Dynamics and Stability

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School on equilibria: existence, selection, dynamics

Santiago, Chile, January 2017

Evolutionary game theory

Classical game theory

- a few agents
- rational, fully understand the game, unbounded computation abilities

Evolutionary game theory

- population(s) of agents
- need not know that there is a game !
- strategies giving good results spread (natural selection, imitation,...)

Aim of lecture : do such models provide support for Nash equilibrium ?

Outline

1. Nash mass action interpretation of equilibria.
2. Evolutionary game dynamics
3. Folk Theorem of evolutionary game theory
4. Convergence results
5. Divergence results
6. Time-allowing : learning dynamics, stochastic stability, equilibrium selection, saving planet Earth,...

Justification of the Nash equilibrium concept

Nash equilibrium in two-player games : each player plays a best-reply to the other player's strategy.

In a one-shot game, if everybody is rational and CAN PREDICT the strategies of the other, then this results in a Nash equilibrium.

Convincing in case of pre-play agreement on what should be played, or social norm.

But in general, how to predict ?

Nash's mass action interpretation

- populations of individual for each player for the game

- interactions between randomly drawn agents

- accumulation of information on the behavior of others, leading to modification of own behavior

- if players are rational AND this process leads to stable mean strategies, the resulting mean strategy profile should be a Nash equilibrium.

Evolutionary dynamics are the right tool to formalize these ideas

Evolutionary game dynamics

- strategic interaction within a single population, with n pure strategies
- individuals play pure strategies

- $x_i(t)$: proportion of the population playing strategy i at time t.
- $x(t) = (x_1(t), ..., x_n(t))$: population state
- evolves in the simplex $S = \Delta(I) = \{x \in \mathbb{R}^n_+ : \sum_i x_i = 1\}$

- $u_i(x)$: payoff of i in state x ; $\quad u(x) = (u_1(x), ..., u_n(x))$

- evolutionary game dynamics : $\dot{x} = f(x, u(x))$.

Biology : Replicator dynamics (Taylor & Jonker, 78)

Assume fitness of i-strategist $= C + u_i(x)$, where C background fitness.

So density $X_i(t)$ of individuals playing i follows : $\dot{X}_i = X_i[C + u_i(x)]$

Then frequency $x_i = X_i / \sum_i X_i$ follows "Replicator dynamics" :

$$\dot{x}_i = x_i \left[u_i(x) - x \cdot u(x)\right] \qquad (REP)$$

where $x \cdot u(x) = \sum_i x_i u_i(x) =$ average payoff.

For pairwise interactions with random matching : $u_i(x) = (Ax)_i$. Thus :

$$\dot{x}_i = x_i \left[(Ax)_i - x \cdot Ax\right] \qquad (REP)$$

Two-population replicator dynamics

- evolution in finite (non necessarily symmetric) two-player game

- pairwise interaction, etc. ; no-self interaction.

- pure strategy sets $I = \{1, ..., n\}$, $J = \{1, ..., m\}$

- population profiles : $x(t) = (x_1(t), .., x_n(t))$, $y(t) = (y_1(t), .., y_m(t))$.

- payoff of strategy i in pop 1 and j in pop 2 : $(Ay)_i$, $(Bx)_j$

- (Taylor version of the two-population) replicator dynamics :

$$\dot{x}_i = x_i \left[(Ay)_i - x \cdot Ay \right], \qquad \dot{y}_j = y_j \left[(Bx)_j - y \cdot Bx \right]$$

Social dynamics : revision protocol

Large ("infinite") population of agents

In each small time interval, a fraction of the agents revise their strategies.

An i-strategist switches to j at rate $\rho_{i \to j}$.

Leads to : $\dot{x}_i = \sum_j \lambda x_j \rho_{j \to i} - \lambda x_i \sum_j \rho_{i \to j}$

Different specifications of $\rho_{i \to j}$ $\to$ different dynamics.

Replicator dynamics

An agent playing i meets a uniformly drawn agent.

Consider adopting its strategy j only if $u_j > u_i$.

Then does it with proba proportional to $u_j - u_i$

Thus $\rho_{i \to j} = x_j [u_j - u_i]_+$ where $[u_j - u_i]_+ = \max(0, u_j - u_i)$.

Leads to replicator dynamics : $\dot{x}_i = x_i(u_i - \sum_j x_j u_j)$

Imitative dynamics ; rest-points $\supset$ Nash equilibria.

Smith dynamics

As previous, but agents gather information on uniformly drawn strategies.

That is, strategy j considered with probability $1/N$, not x_j.

Thus $\rho_{i\to j} = \frac{1}{N}[u_j - u_i]_+$ or simply $\rho_{i\to j} = [u_j - u_i]_+$

Leads to Smith dynamics (from transportation science) :

$$\dot{x}_i = \sum_j x_j[u_i - u_j]_+ - x_i \sum_j [u_i - u_j]_+$$

Innovative (new strategies may appear).

Rest-point = (symmetric) Nash equilibria.

Best-reply dynamics (Gilboa & Matsui, 91)

Let $BR(x)$ denote the set of (mixed) best replies to x :

$$BR(x) := \{y \in S, y \cdot Ax = \max_{z \in S} z \cdot Ax\}$$

Assume that in each time interval, a fraction of the population revise its strategy and choose a best-reply to current mean behavior. Leads to :

$$\dot{x} \in BR(x) - x \qquad (BRD)$$

Solution : absolutely continuous function satisfying (BRD) for almost all t

Solutions exists, but several solutions with same initial condition.

Innovative ; rest points = (symmetric) Nash equilibria.

Classes of dynamics

Studying specific dynamics useful to get an idea on possible behaviors

and when rule programmed (distributed optimization), controlled

But when modeling social adaptation, no reason to expect a specific rule.

Better to prove results for large classes of dynamics

E.g., Myopic Adaptive Dynamics (MAD) : such that $\dot{x} \cdot Ax > 0$ whenever x not a rest point of REP.

Other classes : payoff functional dynamics, sign-preserving dynamics...

Evolutionary folk theorem

Two-population replicator dynamics. Consider a state (x, y).

- if (x, y) is an interior rest point, then this is a Nash equilibrium
- If (x, y) is Lyapunov stable, then this is a Nash equilibrium
- If (x, y) is the limit as $t \to +\infty$ of an interior trajectory $(x(t), y(t))$, then this is a Nash equilibrium

Similar results for single population dynamics. Extends to all dynamics we saw and many more. But do dynamics converge ?

Stability and convergence in some classes of games

- Dominance solvable games
- Evolutionary stable strategies and states
- Zero-sum games and dissipative games
- Potential games

Dominance solvable games

Recall : strategy i strictly dominated by mixed strategy $p \in \Delta(I)$ if for every population state x, $(Ax)_i < p \cdot Ax$.

Dominance solvable : after iterative elimination of strictly dominated strategies, a single strategy profile remains (the unique Nash equilibrium).

Proposition 1

In dominance solvable games, any interior solution of REP converges to the unique Nash equilibrium.

Also true for BRD (open problem for Smith)

Lemma 2

Along any solution of BRD and interior solution of REP, strictly dominated pure strategies get extinct ($x_i(t) \to 0$ as $t \to +\infty$).

Evolutionarily Stable Strategy (ESS)

A strategy is *evolutionarily stable* if, when all the members of the population adopt it, no rare mutant can invade. In our framework :

Definition 3

Let x^*, x be mixed strategies. Then x^* has an invasion barrier against x if for any $\varepsilon > 0$ small enough, in a population composed of a proportion $(1-\varepsilon)$ of x^* and ε of x, strategy x^* has a strictly higher payoff than x.

x^* is evolutionary stable if it has an invasion barrier against all $x \neq x^*$.

For pairwise interactions, boils down to : for all $x \neq x^*$,

(i) $x \cdot Ax^* \leq x^* \cdot Ax^*$

(ii) $x \cdot Ax^* = x^* \cdot Ax^* \Rightarrow x \cdot Ax < x^* \cdot Ax$

Note : symmetric Nash eq. $\subset$ ESS $\subset$ strict symmetric Nash eq.

Examples

1) A game with a dominated equilibrium.

$$\begin{array}{cc} & \begin{array}{cc} T & B \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

Two symmetric Nash equilibria : T, B. Only T is ESS.

2) Hawk-Dove game :

$$\begin{array}{cc} & \begin{array}{cc} H & D \end{array} \\ \begin{array}{c} H \\ D \end{array} & \begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix} \end{array}$$

Unique symmetric Nash equilibrium : $(V/C; (C-V)/C)$. This is an ESS.

More examples

3) Rock-Paper-Scissors game :

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \quad \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

A unique symmetric Nash equilibrium : $(1/3, 1/3, 1/3)$. No ESS.

4) Coordination game :

$$\begin{array}{c} \\ T \\ B \end{array} \quad \begin{array}{c} \begin{array}{cc} T & B \end{array} \\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \end{array} \qquad a, b > 0$$

3 symmetric Nash equilibria : 2 pure + 1 completely mixed. Only the pure Nash are ESS.

Evolutionarily Stable State (ESState)

Consider interaction modeled by symmetric bimatrix game.

Consider a population in which all agents play pure strategies : proportion x_i play strategy i.

Call $x = (x_1, ..., x_n)$ the population state.

Mathematically, analogous to mixed strategy, but interpretation differs.

We say that the population state x is an *evolutionary stable state*(ESState) if the mixed strategy x is an evolutionary stable strategy.

It is an interior evolutionary stable state if, moreover, $x_i > 0$ for all i.

Evolutionarily Stable States and dynamics

Proposition 4

Under REP, BRD, Smith, and others :

- *ESS states are (locally) asymptotically stable.*
- *Interior ESS states are globally asymptotically stable*

(For REP, “globally” refers to interior initial conditions)

Proof : Lyapunov functions, adapted to each dynamics.

For REP : $V(x) = \prod_{i:p_i>0} x_i^{p_i}$ where p ESS.

Zero-sum games

Proposition 5

In zero-sum games, all solutions of BR, Smith - and others - converge to the set of Nash equilibria.

Proof : Lyapunov functions.

Proposition 6

In zero-sum games with an interior equilibrium, interior solutions of REP cycle, but their time-average converges to the set of Nash equilibria.

Time-average of a solution $x(\cdot)$ of REP : $\overline{x}(t) = \frac{1}{t} \int_0^t x(s)ds$.

Sketch of proof for 2-populations REP

Def : a solution of REP is *persistent* if there exists $\delta > 0$ such that for all $t > 0$ and all $i \in I$, $j \in J$, $x_i(t) \geq \delta$, $y_j(t) \geq \delta$.

Lemma : for any persistent solution of REP, the time-average converges to the set of Nash equilibria.

Proof of lemma : integrate the quotient rule

$$\frac{\dot{x}_i}{x_i} - \frac{\dot{x}_j}{x_j} = (Ax)_i - (Ax)_j \text{ with } x = x(t)$$

Proof of result on zero-sum games : lemma + constant of movement

$$H(x, y) = \sum_i p_i \ln x_i + \sum_j q_j \ln y_j$$

REP in zero-sum games : a slightly stronger result

Previous result for zero-sum games *with an interior equilibrium*.

But :

- there is a link between BRD and time-average of REP
- set of Nash equilibria is global attractor for BRD in *all* zero-sum games.

Implies that time-average of REP converges to the set of Nash equilibria in *all* zero-sum games

Dissipative games (also called stable or weakly contractive)

Class of games that generalizes both zero-sum games and games with an interior ESS.

For single population dynamics, payoff matrix A satisfies that for any mixed strategies x, x' :

$$(x - x') \cdot A(x - x') \leq 0$$

In dissipative games, solutions of many dynamics converge to the set of Nash equilibria : BRD, Smith, time-average of REP,...

Holds for more general dissipative games.

Identical interest games (partnership games)

Payoff of player 1 = payoff of player 2. E.g. : $\begin{pmatrix} 0,0 & 4,4 \\ 1,1 & 2,2 \end{pmatrix}$

Proposition 7

In identical interest games all (interior) solutions of all the dynamics we saw converge to the set of Nash equilibria.

Idea of the proof : the common payoff $P(x,y) = x \cdot Ay = y \cdot Bx$ increases along trajectories.

Extends to potential games and any Myopic adaptive dynamics.

Divergence result

Consider Generalized Rock-Paper-Scissors (Cachipún)

$$\begin{matrix} R \\ P \\ S \end{matrix} \begin{pmatrix} 0 & -b & a \\ a & 0 & -b \\ -b & a & 0 \end{pmatrix} \quad \text{with } a > 0, b > 0.$$

◇ unique NE : $p = (1/3, 1/3, 1/3)$. Moreover :

◇ If $0 < b < a$ ("good RPS game"), p global attractor for REP.

◇ If $0 < a < b$, ("bad RPS game"), p global repellor.

REP in Rock-Paper-Scissors

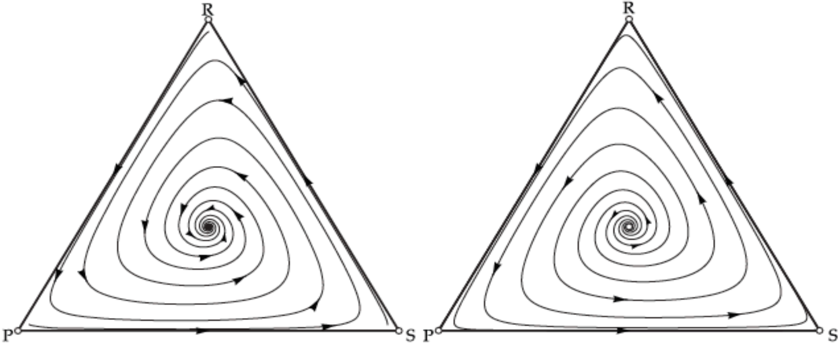

FIGURE 1. Replicator dynamics for Rock-Paper-Scissors games: $a > b$ versus $a < b$

Divergence results - II

In bad Rock-Paper-Scissors, equilibrium repellor for REP, BRD, Smith. Solutions cycle.

Such cycling behavior is "universal" (Hofbauer and Swinkels). Arises in any Myopic Adaptive Dynamics that depends smoothly on payoffs.

Still, in Rock-Paper-Scissors, solutions of REP cycle around the equilibrium ; a modified time average would converge.

Can we have stronger forms of divergence ?

Yes : there are games in which most solutions of REP cycle between pure strategies that do not belong to the support of any equilibrium.

How to build such a game ?

Step 1 : Take a game with cyclic dynamics, e.g. Rock-Paper-Scissors

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{pmatrix} 0 & -1 & \epsilon \\ \epsilon & 0 & -1 \\ -1 & \epsilon & 0 \end{pmatrix} \text{ with } 0 < \epsilon < 1$$

$\diamond$ unique Nash Equilibrium : $p = (1/3, 1/3, 1/3)$

$\diamond$ p globally inferior : $x \neq p \Rightarrow p \cdot Ax < x \cdot Ax$

$\diamond$ p global repellor, solutions converge to the boundary.

Step 2 : Add a strategy equivalent to p

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \left(\begin{array}{ccc|c} 0 & -1 & \epsilon & 0 \\ \epsilon & 0 & -1 & 0 \\ -1 & \epsilon & 0 & 0 \\ \hline \frac{-1+\epsilon}{3} & \frac{-1+\epsilon}{3} & \frac{-1+\epsilon}{3} & 0 \end{array}\right)$$

$\diamond$ strategy 4 identical to $p = (1/3, 1/3, 1/3, 0)$

$\diamond$ symmetric NE $= [p, e_4]$, with $e_4 = (0, 0, 0, 1)$

$\diamond$ If $x \notin [p, e_4]$, 4 earns less than mean payoff

$\diamond$ For ϵ small, solutions cycle outward and down, towards the best-response cycle.

Step 3 : add small bonus to strategy 4

$$\left(\begin{array}{ccc|c} 0 & -1 & \epsilon & 0 \\ \epsilon & 0 & -1 & 0 \\ -1 & \epsilon & 0 & 0 \\ \hline \frac{-1+\epsilon}{3}+\alpha & \frac{-1+\epsilon}{3}+\alpha & \frac{-1+\epsilon}{3}+\alpha & \alpha \end{array}\right)$$

$\diamond$ $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ not NE

$\diamond$ (e_4, e_4) unique NE, and a strict one.

But away from $[p, e_4]$, strategy 4 still bad and solutions still spiral outward and down.

Extensions

- multi population dynamics, wide classes of dynamics

- in previous game, (4,4) is actually the unique correlated equilibrium

- almost all initial conditions : for single-population REP and BRD, there are games with a unique equilibrium but such that any strategy in the support of the equilibrium is eliminated *for almost all initial conditions.*

Idea : replace NE of previous 4×4 game by a Rock-Paper-Scissors game

Example

$$\left(\begin{array}{ccc|ccc} 0 & -3 & 1 & -1 & -1 & -1 \\ 1 & 0 & -3 & -1 & -1 & -1 \\ -3 & 1 & 0 & -1 & -1 & -1 \\ \hline -4 & -4 & 3 & 0 & -5 & 1 \\ -1 & -1 & -3 & 1 & 0 & -5 \\ -1 & -1 & -3 & -5 & 1 & 0 \end{array}\right)$$

Unique NE $(1/3, 1/3, 1/3, 0, 0, 0)$, but under BR, $x_1 + x_2 + x_3 \to 0$ from almost all initial conditions

Idea : first cycle outward, then strategy 4 becomes a best-response, then goes to attractor of bottom Rock-Paper-Scissors.

Robust to perturbation of payoffs, similar examples for REP

Conclusion

Link between outcome of dynamics and Nash equilibrium not so clear.

Nash mass action intuition is correct : if stable frequencies of play emerge, they correspond to a Nash equilibrium.

Dynamics lead to Nash equilibria in some important classes of games.

But not so many such classes known ; in general, dynamics may cycle, and sometimes very far from Nash equilibria...

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