Structure of the set of equilibria

Yannick Viossat (Université Paris-Dauphine)

School on equilibria: existence, selection, dynamics

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Aim

Understand the structure of the set of Nash equilibria, of correlated equilibria, and also of equilibrium payoffs, in FINITE games.

- Natural mathematical question
- May allow to build games with interesting properties or to show that no such game exist

The aim is that YOU understand, not the old guys.

Outline

- Nash equilibria
  - Zero-sum games
  - Bimatrix games
  - N-player games
- Correlated equilibria and links with Nash equilibria
- Evolution of equilibrium set as payoffs vary

Rk: highly personal choice of topics.

Notation for sets of equilibria

NE: set of Nash equilibria

NEP: set of Nash equilibrium payoffs

NEP(G): of game G

CE: set of correlated equilibria

CEP, CEP(G): set of correlated equilibrium payoffs.

NE / CE also used as abbreviations of Nash / correlated equilibrium.

Notation for bimatrix games

- Payoff matrices: A, B
- Pure strategies: $i \in I$, $j \in J$
- Mixed strategies: $x \in \Delta(I)$, $y \in \Delta(J)$
- Supports: $Supp(x) := \{i \in I, x_i > 0\}$
- Payoff for player 1: $x \cdot Ay = \sum_{i,j} x_i y_j a_{ij}$.
- Best-replies: $BR_1(y) \subset \Delta(I)$, $BR_2(x) \subset \Delta(J)$

Preliminaries - bimatrix games

Let $g_1(i,j) = a_{ij}$. Then $g_1(i,y) = \sum_{j \in J} y_j a_{ij} = (Ay)_i$ with $A = (a_{ij})$

Thus, $g_1(x,y) = \sum_i x_i g_1(i,y) = \sum_i x_i (Ay)_i = x \cdot Ay$

So $\max_x g_1(x,y) = \max_i g_1(i,y)$. Moreover:

x best-reply to $y \Leftrightarrow$ for all $i \in I$, $x_i = 0$ or i best-reply to y. Thus:

$$(x,y) \in NE \Leftrightarrow \begin{cases} \sum_i x_i = 1 \text{ and } \forall i \in I, x_i \geq 0 \\ \forall i \in I, [x_i = 0 \text{ or } \forall i' \in I, (Ay)_i \geq (Ay)_{i'}] \\ \text{similar conditions for player 2} \end{cases}$$

Set NE given by unions and intersections of sets defined by LINEAR conditions on x, y.

Why 2 is different from 3?

3-player game with pure strategy sets I, J, K.

Pure strategies: i, j, k ; mixed strategies: x, y, z.

Then $g_1(i, y, z) = \sum_{j,k} y_j z_k a_{ijk}$: quadratic expression.

So best-reply condition: $g_1(i, y, z) \geq g_1(i', y, z)$ for all i', now given by quadratic conditions. For n-player game, polynomial of degree $n - 1$.

Set NE still semi-algebraic (and moreover nonempty, bounded, compact), but we expect way more complications than for bimatrix games.

Zero-sum games: reminder

Player 1 maximizes $g(x,y) = x \cdot Ay$, player 2 minimizes.

Proposition 1 (von Neumann)

$\min_{y\in\Delta(J)} \max_{x\in\Delta(I)} x \cdot Ay = \max_x \min_y x \cdot Ay =$ *value* v

Definition 2

A strategy x of player 1 is optimal if for all y, $g(x,y) \geq v$.

A strategy y of player 2 is optimal if for all x, $g(x,y) \leq v$.

Rk: If both x and y are optimal, then $g(x,y) = v$.

Zero-sum games: properties

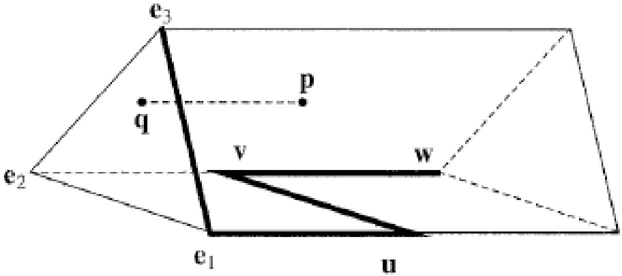
Let O_i denote the set of optimal strategies of player i. E.g.:

$$O_1 = \{x \in \Delta(I) \,|\, \forall y \in \Delta(J), x \cdot Ay \geq v\}$$

Proposition 3

- *O_1 and O_2 are convex polytopes*
- $NE = O_1 \times O_2 = \{(x, y), x \in O_1, y \in O_2\}$
- *Nash equilibria are exchangeable: if (x, y), $(x', y') \in NE$, then $(x, y') \in NE$*
- *In any Nash equilibrium, the payoffs are $(v, -v)$.*
- *(normal-form) generically, unique Nash equilibrium*

Bimatrix games: an example



Picture from Raghavan (02)

Consider bimatrix game $\begin{pmatrix} 2,1 & 1,0 & 1,1 \\ 2,0 & 1,1 & 0,0 \end{pmatrix}$

Picture represents $\Delta(I) \times \Delta(J)$. In bold: equilibria.

We see a "special structure". What can be said in general?

Bimatrix games I - Exchangeability

Proposition 4 (Equilibria with the same support are exchangeable)

If (x, y) *and* (x', y') *are equilibria with same support, then* $(x, y') \in NE$.

Proof :

i) Since x' best-reply to y', any pure strategy in $Supp(x')$ best-reply to y'.

ii) So any mixed strategy with support in $Supp(x')$ best-reply to y'

iii) So x best-reply to y', and similarly, y' best-reply to x.

Bimatrix games - polyhedral structure

Proposition 5

NE is a finite union of products of convex polytopes

Proof. Let $NE(I', J')$ denote the set of equilibria with support $I' \times J'$.

Let $\hat{NE}(I', J') = NE_1(I', J') \times NE_2(I', J')$ where

$NE_1(I', J') = \{x \in \Delta(I') : \Delta(J') \subset BR_2(x)\}$, $NE_2(I', J')$ symmetric

We have:

i) $NE(I', J') \subset \hat{NE}(I', J') \subset NE$, hence $NE = \cup_{I' \subset I, J' \subset J} \hat{NE}(I', J')$

ii) $NE_i(I', J')$ convex polytope.

Note also: equilibria in $\hat{NE}(I', J')$ are exchangeable.

Bimatrix games - Extreme Nash equilibria

Definition 6

A set $S \subset NE$ is a Nash subset if equilibria in S are exchangeable. A Nash subset is maximal if not properly contained in another Nash subset.

Proposition 7 (e.g., Jansen, 1981a)

a) *A maximal Nash subset is a product of convex polytopes.*
b) *NE is the union of finitely many maximal Nash subsets.*

Proof of b): each $\hat{NE}(I', J')$ contained in a maximal Nash subset.

Definition 8

Extreme Nash equilibria are extreme points of maximal Nash subsets.

Such extreme points may be characterized, and allow to recover all NE.

Bimatrix games IV - Payoffs

A subset E of $\mathbb{R}^2$ is a nice rectangle if E of the form $[a, b] \times [c, d]$

Recall notation: NEP, NEP(G)

Proposition 9

a) *NEP is a finite union of nice rectangles*

b) *for any nonempty finite union of rectangles U, there is a bimatrix game G with $NEP(G) = U$.*

a) set of payoffs associated to each subset of NE which is a product of compact convex sets (hence to each set $\hat{NE}(I', J')$) is a nice rectangle.

b) constructive, idea later.

N-player games

(Normal-form) generic case: finite and odd number of equilibria.

But structure may be much richer... Any bound?

A closed subset E of $\mathbb{R}^n$ is semi-algebraic if $\exists$ integers A, B, and polynomials in n variables P_{ab} so that:

$$E = \cup_{a=1}^{A} \cap_{b=1}^{B} \{x \in \mathbb{R}^n, P_{ab}(x) \leq 0\}$$

Proposition 10

For any finite game G, $NE(G)$ and $NEP(G)$ are nonempty compact semi-algebraic sets.

Corollary 11

The sets of Nash equilibria and Nash equilibrium payoffs have a finite number of connected components

Richness of the set of possible equilibria

Datta (03): Any real algebraic variety is *isomorphic* to the set of *completely mixed* Nash equilibria of a 3-player game (and of a N-player game with 2 actions per player)

Balkenborg & Vermeulen (14): any nonempty connected compact semi-algebraic set is *homeomorphic* to a connected *component* of the set of Nash equilibria of a "binary" game

Levy (16), Vigeral & V. (16) : any nonempty, compact, semi-algebraic set in $[0, 1]^n$ is the *projection* of the set of equilibria of a finite game with 2 actions per player on its first n coordinates.

Characterization of the set of equilibrium payoffs

Proposition 12 (Vigeral, 2015, unpublished)

For any $n \geq 3$, any nonempty compact semi-algebraic subset of $\mathbb{R}^n$ is the set of Nash equilibrium payoffs of a n-player game.

Proof: constructive, lots of "gadget games".

Remark: for $n = 2$, not true. NEP then finite union of rectangles.

Correlated equilibria and links with Nash equilibria

CHANGE OF NOTATION!

- N-player finite game ;
- pure strategies: s_i, $t_i \in S_i$; pure strategy profile $s \in S = \prod_i S_i$.
- $\mu \in \Delta(S)$ is a correlated equilibrium (distribution) if for all i, s_i, t_i,

$$\sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) \left[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i}) \right] \geq 0$$

  (incentive contraint for not defecting from s_i to t_i)

- recommandation interpretation

(Product distributions induced by) Nash equilibria are correlated equilibria

The sets of correlated equilibria and of correlated equilibrium payoffs are *convex polytopes*. Notation: CE(G), CEP(G).

Correlated equilibria in zero-sum games

Proposition 13

Let μ be a correlated equilibrium, $s_1 \in S_1$, $g(\cdot)$ payoff of player 1:

a) $g(\mu) = \sum_s \mu(s) g(s) = v$

b) *If $\mu(s_1 \times S_2) > 0$, then $\mu(\cdot | s_1)$ is an optimal strategy of player 2.*

c) *A zero-sum game with a unique NE has a unique CE.*

d) *Normal-form generically, zero-sum games have a unique CE.*

Bimatrix games

Recall: *NE* union of maximal Nash subsets, which are convex polytopes

Extreme points of these polytopes called "extreme Nash equilibria"

In what follows, we see $\Delta(S_1) \times \Delta(S_2)$ as a subset of $\Delta(S)$.

Proposition 14 (Cripps; Evangelista & Raghavan; Gomez-Canovas et al.)

Extreme Nash equilibria are extreme points of the CE polytope.

Rk: not true in payoff space ; all Nash equilibrium payoffs may be in the interior of the polytope of correlated equilibrium payoffs (see later).

N-player games

A game is *tight* if in any CE, all incentive constraints concerning two strategies with positive probability in a CE are satisfied with equality.

Proposition 15 (Nau et al., 03)

a) *If the correlated equilibrium polytope has the dimension of the simplex, then all Nash equilibria belong to its boundary.*

b) *If there is a Nash equilibrium in the relative interior of the CE polytope, then the game is tight.*

Proposition 16 (V., 04)

c) *All tight games have NE in the relative interior of the CE polytope*

d) *2-player tight games include and "generalize" zero-sum games.*

Correlated equilibrium payoffs

Recall sets CE and CEP convex polytopes. Below, polytope means nonempty convex polytope.

Proposition 17

For any polytope $P \subset \mathbb{R}^N$, there exists a N-player game G such that:

$$P = CEP(G) = Conv(NEP(G))$$

Idea: consider game with dominant strategy

$$\begin{pmatrix} x_1, y_1 & 0,0 & 0,0 & 0, y_1 \\ 0,0 & x_2, y_2 & 0,0 & 0, y_2 \\ 0,0 & 0,0 & x_3, y_3 & 0, y_3 \\ x_1, 0 & x_2, 0 & x_3, 0 & x_4, y_4 \end{pmatrix}.$$

Then $CEP(G) = Conv\{(x_i, y_i), 1 \leq i \leq 4\} = Conv(NEP(G))$

Flash back on Nash equilibrium payoffs

How to get any finite union of nice rectangles as set of NEP?

Let $U = \cup_{1 \leq k \leq m} [a_k, b_k] \times [c_k, d_k]$

Let $G_k = (A_k, B_k) = \begin{pmatrix} a_k, c_k & b_k, c_k \\ a_k, d_k & b_k, d_k \end{pmatrix}$.

We have: $NEP(G_k) = [a_k, b_k] \times [c_k, d_k]$

Replacing payoffs x_k, y_k and 0 in previous slide by blocks A_k, B_k, and blocks of 0, gives game G with $NEP(G) = U$ and $CEP(G) = Conv(U)$.

Nash and Correlated equilibrium payoffs

Proposition 18

Let G be a N-player game. For any polytope $P \subset \mathbb{R}^n$ containing $CEP(G)$, $\exists$ game G' such that $NEP(G') = NEP(G)$ and $CEP(G') = P$.

Idea: using similar domination trick and auxiliary 3×3 game, possible to add extreme correlated equilibrium payoffs one by one.

How to add (x, y) to CE payoffs of game G?

Assume for simplicity $x > 1$, $y > 1$. Let:

$$\Gamma = \left(\begin{array}{ccc|c} 0,0 & x+1, y-1 & x-1, y+1 & \\ x-1, y+1 & 0,0 & x+1, y-1 & [0,y] \\ x+1, y-1 & x-1, y+1 & 0,0 & \\ \hline & [x,0] & & G \end{array} \right)$$

where $[x, 0]$, $[0, y]$ are appropriate blocks of such payoffs.

Proposition 19

$NEP(\Gamma) = NEP(G)$ *and* $CEP(\Gamma) = Conv\{CEP(G), (x, y)\}$

A slightly stronger result for bimatrix games

Recall: for any bimatrix game G, $NEP(G)$ nonempty finite union of "nice" rectangles and $CEP(G)$ polytope containing $NEP(G)$. Conversely:

Proposition 20 (joint characterization of equilibrium payoffs)

For any nonempty finite union of "nice" rectangles U and polytope $P \supset U$, $\exists$ bimatrix game G such that $U = NEP(G)$ and $P = CEP(G)$.

Idea: note that $Conv(U) \subset P$. Using domination trick, first build game with $NEP(G) = U$ and $CEP(G) = Conv(U)$, then use N-player result.

Generic case

Games used for previous results highly non-generic. Normal-form generically, finite number of Nash equilibria.

Proposition 21

For any nonempty finite set $U \subset \mathbb{R}^n$ and polytope P containing U, $\exists$ open set of games for which NEP ε-close to U and CEP ε-close to P.

Idea: modification of previously used games, with small bonus to initially dominated equilibria.

Evolution of the set of Nash equilibria as payoffs vary

Normal-form generically, nothing much happens: finite number of equilibria, all "stable" (existing closeby for small payoff perturbations).

But what may happen in full generality?

A first basic fact: upper-semi continuity of the NE correspondence.

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of games of a fixed size.

Assume $G_n \to G$ as $n \to +\infty$.

Let $\sigma_n \in NE(G_n)$. Then any limit point of σ_n is in $NE(G)$.

"The set of equilibria may explode at the limit, but not implode."

An example

Consider the 2×2 game G_ε:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 0,0 & \varepsilon,-\varepsilon \\ -\varepsilon,\varepsilon & 0,0 \end{pmatrix} \end{array}$$

We have: $NEP(G_\varepsilon) = \begin{cases} \{T, L\} & \text{for } \varepsilon > 0 \\ \Delta(I) \times \Delta(J) & \text{for } \varepsilon = 0 \\ \{B, R\} & \text{for } \varepsilon < 0 \end{cases}$

Rk: $\varepsilon = 0$ "tilting point". Though the set of equilibria does not implode at the limit, it may vary a lot under small perturbations.

Close games with very different equilibrium payoffs

Let $G = (A, B)$ and $G' = (A', B')$ be bimatrix games of same size. Let:

$$\Gamma_\varepsilon = \begin{pmatrix} A, B & A' + \varepsilon, B - \varepsilon \\ A - \varepsilon, B' + \varepsilon & A', B' \end{pmatrix}$$

Then $NEP(\Gamma_\varepsilon) = \begin{cases} NEP(G) \text{ for } \varepsilon > 0 \\ NEP(G') \text{ for } \varepsilon < 0 \end{cases}$

Building on this idea, we get:

Proposition 22

Let U and U' be nonempty finite unions of nice rectangles. Let $P \supset U$ and $P' \supset U'$ be polytopes in $\mathbb{R}^2$. Let $EP = (NEP, CEP)$. We have: $\forall \varepsilon > 0$, $\exists$ ε-close games Γ and Γ', $EP(\Gamma) = (U, P)$, $EP(\Gamma') = (U', P')$.

Similar results for N-player games.

Is having a unique equilibrium robust?

Fixing the number of players N and the set of strategy profiles S, a game may be seen as a subset of $\mathbb{R}^{N|S|}$.

We may talk, e.g., of open sets of games.

Proposition 23 (Jansen, 1981b, V. 08)

a) The set of bimatrix games with a unique Nash equilibrium is open.

b) The set of 3-player games with a unique equilibrium is not open, nor is the set of symmetric bimatrix games with a unique symmetric equilibrium.

c) The set of N-player games with a unique correlated equil. is open.

Quasi-strict equilibria

Recall: σ is an equilibrium if for any player i, any $s_i \in S_i$:

$$\sigma_i(s_i) > 0 \Rightarrow s_i \text{ best-reply to } \sigma_{-i}$$

That is: $Supp(\sigma_i) \subset \{$ Pure best replies to $\sigma_{-i}\}$

Definition 24

An equilibrium σ is quasi-strict if for any i, any $s_i \in S_i$:

$$s_i \textit{ best-reply to } \sigma_{-i} \Rightarrow \sigma_i(s_i) > 0$$

That is: $Supp(\sigma_i) \supset \{$ Pure best replies to $\sigma_{-i}\}$

Examples

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \left(\begin{array}{cc} 1,1 & 0,0 \\ 1,0 & 1,1 \end{array}\right) \end{array}$$

(T, L) not quasi-strict ; (B,R) quasi-strict, actually strict.

$$\begin{array}{c} \\ T \\ B_1 \\ B_2 \end{array} \begin{array}{c} \begin{array}{ccc} L & R_1 & R_2 \end{array} \\ \left(\begin{array}{ccc} 1,1 & 0,0 & 0,0 \\ 1,0 & 2,0 & 0,2 \\ 1,0 & 0,2 & 2,0 \end{array}\right) \end{array}$$

$(\frac{1}{2}B_1 + \frac{1}{2}B_2, \frac{1}{2}R_1 + \frac{1}{2}R_2)$ not strict but quasi-strict.

Some lemmas

Lemma 25 (Jansen, 1981b), Norde)

A unique equilibrium of a bimatrix game is quasi-strict.

Lemma 26 (Jansen, 1981b)

If a bimatrix game has two equilibria with the same support, it has a non quasi-strict equilibrium

Let σ be unique NE of bimatrix game G. Consider sequence of games $G_k \to G$ and of equilibria σ_k of G_K such that $\sigma_k \to \sigma$.

Lemma 27 (Jansen, 1981b)

If σ is quasi-strict, then for k large enough, σ_k has the same support as σ, is quasi-strict, and the unique equilibrium of G_k.

Summing up

Normal-form generically, finite number of Nash equilibria, with nice properties. In full generality, more complicated.

In zero-sum games and bimatrix games, Nash equilibria have a special structure.

In n-player games, seemingly little structure beyond semi-algebraicity.

Correlated equilibria simpler geometrically.

Some relations: Nash equilibria on the boundary of correlated equilibrium polytope, though not true in payoff space.

Sets of equilibria may vary under small perturbations, but some properties robust, e.g., having a unique Nash equilibrium for bimatrix games.

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