
- Dynamics: $\dot{\mathbf{x}} = f(\mathbf{x}, \textit{payoffs})$

Replicator dynamics: $\dot{x}_i = x_i\,(u_i - \overline{u})$ with $\overline{u} = \sum_i x_i u_i$.

Domination and extinction

Strategy i dominated by j if: $\forall \mathbf{x} \in \Delta(I), u_i(\mathbf{x}) < u_j(\mathbf{x})$

Pure strategy i goes extinct if $x_i(t) \to 0$ as $t \to +\infty$

Issue: do pure strategies dominated by other pure strategies go extinct?

Literature

In current literature, two big classes: *imitative* and *innovative* dynamics

Imitative dynamics eliminate pure strategies dominated by other pure strategies; innovative dynamics need not.

Claim: misleading picture. Studied imitative dynamics are special.

Dynamics based on imitation need not eliminate dominated strategies

Derivation of dynamics in economics

Idea: from time to time, agents revise their strategies.

i-strategists switch to strategy j at rate $\rho_{ij}(\mathbf{x}, \mathit{payoffs})$

Leads to: $\dot{x}_i = \text{inflow - outflow} = \sum_j x_j \rho_{ji}(x) - x_i \sum_j \rho_{ij}(x)$

Specification of ρ_{ij} called "revision protocol", defines dynamics.

E.g., Replicator dynamics arise from:

$\rho_{ij}(x) = x_j(K + u_j(x))$ (imitation of success)

$\rho_{ij}(x) = x_j[u_j(x) - u_i(x)]_+$ (proportional pairwise imitation rule)

Imitative dynamics

Definition (Sandholm, 2010)

Dynamics *imitative* if $\rho_{ij} = x_j r_{ij}$ with $u_i(x) < u_j(x) \Leftrightarrow r_{ij}(x) > r_{ji}(x)$

◇ models two-step process:

Step 1: revising i-strategist meets j-strategist with probability x_j

Step 2: imitate him with "probability" r_{ij} favouring successful strategies

◇ coincide with monotone dynamics: $\dot{x}_i = x_i g_i(x)$ with $g_i < g_j \Leftrightarrow u_i < u_j$

Theorem (Akin 1980, Nachbar, 1990)

Assume strategy i strictly dominated by strategy j. Then under any imitative dynamics, $x_i(t) \to 0$ as $t \to +\infty$.

Key: $u_i(x) < u_j(x) \Rightarrow \dot{x}_i/x_i < \dot{x}_j/x_j$

Innovative dynamics

In innovative dynamics, strategies initially not played may appear.

Smith dynamics: revising i-strategists pick a strategy j at random, and adopt it with probability proportional to $[u_j - u_i]_+$. So $\rho_{ij} = \frac{1}{N}[u_j - u_i]_+$

Theorem (Hofbauer-Sandholm, 2011)

Under Smith and all innovative dynamics satisfying 4 natural conditions (Positive correlation, Continuity, Innovation, Nash stationarity), pure strategies dominated by other pure strategies may survive!

Innovative dynamics favour rare strategies

Innovation: if i unused best-reply, then $\dot{x}_i > 0$. Hence $\dot{x}_i / x_i = +\infty$.

So by Continuity: if i almost best-reply and $x_i << 1$, then $\dot{x}_i / x_i$ huge.

Thus, if $x_i << 1$, we may have: $u_i < u_j$ but $\dfrac{\dot{x}_i}{x_i} > \dfrac{\dot{x}_j}{x_j}$

$\hookrightarrow$ favours rare strategies.

Imitative dynamics neutral: $u_i < u_j \Rightarrow \frac{\dot{x}_i}{x_i} < \frac{\dot{x}_j}{x_j}$ whatever $x_i, x_j > 0$.

But imitation dynamics might favour rare/frequent strategies; then same survival results should hold.

Imitation protocols favouring rare/frequent strategies

Step 1:

1a) a revising agent meets m randomly drawn agents, with $P(m \geq 3) > 0$. E.g, meets 3 agents, playing (j, k, k)

1b) make a list of strategies played by these agents; here: {j, k}

1c) pick one at random: here j with probability 1/2

Step 2: decides whether to imitate him according to standard r_{ij}.

Leads to: $\rho_{ij} = p_j(x) r_{ij}$ where $p_j(x)$ proba of picking j in step 1.

Step 1 favours rare strategies: $x_i < x_j \Rightarrow p_i/x_i > p_j/x_j$.

If instead, when meeting (j, k, k), agent 1 focuses on the "majoritarian choice" k, favours frequent strategies: $x_i < x_j \Rightarrow p_i/x_i < p_j/x_j$.

Another imitation protocol favouring rare strategies

Step 1:

1a) a revising i-strategist meets a j-strategist with proba x_j.

1b) if $j \neq i$ considers switching to j; otherwise meets someone else.

1c) if meets i again and again, gives up after m meetings.

Step 2: standard r_{ij}.

Probability p_{ij} that i-strategists consider strategy j depends on i and j.

For $i \neq j$, $p_{ij} = x_j + x_i x_j + \ldots + x_i^{m-1} x_j = f(x_i) x_j$, with f increasing.

Rare strategies not imitated more, but frequent strategies imitate more.

Towards a general definition: being more imitated

Let $p_{ij}(x)$ denote probability that a revising i-strategist considers switching to j.

Rare strategies are more imitated per capita if there exists functions ε_{ij} such that:

(0) $\forall x, \forall i \neq j, p_{ij}(x) = x_j(1 + \varepsilon_{ij}(x))$

(1) $\forall x, \forall (i, j, k), x_i \leq x_j \Rightarrow \varepsilon_{ki}(x) \geq \varepsilon_{kj}(x)$

Rare strategies are strictly more imitated if moreover:

(2) $\forall x, \forall (i, j, k), x_i < x_j \Rightarrow \varepsilon_{ki}(x) > \varepsilon_{kj}(x)$

Frequent strategies are more imitated, or strictly more imitated, if inequalities reversed.

Imitating less

Rare strategies imitate less if there exists functions ε_{ij} such that:

(0) $\forall x, \forall i \neq j, p_{ij}(x) = x_j(1 + \varepsilon_{ij}(x))$

(1) $\forall x, \forall (i, j, k), x_i \leq x_j \Rightarrow \varepsilon_{ik}(x) \leq \varepsilon_{jk}(x)$

Rare strategies imitate strictly less if moreover:

(2) $\forall x, \forall (i, j, k), x_i < x_j \Rightarrow \varepsilon_{ik}(x) < \varepsilon_{jk}(x)$

Frequent strategies imitate less, or strictly less if inequalities reversed.

Favouring rare or frequent strategies

Consider a two-step process as explained before.

Definition

The first step favours rare strategies if rare strategies are more imitated, imitate less, and at least one of them strictly.

It favours frequent strategies if frequent strategies are more imitated, imitate less, and at least one of them strictly.

Results

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare strategies.

- in step 2, one of the following possibilities:
  - $r_{ij} = f(u_j)$, with f positive increasing.
  - $r_{ij} = [u_j - u_i]_+$, or same sign .
  - $r_{ij} = [u_j - \overline{u}]_+$, or same sign.

Theorem

There are games such that for many initial conditions, a pure strategy dominated by another pure strategy survives in proportion roughly $1/2$

With advantage to frequent strategies, survival in proportion almost 1!

A simple case: distorted imitation of success

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare strategies.
- in step 2, $r_{ij} = K + u_j$, or $r_{ij} = f(u_j)$, with f positive increasing.

Theorem

There are two-strategy games with a strictly dominated strategy that survives in proportion almost $1/2$ for most initial conditions.

With advantage to frequent strategies, survival in proportion almost 1!

Proof (advantage to rare strategies)

For simplicity, assume $p_{ij}(x) = p_j(x)$. The dynamics are:

$$\dot{x}_i = \sum_j x_j p_i f(u_i) - x_i \sum_j p_j f(u_j)$$

Let $p_i = x_i(1+\varepsilon_i)$. For two strategies i and j with same payoff u:

$$\frac{\dot{x}_i}{x_i} - \frac{\dot{x}_j}{x_j} = (\varepsilon_i - \varepsilon_j) f(u)$$

With only these strategies: since $x_i < x_j \Rightarrow \varepsilon_i > \varepsilon_j$, $x_i \to 1/2$

Now perturb: assume payoff of x_i is $u - \alpha$ so i dominated

We get: $\forall \eta > 0$, $\exists \overline{\alpha} > 0$, $\forall \alpha < \overline{\alpha}$, $x_i > \eta \Rightarrow \liminf x_i > \frac{1}{2} - \eta$.

With advantage to frequent strategies: $x_i > 1/2 + \eta \Rightarrow \liminf x_i > 1 - \eta$.

More general dynamics

Dynamics satisfy Positive Correlation if for all x in $\Delta(I)$:

$$(PC) \qquad \dot{x} \neq 0 \Rightarrow \dot{x} \cdot u(x) > 0$$

where $u(x) = (u_i(x))_{1 \leq i \leq n}$

Distorted imitation of success does not satisfy Positive Correlation, has non Nash interior rest-points, and not all Nash equilibria are rest-points!

Can we have similar results for dynamics with more usual properties?

$\hookrightarrow$ Yes, but requires more elaborate examples.

More general dynamics

Dynamics satisfy Positive Correlation if for all x in $\Delta(I)$:

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Can we have similar results for dynamics with more usual properties?

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Distorted pairwise imitation

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare (resp. frequent) strategies.
- in step 2, $r_{ij} = [u_j - u_i]_+$, or same sign.

Theorem

There are 4 strategy games such that for large sets of initial conditions, a pure strategy dominated by another pure strategy survives in proportion roughly 1/6 (resp. 1/3).

Same results if $r_{ij} = [u_j - \overline{u}]$, or same sign.

Proportion may be increased to $1/2 - \eta$ (resp. $1 - \eta$).

Sketch of proof

We mimick Hofbauer and Sandholm (2011).

They consider dynamics satisfying Positive Correlation (PC):

$$\dot{x} \neq 0 \Rightarrow \dot{x} \cdot u(x) > 0.$$

Geometrically: acute angle between $\dot{x}$ and payoff vector $u(x)$.

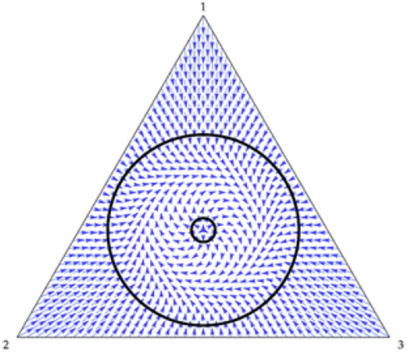
Also: acute angle between $\dot{x}$ and projection of payoff vector on simplex

Lemma

Under theorem's assumptions, our imitation dynamics satisfy (PC)

Hypnodisk game (Hofbauer and Sandholm)

3-strategy game with projected payoff vector field:



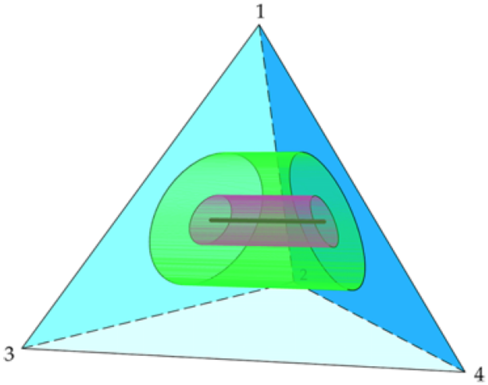
Projected payoff vector field for the hypnodisk game

Due to (PC), all interior solutions enter annulus, except Nash equilibrium.

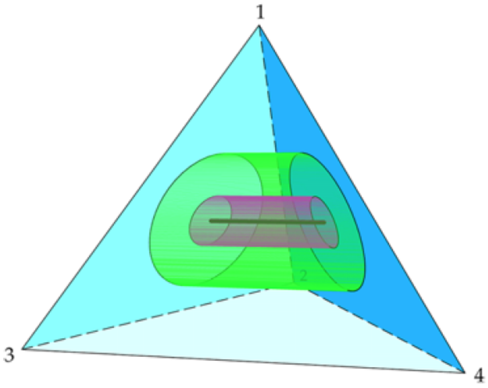
Hypnodisk game with a twin (Hofbauer and Sandholm)

Add as strategy 4 a twin of strategy 3.

- Segment of equilibrium: $x_1 = x_2 = x_3 + x_4 = 1/3$.
- Attracting annulus becomes attracting "intercylinder zone"



Effect of advantage to rare strategies



Advantage to rare strategies: $x_3/x_4 \to 1$.

Attractor A in intersection of intercylinder zone and plane $x_3 = x_4$.

Basin of attraction $B(A) = int(S_4) \backslash Nash\ equilibria$

Continuation of attractors (Hofbauer and Sandholm)

Substract ε to strategy 4 $\rightarrow$ makes it dominated

By standard results on continuation of attractors, for ε small enough, most solutions still converge to an attractor A_ε in the neighborhood of A.

$\hookrightarrow$ under most solutions, strategy 4 survives, and $\liminf x_4 \geq 1/6 - r$, with r radius of outer cylinder

Rk: $1/6$ may be changed to anything $< 1/2$ by modifying base game.

Comments

◇ Dominated strategies may not only survive under innovative dynamics, but also under dynamics arising from imitation

◇ Not so surprising. Elimination of dominated strategies "requires":

$$\forall x, u_i = u_j \Rightarrow \dot{x}_i / x_i = \dot{x}_j / x_j .$$

Fragile property, destroyed by appropriate small perturbation.

◇ If dominated strategies survive, no convergence to equilibrium.

◇ A large scale natural experiment: Rock-Paper-Scissors-Well

Literature (non exhaustive)

- Akin (80): replicator dynamics; Nachbar (90): monotone dynamics
- Samuelson & Zhang (92): aggregate monotone dynamics ; Hofbauer & Weibull (96): convex monotone dynamics
- Dekel & Scotchmer (92), Cabrales & Sobel (92), Björnerstedt et al (96): discrete-time dynamics

More recently:

- Cressman & Hofbauer (05); Cressman et al (06); Heifetz et al (07a, 07b), Jouini et al. (13): continuum of pure strategies
- Fudenberg & Harris (92), Cabrales (00), Imhof (05), Hofbauer & Imhof (09); Mertikopoulos & Moustakas (10), Mertikopoulos & Viossat (16): stochastic dynamics
- Berger & Hofbauer (06), Hofbauer & Sandholm (11): innovative dynamics

A survey: Viossat (15), in Economic Theory Bulletin