# CUTTING PLANES IN INTEGER PROGRAMMING

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# Lecture 1

# Introduction

## 1.1 Integer programming

A *mixed integer linear program* is a problem of the form

$$\begin{array}{rrcl} \max & cx + hy & & \\ \text{subject to} & Ax + Gy & \leq & b \\ & x & \geq & 0 \ \text{integral} \\ & y & \geq & 0, \end{array} \tag{1.1}$$

where the data are row vectors $c = (c_1, \ldots, c_n)$, $h = (h_1, \ldots, h_p)$, an $m \times n$ matrix $A = (a_{ij})$, an $m \times p$ matrix $G = (g_{ij})$ and a column vector $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. We will usually assume that all entries of $c,\ h,\ A,\ G,\ b$ are rational. The column vectors $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$ contain the variables to be optimized. The variables $x_j$ are constrained to be nonnegative integers while the variables $y_j$ are allowed to take any nonnegative real value. We will always assume that there is at least one integer variable, i.e. $n \geq 1$. A *pure integer linear program* is the special case of the mixed integer linear program (1.1) where $p = 0$. For convenience, we refer to mixed integer linear programs simply as *integer programs* in this course.

The set of feasible solutions to (1.1)

$$S := \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ \ : \ Ax + Gy \leq b\} \tag{1.2}$$

is called a *mixed integer linear set*.

Solving integer programs is a difficult task in general. One approach that is commonly used in computational mathematics is to find a relaxation that is easier to solve numerically and gives a good approximation. In this course we focus mostly on *linear programming relaxations*.

We will use the symbols "$\subseteq$" to denote inclusion and "$\subset$" to denote strict inclusion. Given a mixed integer set $S \subseteq \mathbb{Z}^n \times \mathbb{R}^p$, a *linear relaxation* of $S$ is a set of the form $P' := \{(x, y) \in$

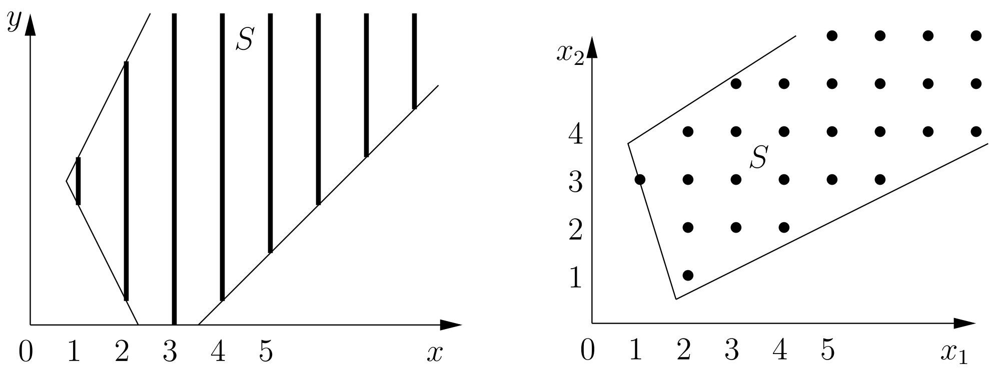


Figure 1.1: A mixed integer linear set and a pure integer linear set.

$\mathbb{R}^n \times \mathbb{R}^p : A'x + G'y \leq b'\}$ that contains $S$. A *linear programming relaxation* of (1.1) is a linear program $\max\{cx+hy : (x,y) \in P'\}$. Why linear programming relaxations? Mainly for two reasons. First, solving linear programs is one of the greatest successes in computational mathematics. There are algorithms that are efficient in theory and practice and therefore one can solve these relaxations in a reasonable amount of time. Second, one can generate a sequence of linear relaxations of $S$ that provide increasingly tighter approximations of the set $S$.

For the mixed integer linear set $S$ defined in (1.2), there is a *natural linear relaxation*, namely the relaxation

$$P_0 := \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$$

obtained from $S$ by discarding the integrality requirement on the vector $x$. The *natural linear programming relaxation* of (1.1) is the linear program $\max\{cx + hy : (x,y) \in P_0\}$.

For example, the 2-variable pure integer program

$$\begin{array}{ll} \max & 5.5x_1 + 2.1x_2 \\ & -x_1 + \phantom{2}x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 17 \\ & x_1,\ x_2 \geq 0 \\ & x_1,\ x_2 \text{ integer} \end{array}$$

has eight feasible solutions represented by the dots in Figure 1.2. One can verify that the optimal solution of this integer program is $x_1 = 1$, $x_2 = 3$ with objective value 11.8. The solution of the natural linear programming relaxation is $x_1 = 1.3$, $x_2 = 3.3$, with objective value 14.08.

## 1.2 The cutting plane method

In this section we introduce an algorithmic principle that is at the heart of the state-of-the-art software for integer programming.

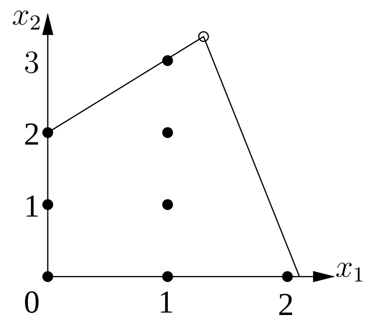


Figure 1.2: A 2-variable integer program

The integer programming formulation (1.1) will be denoted by MILP here for easy reference.

$$\text{MILP :} \qquad \max\{cx + hy : (x, y) \in S\}$$

where $S := \{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$. For ease of exposition, we assume in this section that MILP admits a finite optimum. Let $(x^*, y^*)$ denote an optimal solution and $z^*$ the optimal value of MILP. These are the unknowns that we are looking for.

Let $(x^0, y^0)$ and $z_0$ be, respectively, an optimal solution and the optimal value of the natural linear programming relaxation

$$\max\{cx + hy : (x, y) \in P_0\} \tag{1.3}$$

where $P_0$ is the natural linear relaxation of $S$ (the existence of an optimal rational solution $(x^0, y^0)$ to (1.3) follows from our assumption on the existence of $(x^*, y^*)$ and the rationality of the data, Meyer [50]). We will assume that we have a linear programming solver at our disposal, thus $(x^0, y^0)$ and $z_0$ are available to us. We also assume that $(x^0, y^0)$ is a *basic* optimal solution of (1.3), which can be computed by standard linear programming algorithms. Since $S \subseteq P_0$, it follows that $z^* \leq z_0$. Furthermore, if $x^0$ is an integral vector, then $(x^0, y^0) \in S$ and therefore $z^* = z_0$; in this case MILP is solved.

We present now a strategy for dealing with the case when the solution $(x^0, y^0)$ is not in $S$. The idea is to find an inequality $\alpha x + \gamma y \leq \beta$ that is satisfied by every point in $S$ and such that $\alpha x^0 + \gamma y^0 > \beta$. The existence of such an inequality is guaranteed when $(x^0, y^0)$ is a basic solution of (1.3).

An inequality $\alpha u \leq \beta$ is *valid* for a set $K \subseteq \mathbb{R}^d$ if it is satisfied by every point $\bar{u} \in K$. A valid inequality $\alpha x + \gamma y \leq \beta$ for $S$ that is violated by $(x^0, y^0)$ is a *cutting plane* separating $(x^0, y^0)$ from $S$. Let $\alpha x + \gamma y \leq \beta$ be a cutting plane and define

$$P_1 := P_0 \cap \{(x, y) : \alpha x + \gamma y \leq \beta\}.$$

Since $S \subseteq P_1 \subset P_0$, the linear programming relaxation of MILP based on $P_1$ is stronger than the natural linear programming relaxation (1.3), in the sense that the optimal value of the linear program

$$\max\{cx + hy : (x, y) \in P_1\}$$

is at least as good an upper-bound on the value $z^*$ as $z_0$, while the optimal solution $(x^0, y^0)$ of the natural linear programming relaxation does not belong to $P_1$. The recursive application of this idea leads to the *cutting plane approach*.

### Cutting plane algorithm

Starting with $i = 0$, repeat:

**Recursive Step.** Solve the linear program $\max\{cx + hy : (x, y) \in P_i\}$.

- If the associated optimal basic solution $(x^i, y^i)$ belongs to $S$, stop.
- Otherwise solve the *separation problem*

  *Find a cutting plane* $\alpha x + \gamma y \leq \beta$ *separating* $(x^i, y^i)$ *from* $S$.

  Set $P_{i+1} := P_i \cap \{(x, y) : \alpha x + \gamma y \leq \beta\}$ and repeat the recursive step.

The separation problem that needs to be solved in the cutting plane algorithm is a central issue in integer programming. If the basic solution $(x^i, y^i)$ is not in $S$, there are infinitely many cutting planes separating $(x^i, y^i)$ from $S$. How does one produce effective cuts? Usually, there is a tradeoff between the running time of a separation procedure and the quality of the cutting planes it produces. In practice, it may also be preferable to generate several cutting planes separating $(x^i, y^i)$ from $S$, instead of a single cut as suggested in the above algorithm, and to add them all to $P_i$ to create problem $P_{i+1}$. We will study several separation procedures in this course.

For now, we illustrate the cutting plane approach on our two-variable example:

$$\begin{array}{rl} \max & 5.5x_1 + 2.1x_2 \\ & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 17 \\ & x_1,\, x_2 \geq 0 \\ & x_1,\, x_2 \text{ integer.} \end{array} \tag{1.4}$$

We first introduce a variable $z$ representing the objective function and slack variables $x_3$ and $x_4$ to turn the inequality constraints into equalities. The problem becomes to maximize $z$ subject to

$$\begin{array}{cccccccc} z & -5.5x_1 & -2.1x_2 & & & = & 0 \\ & -x_1 & +x_2 & +x_3 & & = & 2 \\ & 8x_1 & +2x_2 & & +x_4 & = & 17 \\ & \multicolumn{5}{l}{x_1, x_2, x_3, x_4 \geq 0 \text{ integer.}} \end{array}$$

Note that $x_3$ and $x_4$ can be constrained to be integer because the data in the constraints of (1.4) are all integers.

Solving the linear programming relaxation using standard techniques, we get the optimal tableau:

$$\begin{array}{rrrrcl} z & & +0.58x_3 & +0.76x_4 & = & 14.08 \\ & x_2 & +0.8x_3 & +0.1x_4 & = & 3.3 \\ x_1 & & -0.2x_3 & +0.1x_4 & = & 1.3 \\ & & x_1, x_2, x_3, x_4 \geq 0. & & & \end{array}$$

The corresponding basic solution is $x_3 = x_4 = 0$, $x_1 = 1.3$, $x_2 = 3.3$ with objective value $z = 14.08$. Since the values of $x_1$ and $x_2$ are not integer, this is not a solution of (1.4). We can generate a cut from the constraint $x_2 + 0.8x_3 + 0.1x_4 = 3.3$ in the above tableau by using the following reasoning. Since $x_2$ is an integer variable, we have

$$0.8x_3 + 0.1x_4 = 0.3 + k \quad \text{where } k \in \mathbb{Z}.$$

Since the left-hand-side is nonnegative for every feasible solution of (1.4), we must have $k \geq 0$, which implies

$$0.8x_3 + 0.1x_4 \geq 0.3 \tag{1.5}$$

This is the famous Gomory fractional cut [36]. Note that it cuts off the above fractional solution $x_3 = x_4 = 0$, $x_1 = 1.3$, $x_2 = 3.3$. More generally, if nonnegative integer variables $x_1, \ldots, x_n$ satisfy the equation

$$\sum_{j=1}^{n} a_j x_j = a_0$$

where $a_0 \notin \mathbb{Z}$, the Gomory fractional cut is

$$\sum_{j=1}^{n} (a_j - \lfloor a_j \rfloor) x_j \geq a_0 - \lfloor a_0 \rfloor. \tag{1.6}$$

This inequality is satisfied by any $x \in \mathbb{Z}^n_+$ satisfying the equation $\sum_{j=1}^n a_j x_j = a_0$ because $\sum_{j=1}^n a_j x_j = a_0$ implies $\sum_{j=1}^n (a_j - \lfloor a_j \rfloor) x_j = a_0 - \lfloor a_0 \rfloor + k$ for some integer $k$; furthermore $k \geq 0$ since the left-hand-side is nonnegative.

Let us return to our example. Since $x_3 = 2 + x_1 - x_2$ and $x_4 = 17 - 8x_1 - 2x_2$, we can express Gomory's fractional cut (1.5) in the space $(x_1, x_2)$. This yields $x_2 \leq 3$ (see Cut 1 in Figure 1.3).

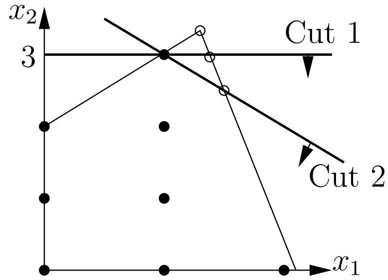


Figure 1.3: The first two cuts in the cutting plane algorithm

Adding this cut to the linear programming relaxation, we get:

$$\begin{array}{rrrl} \max & 5.5x_1 + & 2.1x_2 & \\ & -x_1 + & x_2 & \leq 2 \\ & 8x_1 + & 2x_2 & \leq 17 \\ & & x_2 & \leq 3 \\ & x_1,\ x_2 \geq 0. & & \end{array}$$

We introduce a slack variable $x_5$ for the inequality $x_2 \leq 3$, and observe as before that $x_5$ can be assumed to be integer. Solving this linear program, we find the optimal tableau

$$\begin{array}{cccccccl} z & & & & +0.6875x_4 & +0.725x_5 & = & 13.8625 \\ & x_1 & & x_3 & +0.125x_4 & -1.25x_5 & = & 0.375 \\ & x_1 & & & +0.125x_4 & -0.25x_5 & = & 1.375 \\ & & x_2 & & & +x_5 & = & 3 \\ & & & x_1, x_2, x_3, x_4, x_5 \geq 0. & & & & \end{array}$$

The corresponding basic solution in the $(x_1, x_2)$-space is $x_1 = 1.375$, $x_2 = 3$ with value $z = 13.8625$. Since $x_1$ is fractional, we need to generate another cut. From the row of the tableau $x_1 + 0.125x_4 - 0.25x_5 = 1.375$, we generate the new fractional cut according to (1.6), namely $(0.125 - \lfloor 0.125 \rfloor)x_4 + (-0.25 - \lfloor -0.25 \rfloor)x_5 \geq 1.375 - \lfloor 1.375 \rfloor$, which is $0.125x_4 + 0.75x_5 \geq 0.375$. Replacing $x_4 = 17 - 8x_1 - 2x_2$ and $x_5 = 3 - x_2$ we get (see Cut 2 in Figure 1.3):

$$x_1 + x_2 \leq 4.$$

Adding this cut and solving the updated linear program, we find a new optimal solution $x_1 = 1.5$, $x_2 = 2.5$ with value $z = 13.5$. This solution is again fractional. Two more iterations are needed to obtain the optimal integer solution $x_1 = 1$, $x_2 = 3$ with value $z = 11.8$. We leave the last two iterations as an exercise for the reader (Exercise 1.1).

Next we introduce a general tool that will be useful in the generation of cutting planes.

## 1.3 Modeling disjunctions

Disjunctions occur frequently when one wants to model decisions or alternatives.

As a first example, consider a formulation $P \subseteq [0,1]^n$ where we would like to model the additional restriction that the $j$th variable is binary. This can be done by imposing the disjunction $x_j \leq 0$ or $x_j \geq 1$. In other words, feasible solutions must lie in the union of the two regions $P \cap \{x_j \leq 0\}$ and $P \cap \{x_j \geq 1\}$.

Many applications have disjunctive constraints. For example, when scheduling jobs on a machine, we might need to model that either job $i$ is scheduled before job $j$ or vice versa; if $p_i$ and $p_j$ denote the processing times of these two jobs on the machine, we then need a constraint stating that the starting times $t_i$ and $t_j$ of jobs $i$ and $j$ satisfy $t_j \geq t_i + p_i$ or $t_i \geq t_j + p_j$. In such applications, the feasible solutions lie in the union of two or more polyhedra.

In this section, the goal is to model that a point belongs to the union of $k$ polytopes in $\mathbb{R}^n$, namely bounded sets of the form

$$\begin{aligned} A_i y &\le b_i \\ 0 \le y &\le u_i, \end{aligned} \tag{1.7}$$

for $i = 1, \ldots, k$. The same modeling question is more complicated for unbounded polyhedra and will be discussed in Section 1.4.

A way to model the union of $k$ polytopes in $\mathbb{R}^n$ is to introduce $k$ variables $x_i \in \{0, 1\}$, indicating whether $y$ is in the $i$th polytope, and $k$ vectors of variables $y_i \in \mathbb{R}^n$. The vector $y \in \mathbb{R}^n$ belongs to the union of the $k$ polytopes (1.7) if and only if

$$\begin{array}{rcll} \sum_{i=1}^k y_i & = & y & \\ A_i y_i & \le & b_i x_i & i = 1, \ldots, k \\ 0 \le y_i & \le & u_i x_i & i = 1, \ldots, k \\ \sum_{i=1}^k x_i & = & 1 & \\ x \in \{0,1\}^k. & & & \end{array} \tag{1.8}$$

The next proposition shows that formulation (1.8) is perfect in the sense that the convex hull of its solutions is simply obtained by dropping the integrality restriction.

**Proposition 1.1.** *The convex hull of solutions to* (1.8) *is*

$$\begin{array}{rcll} \sum_{i=1}^k y_i & = & y & \\ A_i y_i & \le & b_i x_i & i = 1, \ldots, k \\ 0 \le y_i & \le & u_i x_i & i = 1, \ldots, k \\ \sum_{i=1}^k x_i & = & 1 & \\ x \in [0,1]^k. & & & \end{array}$$

*Proof.* Let $P \subset \mathbb{R}^n \times \mathbb{R}^{kn} \times \mathbb{R}^k$ be the polytope given in the statement of the proposition. It suffices to show that any point $\bar{z} := (\bar{y}, \bar{y}_1, \ldots, \bar{y}_k, \bar{x}_1, \ldots, \bar{x}_k)$ in $P$ is a convex combination of solutions to (1.8). For $t$ such that $\bar{x}_t \neq 0$, define the point $z^t = (y^t, y_1^t, \ldots, y_k^t, x_1^t, \ldots, x_k^t)$ where

$$y^t := \frac{\bar{y}_t}{\bar{x}_t}, \qquad y_i^t := \begin{cases} \frac{\bar{y}_t}{\bar{x}_t} & \text{for } i = t, \\ 0 & \text{otherwise,} \end{cases} \qquad x_i^t := \begin{cases} 1 & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

The $z^t$s are solutions of (1.8). We claim that $\bar{z}$ is a convex combination of these points, namely $\bar{z} = \sum_{t\,:\,\bar{x}_t \neq 0} \bar{x}_t z^t$. To see this, observe first that $\bar{y} = \sum \bar{y}_i = \sum_{t\,:\,\bar{x}_t \neq 0} \bar{y}_t = \sum_{t\,:\,\bar{x}_t \neq 0} \bar{x}_t y^t$. Second, note that when $\bar{x}_i \neq 0$ we have $\bar{y}_i = \sum_{t\,:\,\bar{x}_t \neq 0} \bar{x}_t y_i^t$. This equality also holds when $\bar{x}_i = 0$ because then $\bar{y}_i = 0$ and $y_i^t = 0$ for all $t$ such that $\bar{x}_t \neq 0$. Finally $\bar{x}_i = \sum_{t\,:\,\bar{x}_t \neq 0} \bar{x}_t x_i^t$ for $i = 1, \ldots, k$. □

## 1.4 Union of polyhedra

In this section, we present a result of Balas [4], [5] about the union of $k$ polyhedra in $\mathbb{R}^n$. See Figure 1.4 for an example. We treated the special case of bounded polyhedra in Section 1.3. We now present the general case.

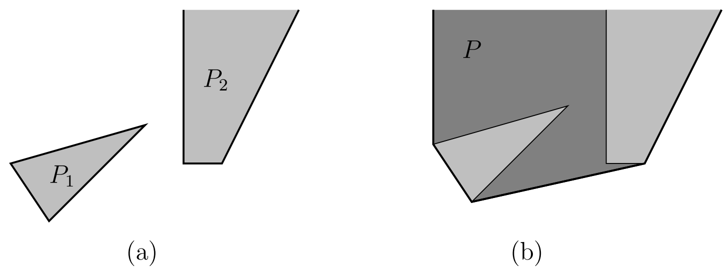


Figure 1.4: (a) Polyhedra $P_1$ and $P_2$ (note that $P_2$ is unbounded) (b) $P := \overline{\text{conv}}(P_1 \cup P_2)$

**Theorem 1.2** (Balas [4], [5])**.** *Given $k$ polyhedra $P_i := \{x \in \mathbb{R}^n : A_i x \le b^i\}$, $i = 1, \dots k$, let $C_i := \{x : A_i x \le 0\}$, and let $R^i \subset \mathbb{R}^n$ be a finite set such that $C_i = \text{cone}(R^i)$. For every $i \in \{1, \dots, k\}$ such that $P_i \neq \emptyset$, let $V^i \subset \mathbb{R}^n$ be a finite set such that $P_i = \text{conv}(V^i) + \text{cone}(R^i)$.*

*Consider the polyhedron*

$$P := \text{conv}(\bigcup_{i:P_i \neq \emptyset} V^i) + \text{cone}(\bigcup_{i=1}^{k} R^i)$$

*and let $Y \subseteq \mathbb{R}^n \times (\mathbb{R}^n)^k \times \mathbb{R}^k$ be the polyhedron described by the following system*

$$\begin{array}{rcll} A_i x^i & \le & \delta_i b^i & i = 1, \dots, k \\ \sum_{i=1}^k x^i & = & x & \\ \sum_{i=1}^k \delta_i & = & 1 & \\ \delta_i & \ge & 0 & i = 1, \dots, k. \end{array} \tag{1.9}$$

*Then $P = \text{proj}_x(Y) := \{x \in \mathbb{R}^n : \exists (x^1, \dots, x^k, \delta) \in (\mathbb{R}^n)^k \times \mathbb{R}^k$ s.t. $(x, x^1, \dots, x^k, \delta) \in Y\}$.*

*Proof.* Assume first that $P = \emptyset$. This implies $P_i = \emptyset$ for all $i = 1, \dots, k$. Since the system describing $Y$, includes $\delta_i \ge 0$, $i = 1, \dots, k$, and $\sum_{i=1}^k \delta_i = 1$, then at least one of the variables $\delta_i$ is positive. Since the system $A_i x^i \le b^i$ is infeasible, the system describing $Y$ is also infeasible. This shows that the theorem holds in this case.

We now assume that $P$ is nonempty, i.e., the index set $K := \{1, \dots, k\}$ can be partitioned into a nonempty set $K^N := \{i : P_i \neq \emptyset\}$ and $K^E := K \setminus K^N$.

Let $x \in P$. Then there exist points $v^i \in \text{conv}(V^i)$ and scalars $\delta_i \ge 0$ for $i \in K^N$, and vectors $r^i \in C^i$ for $i \in K$, such that

$$x = \sum_{i \in K^N} \delta_i v^i + \sum_{i \in K^N} r^i + \sum_{i \in K^E} r^i, \quad \sum_{i \in K^N} \delta_i = 1.$$

For $i \in K^N$ define $x^i := \delta_i v^i + r^i$, and for $i \in K^E$ define $x^i := r^i$ and $\delta_i := 0$. By construction, $x = \sum_{i \in K} x^i$. Since $r^i \in C_i$, it follows that $A_i x^i \leq \delta_i b^i$ for all $i \in K$. Thus $(x, x^1, \ldots, x^k, \delta) \in Y$. This shows $P \subseteq \text{proj}_x(Y)$.

Let $(x, x^1, \ldots, x^k, \delta)$ be a vector in $Y$. Let $K^P := \{i \in K : \delta^i > 0\}$ and let $z^i := \frac{x^i}{\delta_i}$, $i \in K^P$. Then $A_i z^i \leq b^i$. So $P_i \neq \emptyset$ in this case and $z^i \in \text{conv}(V^i) + \text{cone}(R^i)$.
For $i \in K \setminus K^P$, $A_i x^i \leq 0$, and therefore $x^i \in \text{cone}(R^i)$. Since $x = \sum_{i \in K^P} \delta_i z^i + \sum_{i \in K \setminus K^P} x^i$ and $\sum_{i \in K^P} \delta_i = 1, \delta_i \geq 0$, it follows that $x \in \text{conv}(\bigcup_{i \in K^P} V^i) + \text{cone}(\bigcup_{i \in K} R^i)$ and therefore $x \in P$. This shows that $\text{proj}_x(Y) \subseteq P$. $\square$

**Remark 1.3.** *Theorem 1.2 shows that the system of inequalities describing $Y$ gives an extended formulation of the polyhedron $P$ that uses $nk + n + k$ variables and the size of this formulation is approximately the sum of the sizes of the formulations that describe the polyhedra $P_i$.*

The polyhedron $P$ defined in Theorem 1.2 contains the convex hull of $\bigcup_{i=1}^k P_i$ but in general this inclusion is strict. Indeed, the recession cone of $P$ contains cone $C_i, i = 1, \ldots, k$, even if $P_i$ is empty. Furthermore, even if the polyhedra $P_i$ are all nonempty but have different recession cones, the set $\text{conv}(\bigcup_{i=1}^k P_i)$ may not be closed, and therefore it may not be a polyhedron. For example, in $\mathbb{R}^2$, the convex hull of a line $L$ and a point not in $L$ is not a closed set. The next lemma shows that, when the polyhedra $P_i$ are all nonempty, the polyhedron $P$ defined in Theorem 1.2 satisfies $P = \overline{\text{conv}}(\cup_{i=1}^k P_i)$ (where for any set $X \subseteq \mathbb{R}^n$ we denote by $\overline{\text{conv}}(X)$ the topological closure of $\text{conv}(X)$).

**Lemma 1.4.** *Let $P_1, \ldots, P_k \subseteq \mathbb{R}^n$ be nonempty polyhedra. For $i = 1, \ldots, k$, let $V^i, R^i \subset \mathbb{R}^n$ be finite sets such that $P_i = \text{conv}(V^i) + \text{cone}(R^i)$. Then*

$$\overline{\text{conv}}(\cup_{i=1}^k P_i) = \text{conv}(\cup_{i=1}^k V^i) + \text{cone}(\cup_{i=1}^k R^i).$$

*Proof.* Let $Q := \text{conv}(\cup_{i=1}^k V^i)$ and $C := \text{cone}(\cup_{i=1}^k R^i)$.

We first show $\overline{\text{conv}}(\cup_{i=1}^k P_i) \subseteq Q + C$. Note that we just need to show $\text{conv}(\cup_{i=1}^k P_i) \subseteq Q + C$ because $Q + C$ is a polyhedron, and so it is closed. Let $x \in \text{conv}(\cup_{i=1}^k P_i)$. Then $x$ is a convex combination of a finite number of points in $\cup_{i=1}^k P_i$. Since $P_i$ is convex, we can write $x$ as a convex combination of points $x^i \in P_i$, say $x = \sum_{i=1}^k \lambda_i x^i$ where $\lambda_i \geq 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$. Since $P_i = \text{conv}(V^i) + \text{cone}(R^i)$, $x^i = v^i + r^i$ where $v^i \in \text{conv}(V^i)$, $r^i \in \text{cone}(R^i)$, thus $x = \sum_{i=1}^k \lambda_i v^i + \sum_{i=1}^k \lambda_i r^i$, so $x \in Q + C$ since $\sum_{i=1}^k \lambda_i v^i \in Q$ and $\sum_{i=1}^k \lambda_i r^i \in C$.

We now show $Q + C \subseteq \overline{\text{conv}}(\cup_{i=1}^k P_i)$. Let $x \in Q + C$. Then $x = \sum_{i=1}^k \lambda_i v^i + \sum_{i=1}^k r^i$ where $v^i \in \text{conv}(V^i)$, $r^i \in \text{cone}(R^i)$, $\lambda_i \geq 0$ for $i = 1, \ldots, k$, and $\sum_{i=1}^k \lambda_i = 1$. We need to show that there exist points in $\text{conv}(\cup_{i=1}^k P_i)$ that are arbitrarily close to $x$.

Let $I := \{i : \lambda_i > 0\}$. For all $\epsilon > 0$ small enough so that $\lambda_i - \frac{k}{|I|}\epsilon \geq 0$ for all $i \in I$, define the point

$$x^\epsilon := \sum_{i \in I} (\lambda_i - \frac{k}{|I|}\epsilon) v^i + \sum_{i=1}^k \epsilon (v^i + \frac{1}{\epsilon} r^i).$$

Observe that $x^\epsilon \in \text{conv}(\cup_{i=1}^k P_i)$ because $\sum_{i \in I} (\lambda_i - \frac{k}{|I|}\epsilon) + \sum_{i=1}^k \epsilon = 1$ and $v^i + \frac{1}{\epsilon} r^i \in P_i$. Since $\lim_{\epsilon \to 0^+} x^\epsilon = x$, it follows that $x \in \overline{\text{conv}}(\cup_{i=1}^k P_i)$. $\square$

**Theorem 1.5** (Balas [4], [5]). *Let $P_i := \{x \in \mathbb{R}^n : A_i x \leq b^i\}$ be $k$ polyhedra such that $\cup_{i=1}^k P_i \neq \emptyset$, and let $Y$ be the polyhedron defined in Theorem 1.2. Let $C_i := \{x : A_i x \leq 0\}$ and let $R^i \subset \mathbb{R}^n$ be a finite set such that $C_i = \text{cone}(R^i)$, $i = 1, \dots, k$. Then $\overline{\text{conv}}(\cup_{i=1}^k P_i)$ is the projection of $Y$ onto the $x$-space if and only if $C_j \subseteq \text{cone}(\bigcup_{i\,:\,P_i \neq \emptyset} R^i)$ for every $j = 1, \dots, k$.*

*Proof.* For every $i \in \{1, \dots, k\}$ such that $P_i \neq \emptyset$, let $V^i \subset \mathbb{R}^n$ be a finite set such that $P_i = \text{conv}(V^i) + C_i$. Let $P := \text{conv}(\bigcup_{i:P_i \neq \emptyset} V^i) + \text{cone}(\bigcup_{i=1}^k R^i)$ and let $P' := \text{conv}(\bigcup_{i:P_i \neq \emptyset} V^i) + \text{cone}(\bigcup_{i:P_i \neq \emptyset} R^i)$. By Theorem 1.2 $P = \text{proj}_x(Y)$, whereas by Lemma 1.4 $P' = \overline{\text{conv}}(\cup_{i=1}^k P_i)$.

By definition, $P = P'$ if and only if $\text{cone}(\bigcup_{i=1}^k R^i) = \text{cone}(\bigcup_{i:P_i \neq \emptyset} R^i)$. In turn, this occurs if and only if $C_j \subseteq \text{cone}(\bigcup_{i\,:\,P_i \neq \emptyset} R^i)$ for every $j = 1, \dots, k$. □

**Remark 1.6.** *Note that if the $k$ polyhedra $P_i := \{x \in \mathbb{R}^n : A_i x \leq b^i\}$ are all nonempty, then Theorem 1.5 implies that $\overline{\text{conv}}(\cup_{i=1}^k P_i)$ is the projection of $Y$ onto the $x$-space.*

**Corollary 1.7.** *If $P_1, \dots, P_k$ are nonempty polyhedra with identical recession cones, then $\text{conv}(\cup_{i=1}^k P_i)$ is a polyhedron.*

*Proof.* We leave it as an exercise for the reader to check how the last part of the proof of Lemma 1.4 simplifies to show $Q + C \subseteq \text{conv}(\cup_{i=1}^k P_i)$. □

### 1.4.1 Example: modeling a split disjunction

Suppose we are given a system of linear constraints in $n$ variables $Ax \leq b$, and we want to further impose the following disjunctive constraint, which is known as a *split disjunction*,

$$cx \leq d_1 \quad \text{or} \quad cx \geq d_2,$$

where $c \in \mathbb{R}^n$ and $d_1 < d_2$.

If we define $P := \{x \in \mathbb{R}^n : Ax \leq b\}$, $P_1 := \{x \in P : cx \leq d_1\}$, $P_2 := \{x \in P : cx \geq d_2\}$, the set of feasible solutions is $P_1 \cup P_2$. The next lemma shows that $\text{conv}(P_1 \cup P_2)$ is a polyhedron. Note that, to prove this, we cannot apply Corollary 1.7 because $P_1$ and $P_2$ may have different recession cones.

**Lemma 1.8.** *The set $\text{conv}(P_1 \cup P_2)$ is a polyhedron. Furthermore, $\text{conv}(P_1 \cup P_2)$ is the projection onto the space of $x$ variables of the polyhedron $Q$ described by*

$$\begin{array}{rcl} Ax^1 & \leq & \lambda b \\ cx^1 & \leq & \lambda d_1 \\ Ax^2 & \leq & (1-\lambda) b \\ cx^2 & \geq & (1-\lambda) d_2 \\ x^1 + x^2 & = & x \\ 0 \leq \lambda & \leq & 1. \end{array}$$

*Proof.* The lemma holds when $P_1 = P_2 = \emptyset$ by Theorem 1.2. We assume in the remainder that $P_1 \cup P_2 \neq \emptyset$. Let $C_1 = \{r : Ar \leq 0,\, cr \leq 0\}$ and $C_2 = \{r : Ar \leq 0,\, cr \geq 0\}$. Note that $\text{rec}(P) = C_1 \cup C_2$.

We observe that, given a point $\bar{x} \in P_1$ and a vector $r \in C_2 \setminus C_1$, (resp. $\bar{x} \in P_2$, $r \in C_1 \setminus C_2$), we have $\bar{x} + r \in \text{conv}(P_1 \cup P_2)$ and $P_2 \neq \emptyset$ (resp. $P_1 \neq \emptyset$). Indeed, given $\bar{x} \in P_1$ and $r \in C_2 \setminus C_1$, it follows that $cr > 0$ and, if we let $\lambda := \max(1, \frac{d_2 - c\bar{x}}{cr})$, the point $\bar{x} + \lambda r$ is in $P_2$ and $\bar{x} + r$ is in the line segment joining $\bar{x}$ and $\bar{x} + \lambda r$.

The above observation shows that, if $P_2 = \emptyset$ (resp. $P_1 = \emptyset$), then $C_2 \subseteq C_1$ (resp. $C_1 \subseteq C_2$), therefore the cone condition of Theorem 1.5 holds in this case. It also trivially holds when both $P_1, P_2 \neq \emptyset$. Thus, in all cases, Theorem 1.5 implies that $\overline{\text{conv}}(P_1 \cup P_2)$ is the projection onto the space of $x$ variables of the polyhedron $Q$ defined in the statement of the lemma.

Therefore, to prove the lemma, we only need to show $\overline{\text{conv}}(P_1 \cup P_2) = \text{conv}(P_1 \cup P_2)$. We assume $P_1, P_2 \neq \emptyset$ otherwise the statement is obvious. Let $Q_1, Q_2 \subset \mathbb{R}^n$ be two polytopes such that $P_1 = Q_1 + C_1$ and $P_2 = Q_2 + C_2$. By Lemma 1.4, and because $\text{rec}(P) = C_1 \cup C_2$, $\overline{\text{conv}}(P_1 \cup P_2) = \text{conv}(Q_1 \cup Q_2) + \text{rec}(P)$, thus we only need to show that $\text{conv}(Q_1 \cup Q_2) + \text{rec}(P) \subseteq \text{conv}(P_1 \cup P_2)$. Let $\bar{x} \in \text{conv}(Q_1 \cup Q_2) + \text{rec}(P)$. Then there exist $x^1 \in Q_1$, $x^2 \in Q_2$, $0 \leq \lambda \leq 1$, $r \in \text{rec}(P)$, such that $\bar{x} = \lambda x^1 + (1 - \lambda)x^2 + r$. By symmetry we may assume $\lambda > 0$. By the initial observation, $x^1 + \frac{r}{\lambda} \in \text{conv}(P_1 \cup P_2)$, thus $\bar{x} = \lambda(x^1 + \frac{r}{\lambda}) + (1 - \lambda)x^2 \in \text{conv}(P_1 \cup P_2)$. □

### 1.4.2 Example: all the even subsets of a set

Consider the "all even" set $S_n^{\text{even}} := \{x \in \{0,1\}^n : x \text{ has an even number of ones}\}$. The following theorem, due to Jeroslow [44], characterizes the facet-defining inequalities of $\text{conv}(S_n^{\text{even}})$. We do not give the proof, but we give some further details in Exercise 1.6.

**Theorem 1.9.** *Let $\mathcal{S}$ be the family of subsets of $N = \{1, \ldots, n\}$ having odd cardinality. Then*

$$\text{conv}(S_n^{\text{even}}) = \left\{ x \in \mathbb{R}^n \; : \; \begin{array}{lr} \sum_{i \in S} x_i - \sum_{i \in N \setminus S} x_i \leq |S| - 1, & S \in \mathcal{S} \\ 0 \leq x_i \leq 1, & i \in N \end{array} \right\}.$$

The formulation of $\text{conv}(S_n^{\text{even}})$ given in Theorem 1.9 has exponentially many constraints. However, Theorem 1.2 gives us the means of obtaining a compact extended formulation. We present it next.

Let $S_n^k := \{x \in \{0,1\}^n : x \text{ has } k \text{ ones}\}$. It is easy to show that $\text{conv}(S_n^k) = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1,\ i \in N;\ \sum_{i \in N} x_i = k\}$ (for example, this follows from the total unimodularity of the constraint matrix).

Let $N^{\text{even}} := \{k \; : \; 0 \leq k \leq n,\ k \text{ even}\}$. Consider the polytope $Q$ described by the following system:

$$\begin{array}{rcll} x_i - \displaystyle\sum_{k \in N^{\text{even}}} x_i^k & = & 0 & i \in N \\ \displaystyle\sum_{i \in N} x_i^k & = & k\lambda_k & k \in N^{\text{even}} \\ \displaystyle\sum_{k \in N^{\text{even}}} \lambda_k & = & 1 & \\ x_i^k & \leq & \lambda_k & i \in N,\ k \in N^{\text{even}} \\ x_i^k & \geq & 0 & i \in N,\ k \in N^{\text{even}} \\ \lambda_k & \geq & 0 & k \in N^{\text{even}}. \end{array}$$

Since $S_n^{\text{even}} = \bigcup_{k \in N^{even}} S_n^k$, by Theorem 1.2, we have that $\text{conv}(S_n^{\text{even}}) = \text{proj}_x(Q)$.

## 1.5 Exercises

**Exercise 1.1.** Finish solving the 2-variable integer program of Section 1.2 using fractional cuts.

**Exercise 1.2.** Solve the following pure integer program using fractional cuts.

$$\begin{array}{lrrll} \max & 2x_1 & +x_2 & & \\ & -x_1 & +x_2 & \leq & 0 \\ & 6x_1 & +2x_2 & \leq & 21 \\ & x_1, & x_2 & \geq & 0 \text{ integer.} \end{array}$$

**Exercise 1.3.** Let $(x^0, y^0)$ be an optimal solution of (1.3) that is not in $S$. Suppose that $(x^0, y^0) = \frac{1}{2}(x^1, y^1) + \frac{1}{2}(x^2, y^2)$ where $(x^i, y^i) \in S$ for $i = 1, 2$. Show that there is no cutting plane separating $(x^0, y^0)$ from $S$.

**Exercise 1.4.** Let $S := \{(x, y) \in \mathbb{Z} \times \mathbb{R}_+ : x - y \leq \beta\}$. Show that

$$\text{conv}(S) = \left\{(x, y) \in \mathbb{R}^2 : x - y \leq \beta,\ x - \frac{1}{1-f} y \leq \lfloor \beta \rfloor,\ y \geq 0\right\}.$$

**Exercise 1.5.** Prove Corollary 1.7.

**Exercise 1.6.** The purpose of this exercise is to prove Theorem 1.9. For every $c \in \mathbb{R}^n$, consider the primal-dual pair of problems

$$\begin{array}{lll} \max \sum_{i \in N} c_i x_i & & \\ \sum_{i \in S} x_i - \sum_{i \in N \setminus S} x_i & \leq |S| - 1 & S \in \mathcal{S} \\ x_i & \leq 1 & i \in N \\ x_i & \geq 0 & i \in N \end{array} \qquad \begin{array}{lll} \min \sum_{S \in \mathcal{S}} (|S| - 1) y_S + \sum_{i \in N} z_i & & \\ \sum_{S \in \mathcal{S}, S \ni i} y_S - \sum_{S \in \mathcal{S}, S \not\ni i} y_S + z_i & \geq c_i & i \in N \\ z_i & \geq 0 & i \in N \\ y_S & \geq 0 & S \in \mathcal{S}. \end{array}$$

Characterize the optimal solution $x^*$ of the problem $\max\{cx : x \in S_n^{\text{even}}\}$, and show that the dual problem (on the right) has a feasible solution $(y^*, z^*)$ of value $cx^*$.

**Exercise 1.7.** Let $P = \{A_1 x \leq b_1\}$ be a polytope and $S = \{A_2 x < b_2\}$. Formulate the problem of maximizing a linear function over $P \setminus S$ as a mixed 0,1 program.

**Exercise 1.8.** Consider continuous variables $y_j$ that can take any value between 0 and $u_j$, for $j = 1, \ldots, k$. Write a set of mixed integer linear constraints to impose that at most $\ell$ of the $k$ variables $y_j$ can take a nonzero value. [ Hint: use $k$ binary variables $x_j \in \{0, 1\}$.] Either prove that your formulation is perfect, in the spirit of Proposition 1.1, or give an example showing that it is not.

# Lecture 2

# Split and Gomory inequalities

The addition of Gomory's mixed integer cuts to the linear programming relaxation of integer programming formulations has turned out to be one of the most successful strengthening techniques in practice. These inequalities have a geometric interpretation, in the context of Balas' disjunctive programming. They are known as split inequalities in this context. For pure integer problems and mixed 0,1 problems, iterating this process a finite number of times produces a perfect formulation.

## 2.1 Split inequalities

Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron, let $I \subseteq \{1, \dots, n\}$, and let $S := \{x \in P : x_j \in \mathbb{Z}, j \in I\}$ be a mixed integer set, where $I$ indexes the integer variables. We define $C := \{1, \dots, n\} \setminus I$ to be the index set of the continuous variables. In this lecture, we study a general principle for generating valid inequalities for $\text{conv}(S)$.

Given a vector $\pi \in \mathbb{Z}^n$ such that $\pi_j = 0$ for all $j \in C$, the scalar product $\pi x$ is integer for all $x \in S$. Thus, for any $\pi_0 \in \mathbb{Z}$, it follows that every $x \in S$ satisfies exactly one of the terms of the disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$. We refer to the latter as a *split disjunction*, and say that a vector $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $\pi_j = 0$ for all $j \in C$ is a *split*.

Given $P$ and $I$, an inequality $\alpha x \leq \beta$ is a *split inequality* [28] if there exists a split $(\pi, \pi_0)$ such that $\alpha x \leq \beta$ is valid for both sets

$$\begin{array}{rcl} \Pi_1 & := & P \cap \{x : \pi x \leq \pi_0\} \\ \Pi_2 & := & P \cap \{x : \pi x \geq \pi_0 + 1\}. \end{array} \tag{2.1}$$

It follows from the above discussion that $S \subseteq \Pi_1 \cup \Pi_2$, therefore split inequalities are valid for $\text{conv}(S)$. We define $P^{(\pi,\pi_0)} := \text{conv}(\Pi_1 \cup \Pi_2)$. Clearly $\text{conv}(S) \subseteq P^{(\pi,\pi_0)}$ and an inequality is a split inequality if and only if it is valid for $P^{(\pi,\pi_0)}$ for some split $(\pi, \pi_0)$ (see Figure 2.1).

By Lemma 1.8, $P^{(\pi,\pi_0)}$ is a polyhedron. Furthermore $P^{(\pi,\pi_0)}$ is the projection in the

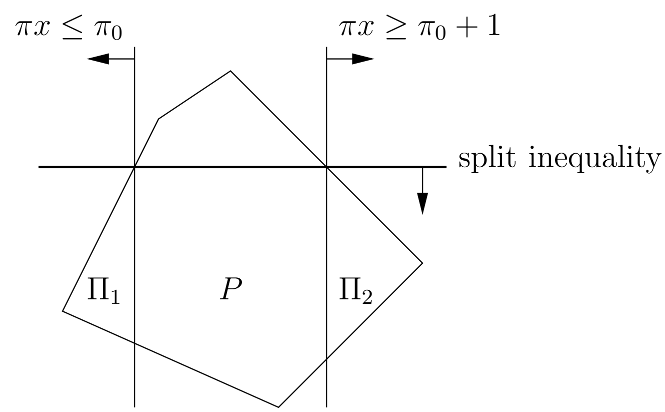


Figure 2.1: A split inequality

$x$-space of the polyhedron defined by the system

$$\begin{array}{rcl} Ax^1 & \le & \lambda b \\ \pi x^1 & \le & \lambda \pi_0 \\ Ax^2 & \le & (1-\lambda) b \\ \pi x^2 & \ge & (1-\lambda)(\pi_0 + 1) \\ x^1 + x^2 & = & x \\ 0 \le \lambda & \le & 1. \end{array} \tag{2.2}$$

The polyhedron $P^{(\pi,\pi_0)}$ could have a large number of facets compared to the number of constraints defining $P$ (see Exercise 2.2). However, the extended formulation (2.2) has only $2m + n + 4$ constraints, where $m$ is the number of constraints in the system $Ax \le b$.

The *split closure* of $P$ is the set defined by

$$P^{\text{split}} := \bigcap P^{(\pi,\pi_0)}, \tag{2.3}$$

where the intersection is taken over all splits $(\pi, \pi_0)$. Clearly $\text{conv}(S) \subseteq P^{\text{split}} \subseteq P$. Although each of the sets $P^{(\pi,\pi_0)}$ is a polyhedron, it is not obvious that $P^{\text{split}}$ itself is a polyhedron, since it is defined as the intersection of an infinite number of polyhedra. Cook, Kannan and Schrijver [28] proved that, if $P$ is a rational polyhedron, then $P^{\text{split}}$ is also a rational polyhedron.

**Remark 2.1.** *Given a split $(\pi, \pi_0)$, let $g$ be the greatest common divisor of the entries of vector $\pi$, and let $(\pi', \pi'_0)$ be the split defined by $\pi'_j = \frac{\pi_j}{g}$ for $j \in I$, and $\pi'_0 = \lfloor \frac{\pi_0}{g} \rfloor$. Since $\pi'_0 \le \frac{\pi_0}{g}$ and $\pi'_0 + 1 \ge \frac{\pi_0 + 1}{g}$, it follows that $P^{(\pi',\pi'_0)} \subseteq P^{(\pi,\pi_0)}$. In particular $P^{\text{split}}$ is the intersection of all polyhedra $P^{(\pi,\pi_0)}$ relative to splits $(\pi, \pi_0)$ such that the entries of $\pi$ are relatively prime.*

The following proposition can be proved using basic polyhedral theory.

**Proposition 2.2.** *Assume that the polyhedron $P$ is nonempty.*

*i) Let $(\pi, \pi_0)$ be a split. If $\Pi_1, \Pi_2 \ne \emptyset$ and $V_1, V_2 \subseteq \mathbb{R}^n$ are finite sets such that $\Pi_1 = \text{conv}(V_1) + \text{rec}(\Pi_1)$ and $\Pi_2 = \text{conv}(V_2) + \text{rec}(\Pi_2)$, then $P^{(\pi,\pi_0)} = \text{conv}(V_1 \cup V_2) + \text{rec}(P)$.*

*ii) Let $(\pi, \pi_0)$ be a split. There is strict inclusion $P^{(\pi,\pi_0)} \subset P$ if and only if some minimal face of $P$ lies in the region defined by $\pi_0 < \pi x < \pi_0 + 1$.*

*iii) If $P$ is a rational polyhedron, $P^{\text{split}} \subset P$ if and only if $\text{conv}(S) \subset P$.*

Let $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ be a split (i.e. $\pi_j = 0$ for all $j \in C$). The next lemma, due to Bonami [17], gives a necessary and sufficient condition for a point to be in $P^{(\pi,\pi_0)}$.

**Lemma 2.3.** *Given $\bar{x} \in P$ such that $\pi_0 < \pi\bar{x} < \pi_0 + 1$, $\bar{x}$ belongs to $P^{(\pi,\pi_0)}$ if and only if there exists $\tilde{x} \in \Pi_2$ such that*

$$b - A\tilde{x} \leq \frac{b - A\bar{x}}{\pi\bar{x} - \pi_0}. \tag{2.4}$$

*Proof.* For the "if" direction, let $\tilde{x}$ be as in the statement and define $\lambda := \frac{\pi\bar{x}-\pi_0}{\pi_0+1-\pi\bar{x}}$. Note that $\lambda > 0$. We show that the point $\hat{x} = \bar{x} + \lambda(\bar{x} - \tilde{x})$ belongs to $\Pi_1$, thus $\bar{x} \in P^{(\pi,\pi_0)}$ since it is a convex combination of $\tilde{x} \in \Pi_2$ and $\hat{x} \in \Pi_1$. Indeed,

$$\pi\hat{x} = \pi\bar{x} + \lambda(\pi\bar{x} - \pi\tilde{x}) \leq \pi\bar{x} + \lambda(\pi\bar{x} - (\pi_0 + 1)) = \pi_0,$$

$$A\hat{x} - b = (1 + \lambda)A\bar{x} - \lambda A\tilde{x} - b = (1 + \lambda)(A\bar{x} - b) + \lambda(b - A\tilde{x}) \leq 0,$$

where the last inequality follows from (2.4) and the fact that $\frac{\lambda}{1+\lambda} = \pi\bar{x} - \pi_0$. Thus $\hat{x} \in \Pi_1$ and $\bar{x} \in P^{(\pi,\pi_0)}$.

For the "only if" part, suppose there exist $x^1 \in \Pi_1$ and $x^2 \in \Pi_2$ such that $\bar{x} = (1-\lambda)x^1 + \lambda x^2$ where $0 < \lambda < 1$. We can choose $x^1$ and $x^2$ so that $\pi x^1 = \pi_0$ and $\pi x^2 = \pi_0 + 1$. It follows that $\pi\bar{x} = \pi_0 + \lambda$, thus $\lambda = \pi\bar{x} - \pi_0$. We show that $\tilde{x} := x^2$ satisfies (2.4). Indeed,

$$b - A\tilde{x} = b - \frac{A\bar{x} - (1 - \lambda)Ax^1}{\lambda} = \frac{b - A\bar{x} - (1 - \lambda)(b - Ax^1)}{\lambda} \leq \frac{b - A\bar{x}}{\pi\bar{x} - \pi_0},$$

where the last inequality follows from $\lambda < 1$ and $Ax^1 \leq b$. □

By Lemma 2.3, given $\bar{x} \in P$ such that $\pi_0 < \pi\bar{x} < \pi_0 + 1$, in order to decide whether $\bar{x}$ is in $P^{(\pi,\pi_0)}$ one can solve the following linear programming problem

$$\begin{aligned} &\max\ \pi x \\ b - \frac{b - A\bar{x}}{\pi\bar{x} - \pi_0} \leq\ &Ax \leq b. \end{aligned} \tag{2.5}$$

If the optimal value is greater than or equal to $\pi_0 + 1$, then $\bar{x} \in P^{(\pi,\pi_0)}$, otherwise $\bar{x} \notin P^{(\pi,\pi_0)}$.

### 2.1.1 Inequality description of the split closure

Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron, let $I \subseteq \{1, \ldots, n\}$ and $C := \{1, \ldots, n\} \setminus I$ index integer and continuous variables respectively, and let $S := \{x \in P : x_j \in \mathbb{Z}, j \in I\}$ be the corresponding mixed integer set.

The main goal of this section is to prove that all split inequalities that are necessary to describe $P^{(\pi,\pi_0)}$ can be written in the following form. Let $m$ be the number of rows of $A$. For $u \in \mathbb{R}^m$, let $u^+$ be defined by $u_i^+ := \max\{0, u_i\}$, $i = 1, \ldots, m$, and let $u^- := (-u)^+$, so

$u = u^+ - u^-$. Throughout this lecture, we denote by $A_I$ and $A_C$ the matrices comprising the columns of $A$ with indices in $I$ and $C$, respectively, and for any vector $\alpha \in \mathbb{R}^n$ we define $\alpha_I$ and $\alpha_C$ accordingly.

Let $u \in \mathbb{R}^m$ such that $uA_I$ is integral, $uA_C = 0$, and $ub \notin \mathbb{Z}$, consider the inequality

$$\frac{u^+(b - Ax)}{f} + \frac{u^-(b - Ax)}{1 - f} \geq 1. \tag{2.6}$$

where $f := ub - \lfloor ub \rfloor$.

**Lemma 2.4.** *Let $u \in \mathbb{R}^m$ such that $uA_I$ is integral, $uA_C = 0$, and $ub \notin \mathbb{Z}$. Define $\pi := uA$ and $\pi_0 := \lfloor ub \rfloor$. The inequality (2.6) is valid for $P^{(\pi,\pi_0)}$, thus it is a split inequality for $P$.*

*Proof.* By definition, $\pi_I$ is integral and $\pi_C = 0$, thus $(\pi, \pi_0)$ is a split. It suffices to show that (2.6) is valid for $P^{(\pi,\pi_0)}$. We show that (2.6) is valid for $\Pi_1$, the argument for $\Pi_2$ being symmetric. Given $\bar{x} \in \Pi_1$, let $s^1 := u^+(b - A\bar{x})$ and $s^2 := u^-(b - A\bar{x})$. Observe that $s^1 - s^2 = ub - uA\bar{x} = ub - \pi\bar{x}$. Thus $(1 - f)s^1 + fs^2 = (1 - f)(s^1 - s^2) + s^2 = (1 - f)(ub - \pi\bar{x}) + s^2 \geq (1 - f)f$, where the last inequality follows from $\pi\bar{x} \leq \pi_0$ and from $s^2 \geq 0$. □

Let $B_\pi \subseteq \mathbb{R}^m$ denote the set of basic solutions to the system $uA = \pi$. (Recall that $u$ is basic if the rows of $A$ corresponding to the nonzero entries of $u$ are linearly independent.)

**Theorem 2.5.** *Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron, let $I \subseteq \{1, \ldots, n\}$, and let $S := \{x \in P : x_j \in \mathbb{Z}, j \in I\}$. Given a split $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, $P^{(\pi,\pi_0)}$ is the set of all points in $P$ satisfying the inequalities*

$$\frac{u^+(b - Ax)}{ub - \pi_0} + \frac{u^-(b - Ax)}{\pi_0 + 1 - ub} \geq 1, \qquad \text{for all } u \in B_\pi \text{ s.t. } \pi_0 < ub < \pi_0 + 1. \tag{2.7}$$

*Proof.* It follows from Lemma 2.4 that the inequalities (2.7) are valid for $P^{(\pi,\pi_0)}$. Thus we only need to show that, given a point $\bar{x} \in P \setminus P^{(\pi,\pi_0)}$, there exists an inequality (2.7) violated by $\bar{x}$. Since $\bar{x} \in P \setminus P^{(\pi,\pi_0)}$, it satisfies $\pi_0 < \pi\bar{x} < \pi_0 + 1$, and it follows from Lemma 2.3 that the linear program (2.5) has optimal value less than $\pi_0 + 1$. The dual of (2.5) is

$$\begin{aligned} \min \quad & (u^1 - u^2)b + u^2 \frac{b - A\bar{x}}{\pi\bar{x} - \pi_0} \\ & (u^1 - u^2)A = \pi \\ & u^1, u^2 \geq 0. \end{aligned} \tag{2.8}$$

Let $(u^1, u^2)$ be an optimal basic solution of (2.8), and let $u := u^1 - u^2$. Since $(u^1, u^2)$ is basic, it follows that, for $i = 1, \ldots, m$, at most one among $u^1_i$ and $u^2_i$ is nonzero, therefore $u^+ = u^1$ and $u^- = u^2$. Furthermore, the rows of $A$ corresponding to nonzero components of $u$ are linearly independent, hence $u \in B_\pi$. By the linear programming duality theorem, problem (2.8) has optimal value less than $\pi_0 + 1$, thus

$$ub + u^- \frac{b - A\bar{x}}{\pi\bar{x} - \pi_0} < \pi_0 + 1. \tag{2.9}$$

Using the fact that $uA = \pi$ and $\pi\bar{x} - \pi_0 > 0$, (2.9) is equivalent to

$$u^-(b - A\bar{x}) < (\pi_0 + 1 - ub)(uA\bar{x} - \pi_0). \tag{2.10}$$

Since $uA\bar{x} - \pi_0 = ub - \pi_0 - u(b - A\bar{x})$, (2.10) can be expressed as

$$(\pi_0 + 1 - ub)u^+(b - A\bar{x}) + (ub - \pi_0)u^-(b - A\bar{x}) < (ub - \pi_0)(\pi_0 + 1 - ub). \tag{2.11}$$

We prove that $\pi_0 < ub < \pi_0 + 1$. Since $u^- \frac{b-A\bar{x}}{\pi\bar{x}-\pi_0} \geq 0$, (2.9) implies that $ub < \pi_0 + 1$. Inequality (2.11) is equivalent to $u^+(b - A\bar{x}) < (ub - \pi_0)(\pi_0 + 1 - ub + u(b - A\bar{x}))$. Since $u^+(b - A\bar{x}) \geq 0$ and $\pi_0 + 1 - ub + u(b - A\bar{x}) = \pi_0 + 1 - \pi\bar{x} > 0$, it follows that $ub - \pi_0 > 0$.

Therefore $(ub - \pi_0)(\pi_0 + 1 - ub) > 0$, which implies that (2.11) can be rewritten as

$$\frac{u^+(b - A\bar{x})}{ub - \pi_0} + \frac{u^-(b - A\bar{x})}{\pi_0 + 1 - ub} < 1,$$

showing that $\bar{x}$ violates the inequality (2.7) relative to $u$. □

Theorem 2.5 implies the following characterization of $P^{\text{split}}$.

**Corollary 2.6.** *$P^{\text{split}}$ is the set of all points in $P$ satisfying the inequalities* (2.6) *for all $u \in \mathbb{R}^m$ such that $uA_I$ is integral, $uA_C = 0$, $ub \notin \mathbb{Z}$, and the rows of $A$ corresponding to nonzero entries of $u$ are linearly independent.*

Another consequence of Theorem 2.5 is the following result of Andersen, Cornuéjols, and Li [1]. Let $a^1, \ldots, a^m$ denote the rows of $A$. Let $k := \text{rank}(A)$, and denote by $\mathcal{B}$ the family of bases of $Ax \leq b$, that is, sets $B \subseteq \{1, \ldots, m\}$ such that $|B| = k$ and the vectors $a^i$, $i \in B$, are linearly independent. For every $B \in \mathcal{B}$, let $P_B := \{x : a^i x \leq b_i,\ i \in B\}$.

**Corollary 2.7.** $P^{\text{split}} = \bigcap_{B \in \mathcal{B}} P_B^{\text{split}}$.

Note that, given $B \in \mathcal{B}$, the polyhedron $P_B := \{x : a^i x \leq b_i,\ i \in B\}$ has a unique minimal face, namely $F_B := \{x : a^i x = b_i,\ i \in B\}$. In particular $P_B$ is a translate of its recession cone, that is, $P_B = v + \text{rec}(P_B)$ for any $v \in F_B$. Theorem 2.5 implies that the polyhedron $P_B^{(\pi,\pi_0)}$ is defined by introducing only one split inequality.

**Remark 2.8.** *Let $(\pi, \pi_0)$ be a split. For all $B \in \mathcal{B}$ such that $P_B^{(\pi,\pi_0)} \neq P_B$,*

$$P_B^{(\pi,\pi_0)} = P_B \cap \left\{ x : \frac{\bar{u}^+(b - Ax)}{\bar{u}b - \pi_0} + \frac{\bar{u}^-(b - Ax)}{\pi_0 + 1 - \bar{u}b} \geq 1 \right\},$$

*where $\bar{u}$ is the unique vector such that $\bar{u}A = \pi$ and $\bar{u}_i = 0$ for all $i \notin B$.*

Any basic solution $u$ in $B_\pi$ needed in Theorem 2.5 is defined by some $B \in \mathcal{B}$ by $uA = \pi$ and $u_i = 0$ for all $i \notin B$. Note that $B$ needs not be a feasible basis of the system $Ax \leq b$ defining $P$. By this we mean that the minimal face $F_B$ of $P_B$ may not be a face of $P$, since it could be that $F_B \cap P = \emptyset$. Figure 2.2 illustrates the fact that the description of $P^{(\pi,\pi_0)}$ may require split inequalities generated from infeasible bases. Indeed, the polyhedron $P$ on the

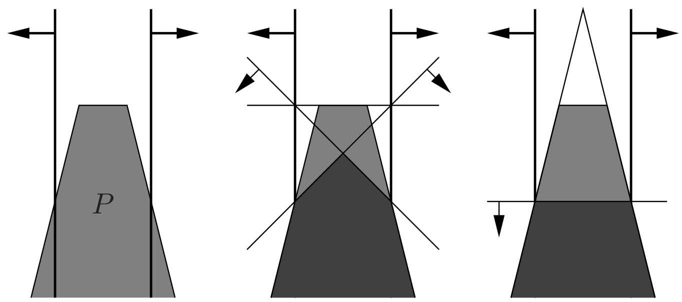


Figure 2.2: A split inequality from an infeasible basis can be stronger than split inequalities from feasible bases

left has two feasible bases an one infeasible one. Split inequalities from the two feasible bases give the dark polyhedron in the middle. The split inequality defined by the infeasible basis gives the dark polyhedron on the right; in this case this single split inequality is sufficient to define $P^{(\pi,\pi_0)}$.

Cook, Kannan and Schrijver [28] showed that, if $P$ is a rational polyhedron, then $P^{\text{split}}$ is also a polyhedron.

A natural question is whether one can optimize a linear function over $P^{\text{split}}$ in polynomial time. It turns out that this problem is NP-hard (Caprara and Letchford [19], Cornuéjols and Li [30]). The equivalence between optimization and separation implies that, given a positive integer $n$, a rational polyhedron $P \subset \mathbb{R}^n$, a set $I \subseteq \{1,\dots,n\}$ of integer variables, and a rational point $\bar{x} \in P$, it is NP-hard to find a split inequality that cuts off $\bar{x}$ or show that none exists.

### 2.1.2 Split rank

Let $P := \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polyhedron, let $I \subseteq \{1,\dots,n\}$, and let $S := \{x \in P : x_j \in \mathbb{Z},\ j \in I\}$. Let us denote the split closure $P^{\text{split}}$ of $P$ by $P^1$ and, for $k \ge 2$, let $P^k$ denote the split closure of $P^{k-1}$. We refer to $P^k$ as the *kth split closure* relative to $P$. Cook, Kannan and Schrijver [28] showed that $P^k$ is a polyhedron for all $k$. One may ask whether, by repeatedly applying the split closure operator, one eventually obtains $\text{conv}(S)$. The next example, due to Cook, Kannan and Schrijver [28], shows that this is not the case.

**Example 2.9.** Let $S := \{(x,y) \in \mathbb{Z}^2_+ \times \mathbb{R}_+ : x_1 \ge y,\ x_2 \ge y,\ x_1 + x_2 + 2y \le 2\}$. Starting from $P := \{(x_1,x_2,y) \in \mathbb{R}^3_+ : x_1 \ge y,\ x_2 \ge y,\ x_1 + x_2 + 2y \le 2\}$, we claim that there is no finite $k$ such that $P^k = \text{conv}(S)$.

To see this, note that $P$ is a simplex with vertices $O = (0,0,0)$, $A = (2,0,0)$, $B = (0,2,0)$ and $C = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$ (see Figure 2.3). $S$ is contained in the plane $y = 0$. So $\text{conv}(S) = P \cap \{(x_1,x_2,y) : y \le 0\}$. More generally, consider a simplex $P$ with vertices $O, A, B$ and $C = (\frac{1}{2},\frac{1}{2},t)$ with $t > 0$. Define $C_1 := C$, let $C_2$ be the point on the edge from $C$ to $A$ with coordinate $x_1 = 1$ and $C_3$ the point on the edge from $C$ to $B$ with coordinate $x_2 = 1$. Observe that no split inequality removes all three points $C_1$, $C_2$, $C_3$ Indeed, the projections of $C_1$, $C_2$, $C_3$ onto the plane $y = 0$ are inner points on the edges of the triangle $T$ with vertices $(1, 0)$, $(0, 1)$, $(1, 1)$. Since these vertices are integral, any split leaves at least one edge of $T$ entirely on one side of the disjunction. It follows that the corresponding $C_i$ is not removed by the split inequality.

Let $Q_i$ be the intersection of all split inequalities that do not cut off $C_i$. All split inequalities belong to at least one of these three sets, thus $P^1 = Q_1 \cap Q_2 \cap Q_3$. Let $S_i$ be the simplex with vertices $O, A, B, C_i$. Clearly, $S_i \subseteq Q_i$. Thus $S_1 \cap S_2 \cap S_3 \subseteq P^1$. It is easy to verify that $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in S_i$ for $i = 1, 2$ and 3. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in P^1$. By induction, $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3^k}) \in P^k$. Therefore $P^k \neq \text{conv}(S)$ for every positive integer $k$. ■

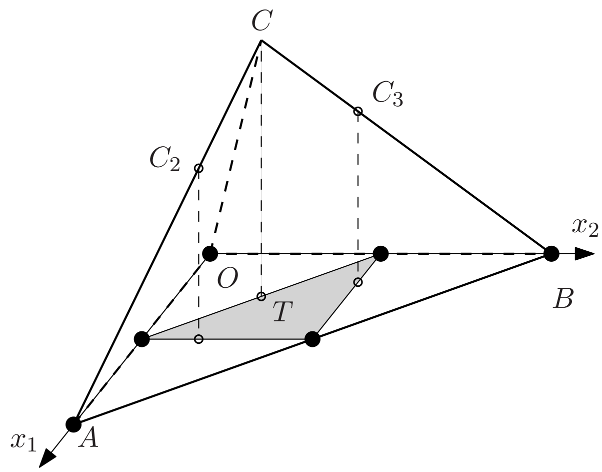


Figure 2.3: Example showing that the split rank can be unbounded, and representation of the proof.

The smallest $k$ such that $P^k = \text{conv}(S)$ is called the *split rank* of $P$, if such an integer $k$ exists. The *split rank of a valid inequality* $\alpha x \leq \beta$ for conv$(S)$ is the smallest $k$ such that $\alpha x \leq \beta$ is valid for $P^k$. In the above example the inequality $y \leq 0$ does not have finite split rank. By contrast, the split rank is always finite for pure integer programs (Chvátal [20]) and at most $|I|$ for mixed 0,1 linear programs (Theorem 2.13).

### 2.1.3 Mixed integer rounding inequalities

Mixed integer rounding inequalities, introduced by Nemhauser and Wolsey [51], offer an alternative, yet equivalent, definition of the split inequalities.

As seen in Exercise 1.4, the convex hull of the 2-dimensional mixed integer set $\{(\xi, \upsilon) \in \mathbb{Z} \times \mathbb{R}_+ : \xi - \upsilon \leq \beta\}$ is defined by the original inequalities $\upsilon \geq 0$, $\xi - \upsilon \leq \beta$ and the *simple rounding inequality*

$$\xi - \frac{1}{1-f}\upsilon \leq \lfloor \beta \rfloor,$$

where $f := \beta - \lfloor \beta \rfloor$. The simple rounding inequality is a split inequality, relative to the split disjunction $(\xi \leq \lfloor \beta \rfloor) \vee (\xi \geq \lceil \beta \rceil)$.

Mixed integer rounding inequalities for general mixed integer linear sets are inequalities that can be derived as a simple rounding inequality using variable aggregation. Formally, let $P := \{x \in \mathbb{R}^n : Ax \le b\}$, $S := \{x \in P : x_j \in \mathbb{Z}, j \in I\}$, and $C := \{1, \dots, n\} \setminus I$. Suppose that a given valid inequality for $P$ can be written, rearranging the variables, in the form

$$\pi x - (\gamma - cx) \le \beta \tag{2.12}$$

such that $\pi_I$ is integral, $\pi_C = 0$, and $cx \le \gamma$ is a valid inequality for $P$. Clearly $\pi x$ is an integer and $\gamma - cx \ge 0$ for all $x \in S$. Deriving the simple mixed integer rounding with the variable substitution $\xi = \pi x$ as an integer variable and $\upsilon = \gamma - cx$ as a nonnegative continuous variable, we obtain the inequality

$$\pi x - \frac{1}{1-f}(\gamma - cx) \le \lfloor \beta \rfloor. \tag{2.13}$$

Inequalities that can be obtained with the above derivation are the *mixed integer rounding inequalities*. Nemhauser-Wolsey [51] showed that mixed integer rounding and split inequalities are equivalent, More formally, define the *mixed integer rounding closure* $P^{\text{MIR}}$ of $P$ as the set of points in $P$ satisfying all mixed integer rounding inequalities.

**Theorem 2.10.** $P^{\text{MIR}} = P^{\text{split}}$.

*Proof.* By construction, the mixed integer rounding inequality (2.13) is a split inequality relative to the split $(\pi, \lfloor \beta \rfloor)$, thus $P^{\text{MIR}} \supseteq P^{\text{split}}$.
To prove the converse, by Theorem 2.5 it suffices to show that any inequality of the form (2.6) is a mixed integer rounding inequality. Let $u \in \mathbb{R}^m$ such that $uA_I$ is integral and $uA_C = 0$. The inequality $u^+ Ax \le u^+ b$ is valid for $P$, and it can be written as

$$uAx - u^-(b - Ax) \le ub.$$

Since $u^- Ax \le u^- b$ is valid for $P$, we can derive the mixed integer rounding inequality

$$uAx - \frac{u^-}{1-f}(b - Ax) \le \lfloor ub \rfloor \tag{2.14}$$

where $f := ub - \lfloor ub \rfloor$. One can readily verify that (2.14) is equivalent to the split inequality (2.6). □

## 2.2 Gomory's mixed integer cuts

Gomory's mixed integer inequalities [37], introduced in 1960, were the first example of general-purpose valid inequalities for mixed integer linear programs. They can be interpreted as split inequalities for problems written in standard equality form. Let $P := \{x \in \mathbb{R}^n : Ax = b, x \ge 0\}$ be a polyhedron expressed in standard equality form. Let $I \subseteq \{1, \dots, n\}$, $C := \{1, \dots, n\} \setminus I$, and $S := \{x \in P : x_j \in \mathbb{Z}, j \in I\}$.

By Corollary 2.6 applied to $P = \{x \in \mathbb{R}^n : Ax \le b, -Ax \le -b, -x \le 0\}$, any undominated split inequality is determined by a vector $(u, v) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $uA_I - v_I \in \mathbb{Z}^I$, $uA_C - v_C = 0$, and $ub \notin \mathbb{Z}$, and can be written in the form

$$\frac{v^+ x}{f} + \frac{v^- x}{1-f} \ge 1, \tag{2.15}$$

where $f := ub - \lfloor ub \rfloor > 0$.

Let $\alpha := uA$, $\beta := ub$, and $f_j := \alpha_j - \lfloor \alpha_j \rfloor$, $j \in I$. Since the variables defining $P$ are all nonnegative, for a given $u$, the choice of $v_j = v_j^+ - v_j^-$, $j \in I$ that gives the strongest inequality is the smallest possible value that satisfies the requirement $\alpha_j - (v_j^+ - v_j^-) \in \mathbb{Z}$. Therefore $v_j = f_j$ whenever $v_j = v_j^+$ and $v_j = f_j - 1$ whenever $v_j = -v_j^-$. This implies that $v_j = v_j^+$ if $\frac{f_j}{f} \leq \frac{1-f_j}{1-f}$ (i.e. if $f_j \leq f$), and $v_j = v_j^-$ otherwise. Finally observe that $uA_C - v_C = 0$ is equivalent to $v_j^+ - v_j^- = \alpha_j$, $j \in C$. Hence $v_j^+ = \alpha_j$ if $\alpha_j \geq 0$, otherwise $v_j^- = -\alpha_j$. It follows that the undominated split inequalities are of the form

$$\sum_{\substack{j \in I \\ f_j \leq f}} \frac{f_j}{f} x_j + \sum_{\substack{j \in I \\ f_j > f}} \frac{1-f_j}{1-f} x_j + \sum_{\substack{j \in C \\ \alpha_j \geq 0}} \frac{\alpha_j}{f} x_j - \sum_{\substack{j \in C \\ \alpha_j < 0}} \frac{\alpha_j}{1-f} x_j \geq 1. \tag{2.16}$$

This is *Gomory's mixed integer inequality* derived from the equation $\alpha x = \beta$ [37]. Note that (2.16) is a split inequality relative to the split $(\pi, \pi_0)$ defined by $\pi_0 = \lfloor ub \rfloor$ and, for $j = 1, \ldots, n$,

$$\pi_j = \begin{cases} \lfloor \alpha_j \rfloor & \text{if } j \in I \text{ and } f_j \leq f_0 \\ \lceil \alpha_j \rceil & \text{if } j \in I \text{ and } f_j > f_0 \\ 0 & \text{if } j \in C. \end{cases} \tag{2.17}$$

In practice, Gomory's mixed integer inequalities have turned out to be effective cutting planes in branch-and-cut algorithms. We now discuss the implementation of these inequalities.

Let $B$ be a feasible basis of the system $Ax = b$, $x \geq 0$. The tableau associated with $B$ is of the form

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i, \quad i \in B.$$

The basic solution associated with such a tableau is $x_i^* = \bar{b}_i$, $i \in B$, $x_j^* = 0$, $j \in N$. This vector belongs to $S$ if and only if $\bar{b}_i \in \mathbb{Z}$ for all $i \in B \cap I$. If not, consider an index $i \in B_I$ such that $f_0 := \bar{b}_i - \lfloor \bar{b}_i \rfloor > 0$, and let $f_j := a_{ij} - \lfloor a_{ij} \rfloor$ for all $j \in N$. The Gomory mixed integer inequality (2.16) derived from the tableau equation relative to $x_i$ is

$$\sum_{\substack{j \in N \cap I \\ f_j \leq f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \in N \cap I \\ f_j > f_0}} \frac{1-f_j}{1-f_0} x_j + \sum_{\substack{j \in N \cap C \\ \bar{a}_{ij} \geq 0}} \frac{\bar{a}_{ij}}{f_0} x_j - \sum_{\substack{j \in N \cap C \\ \bar{a}_{ij} < 0}} \frac{\bar{a}_{ij}}{1-f_0} x_j \geq 1. \tag{2.18}$$

Clearly the above inequality cuts off the basic solution $x^*$ defined by $B$, since $x_j^* = 0$ for all $j \in N$.

**Remark 2.11.** *For pure integer sets (i.e. $C = \emptyset$), the Gomory mixed integer cut* (2.18) *dominates the Gomory fractional cut* (1.6)*, which is $\sum_{j \in N} f_j x_j \geq f_0$ using the above notation, because $\frac{f_j}{f_0} > \frac{1-f_j}{1-f_0}$ whenever $f_j > f_0$.*

**Example 2.12.** Consider the following pure integer programming problem, which we solved in Section 1.2 using Gomory fractional cuts.

$$\begin{aligned} \max\ z = \quad & 5.5x_1 + 2.1x_2 \\ & -x_1 + \quad x_2 \le 2 \\ & 8x_1 + \ 2x_2 \le 17 \\ & x_1,\ x_2 \ge 0 \\ & x_1,\ x_2 \text{ integer.} \end{aligned}$$

We first add slack variables $x_3$ and $x_4$ to turn the inequality constraints into equalities. The problem becomes:

$$\begin{array}{cccccc} z & -5.5x_1 & -2.1x_2 & & & = & 0 \\ & -x_1 & +x_2 & +x_3 & & = & 2 \\ & 8x_1 & +2x_2 & & +x_4 & = & 17 \\ & \multicolumn{5}{l}{x_1, x_2, x_3, x_4 \ge 0 \ \text{ integer.}} \end{array}$$

Solving the linear programming relaxation, we get the optimal tableau:

$$\begin{array}{cccccc} z & & & +0.58x_3 & +0.76x_4 & = & 14.08 \\ & & x_2 & +0.8x_3 & +0.1x_4 & = & 3.3 \\ & x_1 & & -0.2x_3 & +0.1x_4 & = & 1.3 \\ & \multicolumn{5}{c}{x_1, x_2, x_3, x_4 \ge 0.} \end{array}$$

The corresponding basic solution is $x_3^* = x_4^* = 0$, $x_1^* = 1.3$, $x_2^* = 3.3$ and $z^* = 14.08$. This solution is not integer. Let us generate the Gomory mixed integer cut corresponding to the equation

$$x_2 + 0.8x_3 + 0.1x_4 = 3.3$$

found in the above optimal tableau. We have $f_0 = 0.3$, $f_3 = 0.8$ and $f_4 = 0.1$. Applying formula (2.18), we get the Gomory mixed integer cut

$$\frac{1-0.8}{1-0.3}x_3 + \frac{0.1}{0.3}x_4 \ge 1, \quad \text{i.e.} \quad 6x_3 + 7x_4 \ge 21.$$

We could also generate a Gomory mixed integer cut from the other equation in the final tableau $x_1 - 0.2x_3 + 0.1x_4 = 1.3$. It turns out that, in this case, we get exactly the same Gomory mixed integer cut.

Since $x_3 = 2 + x_1 - x_2$ and $x_4 = 17 - 8x_1 - 2x_2$, we can express the above Gomory mixed integer cut in the space $(x_1, x_2)$. This yields

$$5x_1 + 2x_2 \le 11.$$

Adding this cut to the linear programming relaxation, we get the following formulation (see Figure 2.4).

$$\begin{aligned} \max\ z = \ & 5.5x_1 + 2.1x_2 \\ & -x_1 + x_2 \le 2 \\ & 8x_1 + 2x_2 \le 17 \\ & 5x_1 + 2x_2 \le 11 \\ & x_1, x_2 \ge 0. \end{aligned}$$

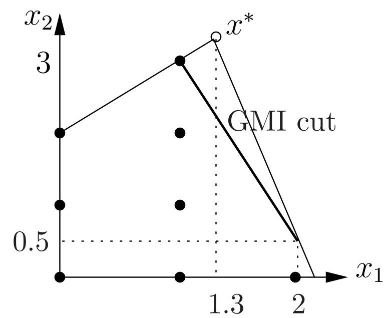


Figure 2.4: Formulation strengthened by a Gomory mixed integer cut

Note that the Gomory mixed integer cut $5x_1 + 2x_2 \leq 11$ is stronger than the Gomory fractional cut $x_2 \leq 3$ generated from the same row (recall the solution of our example in Section 1.2). This is not surprising since, as noted in Remark 2.11, Gomory mixed integer cuts are always at least as strong as the fractional cuts generated from the same rows.

Solving the linear programming relaxation, we get the optimal tableau:

$$\begin{array}{lllllcl} z & & & & +1/12x_4 & +29/30x_5 & = & 12.05 \\ & & & x_3 & +7/6x_4 & -5/3x_5 & = & 3.5 \\ & x_1 & & & +1/3x_4 & -1/3x_5 & = & 2 \\ & & x_2 & & -5/6x_4 & +4/3x_5 & = & 0.5 \\ & & & & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

The equation

$$x_3 + 7/6x_4 - 5/3x_5 = 3.5$$

found in the above tableau produces the following Gomory mixed integer cut.

$$\frac{1/6}{0.5}x_4 + \frac{1/3}{0.5}x_5 \geq 1, \quad \text{i.e.} \quad x_4 + 2x_5 \geq 3.$$

Expressing this cut in the original variables $x_1$, $x_2$ we get the inequality

$$3x_1 + x_2 \leq 6.$$

Adding this inequality and resolving the linear relaxation, we find the basic solution $x_1 = 1$, $x_2 = 3$ and $z = 11.8$. Since $x_1$ and $x_2$ are integer, this is the optimal solution to the integer program. ■

### Implementing Gomory's mixed integer cuts

Gomory presented his results on fractional cuts in 1958 and it had an enormous immediate impact: Reducing integer linear programming to a sequence of linear programs was a great theoretical breakthrough. However, when Gomory programmed his fractional cutting plane algorithm later that year, he was disappointed by the computational results. Convergence was often very slow.

Gomory [37] extended his approach to mixed integer linear programs in 1960, inventing the Gomory mixed integer cuts. Three years later, in 1963, Gomory [38] states that these cuts were "almost completely computationally untested." Surprisingly this statement was still true three decades later! During that period, the general view was that the Gomory cuts are mathematically elegant but impractical, even though there was scant evidence in the literature to justify this negative attitude. Gomory's mixed integer cuts were revived in 1996 [7], based on an implementation that added several cuts from the optimal simplex tableau at a time (instead of just one cut, as tried by Gomory when testing fractional cuts), reoptimized the resulting strengthened linear program, performed a few *rounds* of such cut generation, and incorporated this procedure in a branch-and-cut framework (instead of applying a pure cutting plane approach as Gomory had done). Incorporated in this way, Gomory's mixed integer cuts became an attractive component of integer programming solvers. In addition, linear programming solvers had become more stable by the 1990s.

Commercial integer programming solvers, such as Cplex, started incorporating the Gomory mixed integer cuts in 1999. Bixby, Fenelon, Gu, Rothberg and Wunderling [15] give a fascinating account of the evolution of the Cplex solver. They view 1999 as the transition year from the "old generation" of Cplex to the "new generation". Their paper lists some of the key features in a 2002 "new generation" solver and compares the speedup in computing time obtained by enabling one feature versus disabling it, while keeping everything else unchanged. The average speedups obtained for each feature on a set of 106 instances are summarized in the next table (we refer the reader to [15] for information on the choice of the test set).

| Feature | Speedup factor |
|---|---:|
| Cuts | 54 |
| Preprocessing | 11 |
| Branching variable selection | 3 |
| Heuristics | 1.5 |

The clear winner in these tests was cutting planes. Eight types of cutting planes were implemented in Cplex in 2002. Performing a similar experiment, disabling only one of the cut generators at a time, they obtained the following speedups in computing time.

| Cut type | Speedup factor |
|---|---:|
| Gomory mixed integer | 2.5 |
| Mixed integer rounding | 1.8 |
| Knapsack cover | 1.4 |
| Flow cover | 1.2 |
| Implied bounds | 1.2 |
| Path | 1.04 |
| Clique | 1.02 |
| GUB cover | 1.02 |

Even when all the other cutting planes are used in Cplex (2002 version), the addition of Gomory mixed integer cuts by itself produces a solver that is two and a half times faster! As Bixby and his co-authors conclude "Gomory cuts are the clear winner by this measure". In the above table, the Gomory mixed integer cuts are those generated from rows of optimal simplex tableaux. Note also the excellent performance of the mixed integer rounding inequalities. These are obtained using formula (2.13) where the inequality (2.12) is obtained by aggregating the constraints in $Ax \leq b$ using various heuristics.

Note, however, that the textbook formulas for generating Gomory mixed integer and mixed integer rounding cuts are not used directly in open-source and commercial software that use finite numerical precision in the computations. These solvers perform additional steps to avoid the generation of invalid cuts, and of cuts that could substantially slow down the solution of the linear programs. These steps come in two flavors: some modify the cut coefficients slightly while others simply discard the cut. We will discuss briefly both types of steps, starting with the first type. Consider a bounded variable $x_j$ with upper and lower bounds no greater than $L$ in absolute value (for example $L = 10^4$). When the coefficient of $x_j$ has a very small absolute value (say below $10^{-12}$) in a Gomory mixed integer cut, such a coefficient is set to 0 and the right-hand-side of the cut is adjusted accordingly (using the upper bound when the coefficient of $x_j$ is positive, and the lower bound when it is negative). The resulting inequality is a slight weakening of the Gomory mixed integer cut, but it is numerically more stable. For the second issue, several parameters of a Gomory mixed integer cut are checked before adding it to the formulation. One such parameter is the value of $f$ in formula (2.16): If $f$ or $1 - f$ is too small, the cut is discarded. A reasonable cut off point is $10^{-2}$, i.e. only add Gomory mixed integer cuts for which $0.01 \leq f \leq 0.99$. One also usually discards cuts that have too large a ratio between the absolute values of the largest and smallest nonzero coefficients (this ratio is sometimes called the *dynamism* of the cut). A reasonable rule might be to discard Gomory mixed integer cuts with a dynamism in excess of $10^6$. Furthermore, in order to avoid fill-in of the basis inverse when solving the linear programming relaxations, one also discards cuts that are too dense. The first two parameters help reduce the generation of invalid cuts while the third helps solving the linear programs. A paper of Cook, Dash, Fukasawa, Goycoolea [27] addresses the issue of always rounding coefficients in the "right" direction to keep valid cuts. Despite the various steps to make the Gomory mixed integer cut generation safer, it should be clear that any integer programming solver based on finite precision arithmetic will fail on some instances.

Another issue that has attracted attention but still needs further investigation is the choice of the equations used to generate the Gomory mixed integer cuts: Gomory proposed to use the rows of the optimal simplex tableau but other equations can also be used. Balas and Saxena [9], and Dash, Günlük and Lodi [31] showed that integer programming formulations can typically be strengthened very significantly by generating Gomory cuts from a well chosen set of equations. However, finding such a good family of equations efficiently remains a challenge.

## 2.3 Lift-and-project

In this section, we consider mixed 0,1 linear programs.

Given $I \subseteq \{1, \ldots, n\}$, we consider a polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ which is contained in $\{x \in \mathbb{R}^n : 0 \leq x_j \leq 1, j \in I\}$ and we let $S := P \cap \{x \in \mathbb{R}^n : x_j \in \{0, 1\}, j \in I\}$.

Given $j \in I$, let

$$P_j := \text{conv}\left((P \cap \{x \in \mathbb{R}^n : x_j = 0\}) \cup (P \cap \{x \in \mathbb{R}^n : x_j = 1\})\right).$$

Note that since $0 \leq x_j \leq 1$ are valid inequalities for $P$, the set $P_j$ is the convex hull of two faces of $P$. A *lift-and-project* inequality is a split inequality for $P$ relative to some disjunction $x_j = 0$ or $x_j = 1$, $j \in I$. That is, $\alpha x \leq \beta$ is a lift-and-project inequality if and only if, for some $j \in I$, it is valid for $P_j$.

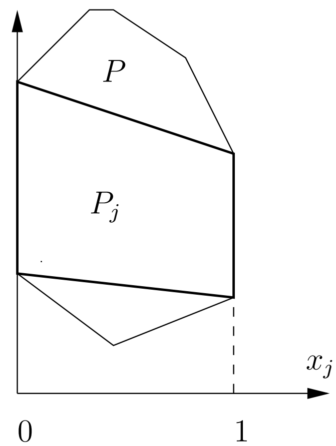


Figure 2.5: $P_j$

The term *lift-and-project* refers to the description of $P_j$ as the projection of a polyhedron in a "lifted" space, namely $P_j$ is the projection of the extended formulation (2.2) where $\pi = e^j$ and $\pi_0 = 0$. The *lift-and-project closure* of $P$ is the set of points satisfying all lift-and-project inequalities, therefore

$$P^{\text{lift}} = \bigcap_{j \in I} P_j.$$

Since $P_j$ is a polyhedron for all $j \in I$, it follows that $P^{\text{lift}}$ is a polyhedron as well. Furthermore, it follows from the definition that $P^{\text{split}} \subseteq P^{\text{lift}}$.

### 2.3.1 Lift-and-project rank for mixed 0,1 linear programs

Unlike general mixed integer linear sets, for which the split rank might not be finite as seen in Example 2.9, mixed $0,1$ linear sets have the nice property that the convex hull can be obtained by iteratively adding lift-and-project inequalities. Indeed, a much stronger property holds, that we describe here.

Given $I \subseteq \{1, \ldots, n\}$, consider a polyhedron $P$ contained in $\{x \in \mathbb{R}^n : 0 \leq x_j \leq 1, j \in I\}$ and let $S := P \cap \{x \in \mathbb{R}^n : x_j \in \{0, 1\}, j \in I\}$. Possibly by permuting the indices, we assume $I = \{1, \ldots, p\}$ and, for $t = 1, \ldots, p$, define

$$(P)^t := ((P_1)_2 \ldots)_t.$$

The next theorem shows that $(P)^p = \text{conv}(S)$. Since $P^{\text{split}} \subseteq P^{\text{lift}} \subseteq (P)^1$, this implies that the split rank and the lift-and-project rank of $P$ are at most equal to the number $p$ of $0,1$ variables.

**Theorem 2.13** (Sequential convexification theorem, Balas [4])**.** *Let* $I = \{1, \dots, p\}$*, and let* $P$ *be a polyhedron contained in* $\{x \in \mathbb{R}^n : 0 \le x_j \le 1, j \in I\}$*. Then, for* $t = 1, \dots, p$*,* $(P)^t = \text{conv}(\{x \in P : x_j \in \{0,1\}, j = 1, \dots, t\})$*. In particular,* $(P)^p = \text{conv}(S)$*.*

*Proof.* Let $S_t := \{x \in P : x_j \in \{0,1\}, j = 1, \dots, t\}$. We need to show $(P)^t = \text{conv}(S_t)$ for $t = 1, \dots, p$. We prove this result by induction. The result holds for $t = 1$ since $(P)^1 = P_1 = \text{conv}(S_1)$ where the second equality follows from the definition of $P_1$. Assume inductively that $(P)^{t-1} = \text{conv}(S_{t-1})$. Then

$$\begin{aligned}(P)^t &= ((P)^{t-1})t = \text{conv}\big(((P)^{t-1} \cap \{x \in \mathbb{R}^n : x_t = 0\}) \cup ((P)^{t-1} \cap \{x \in \mathbb{R}^n : x_t = 1\})\big)\\ &= \text{conv}\big((\text{conv}(S_{t-1}) \cap \{x \in \mathbb{R}^n : x_t = 0\}) \cup (\text{conv}(S_{t-1}) \cap \{x \in \mathbb{R}^n : x_t = 1\})\big).\end{aligned}$$

We will need the following claim.

**Claim.** *Consider a set* $A \subset \mathbb{R}^n$ *and a hyperplane* $H := \{x \in \mathbb{R}^n : \gamma x = \gamma_0\}$ *such that* $\gamma x \le \gamma_0$ *for every* $x \in A$*. Then* $\text{conv}(A) \cap H = \text{conv}(A \cap H)$*.*

Clearly $\text{conv}(A \cap H) \subseteq \text{conv}(A) \cap H$. To show $\text{conv}(A) \cap H \subseteq \text{conv}(A \cap H)$, consider $x \in \text{conv}(A) \cap H$. Then $x = \sum_{i=1}^k \lambda_i x^i$ where $x^i \in A$, $\lambda_i > 0$ for $i = 1, \dots k$, and $\sum_{i=1}^k \lambda_i = 1$. Since $x^i \in A$, we have $\gamma x^i \le \gamma_0$. Since $x \in H$, we have $\gamma x = \gamma_0$. This implies $x^i \in H$. Therefore $x^i \in A \cap H$, proving the claim.

Applying the claim to the set $S_{t-1}$ and the hyperplanes $\{x \in \mathbb{R}^n : x_t = 0\}$ and $\{x \in \mathbb{R}^n : x_t = 1\}$, we obtain

$$(P)^t = \text{conv}\big((\text{conv}(S_{t-1} \cap \{x \in \mathbb{R}^n : x_t = 0\})) \cup (\text{conv}(S_{t-1} \cap \{x \in \mathbb{R}^n : x_t = 1\}))\big).$$

For any two sets $A, B$, it follows from the characterization of convex hulls that $\text{conv}(\text{conv}(A) \cup \text{conv}(B)) = \text{conv}(A \cup B)$. This implies

$$\begin{aligned}(P)^t &= \text{conv}\big((S_{t-1} \cap \{x \in \mathbb{R}^n : x_t = 0\}) \cup (S_{t-1} \cap \{x \in \mathbb{R}^n : x_t = 1\})\big)\\ &= \text{conv}(S_t).\end{aligned}$$

□

Theorem 2.13 does not extend to the general mixed integer case (see Exercise 2.6).

**Example 2.14.** The purpose of this example is to show that the split rank of mixed 0,1 linear programs might indeed be as large as the number of binary variables. The example is due to Cornuéjols and Li [29]. Consider the following polytope, studied by Chvátal, Cook, and Hartmann [21]

$$P := \{x \in [0,1]^n : \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \ge \frac{1}{2}, \text{ for all } J \subseteq \{1, 2, \cdots, n\}\}.$$

Note that $P \cap \{0,1\}^n = \emptyset$. We will show that the $(n-1)$th split closure $P^{n-1}$ of $P$ is nonempty, implying that the split rank of $P$ is $n$.

For $j = 1, \dots, n$, let $F_j$ be the set of all vectors $x \in \mathbb{R}^n$ such that $j$ components of $x$ are $\frac{1}{2}$ and each of the remaining $n - j$ components are in $\{0,1\}$. Note that $F_1 \subseteq P$ (indeed, $P$ is the convex hull of $F_1$). Let $P^0 := P$. We will show that $P^k$ contains $F_{k+1}$, $k = 0, \dots, n-1$. Thus, $P^{n-1} \neq \emptyset$.

We proceed by induction on $k$. The statement holds for $k = 0$, thus we assume that $k \ge 1$ and that $F_k \subseteq P^{k-1}$. We need to show that, for every split $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, $F_{k+1}$ is contained in $(P^{k-1})^{(\pi,\pi_0)}$. Let $v \in F_{k+1}$. We show that $v \in (P^{k-1})^{(\pi,\pi_0)}$. We assume that $\pi_0 < \pi v < \pi_0 + 1$, otherwise $v \in (P^{k-1})^{(\pi,\pi_0)}$ by definition. Since all fractional components of $v$ equal $\frac{1}{2}$, it follows that $\pi v = \pi_0 + \frac{1}{2}$. This implies that there exists $j \in \{1, \dots, n\}$ such that $v_j = \frac{1}{2}$ and $|\pi_j| \geq 1$. Assume $\pi_j \geq 1$ and let $v^0, v^1 \in \mathbb{R}^n$ be equal to $v$ except for the $j$th component, which is 0 and 1 respectively. By construction $v^0, v^1 \in F_k$, therefore $v^0, v^1 \in P^{k-1}$. Observe that $\pi v^0 = \pi v - \frac{1}{2}\pi_j \leq \pi_0$, while $\pi v^1 = \pi v + \frac{1}{2}\pi_j \geq \pi_0 + 1$. Thus $v^0, v^1 \in (P^{k-1})^{(\pi,\pi_0)}$, which implies that $v \in (P^{k-1})^{(\pi,\pi_0)}$ since $v = \frac{v^0+v^1}{2}$. If $\pi_j \leq -1$ the proof is identical, with the roles of $v^0, v^1$ interchanged. ∎

In view of Example 2.9 showing that no bound may exist on the split rank when the integer variables are not restriced to be 0,1, and Theorem 2.13 showing that the rank is always bounded when they are 0,1 valued, one is tempted to convert general integer variables into 0,1 variables. For a bounded integer variable $0 \leq x \leq u$, there are several natural transformations:

(i) a binary expansion of $x$ (see Owen and Mehrotra [52]);

(ii) $x = \sum_{i=1}^{u} i z_i$, $\sum z_i \leq 1$, $z_i \in \{0, 1\}$ (see Sherali and Adams [56] and Köppe, Louveaux and Weismantel [48]);

(iii) $x = \sum_{i=1}^{u} z_i$, $z_i \leq z_{i-1}$, $z_i \in \{0, 1\}$ (see Roy [54] and Bonami and Margot [18]).

More studies are needed to determine whether any practical benefit can be gained from such transformations.

### 2.3.2 A finite cutting plane algorithm for mixed $0, 1$ linear programming

Theorem 2.13 implies that, for mixed $0, 1$ linear programs, the convex hull of the feasible solutions can be described by a finite number of lift-and-project cuts. However, the result does not immediately provide a finite cutting plane algorithm for this type of problems. Next we describe such an algorithm, due to Balas, Ceria and Cornuéjols [6].

We assume that we are given mixed $0, 1$ programming problems in the form

$$\begin{array}{ll} \max & cx \\ & Ax \leq b \\ & x_j \in \{0,1\} \quad j \in I \end{array} \tag{2.19}$$

where $Ax \leq b$ is a linear system of $m$ constraints in $n$ variables which includes the constraints $0 \leq x_j \leq 1$, $j \in I$, and where $I \subseteq \{1, \dots, n\}$.

At each iteration we strengthen the formulation by introducing a lift-and-project cut, until the optimal solution of the linear relaxation satisfies the integrality conditions. We denote by $A^k x \leq b^k$ the system after introducing $k$ cuts, where $A^0 = A$ and $b^0 = b$, and we let $P^k := \{x \in \mathbb{R}^n : A^k x \leq b^k\}$. At iteration $k$, we compute an optimal solution $\bar{x}$ for the linear programming relaxation $\max\{cx : x \in P^k\}$. If $\bar{x}$ does not satisfy the integrality conditions, we select an index $j \in I$ such that $0 < \bar{x}_j < 1$ and a suitable subsystem $\tilde{A}x \leq \tilde{b}$ of the system $A^k x \leq b^k$, and compute an optimal basic solution $(u^1, u^2)$ of the cut-generating linear program (2.8)

$$\begin{array}{ll} \min & (u^1-u^2)\tilde{b} + u^2\dfrac{\tilde{b}-\tilde{A}\bar{x}}{\bar{x}_j} \\ & (u^1-u^2)\tilde{A} = e^j \\ & u^1, u^2 \geq 0. \end{array} \tag{2.20}$$

Let $u := u^1 - u^2$. The lift-and-project inequality

$$u^1\frac{\tilde{b}-\tilde{A}x}{u\tilde{b}} + u^2\frac{\tilde{b}-\tilde{A}x}{1-u\tilde{b}} \geq 1 \tag{2.21}$$

is added to the formulation $A^k x \leq b^k$, and the process repeated.

For every $j \in I$, the cuts generated by solving (2.20) with respect to index $j$ are called *$j$-cuts*. In the remainder, we assume that $I = \{1, \dots, p\}$.

For any iteration $k$ and for $j = 1, \dots, p$, let $A^{k,j}x \leq b^{k,j}$ denote the system of linear inequalities comprising $Ax \leq b$ and all the $h$-cuts of $A^k x \leq b^k$ for $h = 1, \dots, j$. We define $A^{k,0} := A$, $b^{k,0} := b$. Let $P^{k,j} := \{x : A^{k,j}x \leq b^{k,j}\}$, $j = 0, \dots, p$. Note that $P^{k,0} = P$ and $P^{k,p} = P^k$.

**Specialized lift-and-project algorithm**

Start with $k := 0$.

1. Compute an optimal basic solution $\bar{x}$ of the linear program $\max\{cx : x \in P^k\}$.
2. If $\bar{x}_1, \dots, \bar{x}_p \in \{0,1\}$, then $\bar{x}$ is optimal for (2.19). Otherwise
3. Let $j \in I$ be the largest index such that $0 < \bar{x}_j < 1$. Let $\tilde{A} := A^{k,j-1}$ and $\tilde{b} := b^{k,j-1}$. Compute an optimal basic solution $(u^1, u^2)$ to the linear program (2.20), and add the $j$-cut (2.21) to $A^k x \leq b^k$.
4. Set $k := k+1$ and go to 1.

**Theorem 2.15.** *The specialized lift-and-project algorithm terminates after a finite number of iterations for every mixed* $0,1$ *linear program.*

*Proof.* The proof is in two steps.
i) We prove that, at each iteration $k$, the $j$-cut computed in Step 3 of the algorithm cuts off the solution $\bar{x}$ computed in Step 1. We first show that $\bar{x}$ is a vertex of $P^{k,j}$. Let $H := \{x : x_t = \bar{x}_t,\ t = j+1, \dots, p\}$. By the choice of $j$, $\bar{x}_t \in \{0,1\}$ for $t = j+1, \dots, p$, thus $P^k \cap H$ is a face of $P^k$. Let $(P^{k,j})_{j+1,\dots,p} := ((P^{k,j}_{j+1})_{j+2} \dots)_p$ and observe that $(P^{k,j})_{j+1,\dots,p} \subseteq P^k \subseteq P^{k,j}$. By Theorem 2.13, $(P^{k,j})_{j+1,\dots,p} = \mathrm{conv}(P^{k,j} \cap \{x : x_t \in \{0,1\},\ t = j+1, \dots, p\})$, therefore $(P^{k,j})_{j+1,\dots,p} \cap H = P^k \cap H = P^{k,j} \cap H$. Since $\bar{x}$ is a vertex of $P^k$, it follows that it is a vertex of $P^k \cap H = P^{k,j} \cap H$, and thus it is a vertex of $P^{k,j}$. Since $0 < \bar{x}_j < 1$, $\bar{x}$ is not a vertex of $(P^{k,j-1})_j$. Given that $\bar{x}$ is a vertex of $P^{k,j}$ and that $(P^{k,j-1})_j \subseteq P^{k,j}$, it follows that $\bar{x} \notin (P^{k,j-1})_j$. Thus the $j$-cut computed in Step 3 cuts off $\bar{x}$.
ii) We show that, for $j = 1, \dots, p$, the number of $j$-cuts generated is finite. Observe that no cut can be generated twice, since by i) at every iteration we cut off at least one vertex of the

current relaxation. Inductively, it suffices to show that, for $j = 1, \dots, p$ and an iteration $\bar{k}$ such that no $h$-cut with $h \le j-1$ is added after iteration $\bar{k}$, only a finite number of $j$-cuts are added after iteration $\bar{k}$. Indeed, for $k \ge \bar{k}$, $A^{k,j-1} = A^{\bar{k},j-1}$, thus every $j$-cut added after iteration $\bar{k}$ corresponds to a basic solution of the system $uA^{\bar{k},j-1} = e^j$. Since there are only a finite number of such vectors $u$, it follows that a finite number of $j$-cuts are added after iteration $\bar{k}$. □

## 2.4 Exercises

**Exercise 2.1.** Consider $S_1, S_2 \subseteq \mathbb{R}^n_+$. For $i = 1, 2$, let $\sum_{j=1}^n \alpha^i_j x_j \le \alpha^i_0$ be a valid inequality for $S_i$. Prove that $\sum_{j=1}^n \min(\alpha^1_j, \alpha^2_j) x_j \le \max(\alpha^1_0, \alpha^2_0)$ is a valid inequality for $S_1 \cup S_2$.

**Exercise 2.2.** Let $\Pi_1$ and $\Pi_2$ be defined as in (2.1). Assume $\Pi_1$ and $\Pi_2$ have $n^1_k$ and $n^2_k$ faces of dimension $k$ respectively, for $k = 1, \dots, n+p-2$. Give an upper bound on the number of facets of $\text{conv}(\Pi_1 \cup \Pi_2)$. Can you construct a family of polyhedra $P$ with $m$ constraints such that the number of facets of $\text{conv}(\Pi_1 \cup \Pi_2)$ grows more than linearly with $m$?

**Exercise 2.3.** Let $P := \{(x_1, x_2, y) \in \mathbb{R}^3 \,:\, x_1 \ge y,\ x_2 \ge y,\ x_1 + x_2 + 2y \le 2,\ y \ge 0\}$ and $S := P \cap (\mathbb{Z}^2 \times \mathbb{R})$. Prove that $P^{\text{split}} = \{(x_1, x_2, y) \in \mathbb{R}^3 \,:\, x_1 \ge 3y,\ x_2 \ge 3y,\ x_1 + x_2 + 2y \le 2,\ y \ge 0\}$.

**Exercise 2.4.** Let $S := \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p_+ \,:\, \sum_{j=1}^n a_j x_j + \sum_{j=1}^p g_j y_j \le b\}$ where $a_1, \dots, a_n \in \mathbb{Z}$ are not all equal to 0 and are relatively prime, $g_1, \dots, g_p \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \mathbb{Z}$. Let $f := b - \lfloor b \rfloor$ and $J^- := \{j \in \{1, \dots, p\} \,:\, g_j < 0\}$.

1. Prove that the inequality $\sum_{j=1}^n \lfloor a_j \rfloor x_j + \frac{1}{1-f} \sum_{j \in J^-} g_j y_j \le \lfloor b \rfloor$ is a valid for $S$.

2. Prove that the above inequality defines a facet of $\text{conv}(S)$.

**Exercise 2.5.** Consider the following mixed integer linear program

$$
\begin{array}{rrrrrrrrl}
z = \max & 7x_1 & +5x_2 & +x_3 & +y_1 & & & & \\
& x_1 & +3x_2 & & +4y_1 & +y_2 & & & = 11 \\
& 5x_1 & +x_2 & +3x_3 & & & +y_3 & & = 12 \\
& & & 2x_3 & +2y_1 & & & -y_4 & = 3 \\
& x_1, & x_2, & x_3 & \in \mathbb{Z}_+ & & & & \\
& y_1, & y_2, & y_3, & y_4 & \in \mathbb{R}_+. & & &
\end{array}
$$

The optimal tableau of the linear programming relaxation is:

$$
\begin{array}{lrrrrcr}
z & +0.357x_3 & +1.286y_2 & +1.143y_3 & +2.071y_4 & = & 21.643 \\
x_1 & +0.786x_3 & -0.071y_2 & +0.214y_3 & -0.143y_4 & = & 2.214 \\
x_2 & -0.929x_3 & +0.357y_2 & -0.071y_3 & +0.714y_4 & = & 0.929 \\
y_1 & +0.500x_3 & & & -0.500y_4 & = & 1.500
\end{array}
$$

1. The optimal linear programming solution is $x_1 = 2.214$, $x_2 = 0.929$, $y_1 = 1.5$ and $x_3 = y_2 = y_3 = y_4 = 0$. Use the equations where $x_1$ and $x_2$ are basic to derive two Gomory mixed integer inequalities that cut off this fractional solution.

2. The coefficients in the above optimal simplex tableau are rounded to three decimals. Discuss how this may affect the validity of the Gomory mixed integer inequalities you generated above.

**Exercise 2.6.** Show that the sequential convexification theorem does not extend to $0, 1, 2$ variables. Specifically, consider $P := \{x \in \mathbb{R}^{n+p}_+ : Ax \geq b\}$ and $S := \{x \in \{0, 1, 2\}^n \times \mathbb{R}^p_+ : Ax \geq b\}$. Let $P_j := \text{conv}\,((P \cap \{x_j = 0\}) \cup (P \cap \{x_j = 1\}) \cup (P \cap \{x_j = 2\}))$ for $j \leq n$. Give an example where $(P_1)_2 \neq \text{conv}(\{x \in \{0, 1, 2\}^2 \times \mathbb{R}^{n-2+p}_+ : Ax \geq b\})$.

Do we always have $(P_1)_2 = (P_2)_1$?

**Exercise 2.7.** Consider a pure integer set $S := P \cap \mathbb{Z}^n$ where $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ is a rational polyhedron. Define the *two-side-split closure* $S^1$ of $P$ as the intersection of all split inequalities that are not one-side split inequalities, i.e. they are obtained from split disjunctions $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ such that both $\Pi_1 := P \cap \{x : \pi x \leq \pi_0\}$ and $\Pi_2 := P \cap \{x : \pi x \geq \pi_0 + 1\}$ are nonempty. We can iterate the closure process to obtain the *tth two-side-split closure* $S^t$ for $t \geq 2$ integer, by taking the *two-side-split closure* of $S^{t-1}$. Using the following example, show that there is in general no finite $k$ such that $S^k = \text{conv}(S)$.

$S := P \cap \mathbb{Z}^2$ where $P := \{x \in \mathbb{R}^2 : x_1 \geq 0,\ x_2, \geq 0,\ x_2 \leq 1 + \frac{1}{4}x_1,\ x_1 \leq 1 + \frac{1}{4}x_2\}$.

**Exercise 2.8.** Show that the lift-and-project closure is strictly contained in $P$ whenever $P \neq \text{conv}(S)$.

**Exercise 2.9.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $S := P \cap (\{0, 1\}^p \times \mathbb{R}^{n-p})$. Show that the $k$th lift-and-project closure of $P$ is the set

$$\bigcap_{\substack{J \subseteq \{1,\ldots,p\} \\ |J| = k}} \text{conv}\{x \in P : x_j \in \{0, 1\} \text{ for } j \in J\}.$$

**Exercise 2.10.** Consider the polytope $P := \{x \in \mathbb{R}^n_+ : x_i + x_j \leq 1 \text{ for all } 1 \leq i < j \leq n\}$ and $S := P \cap \{0, 1\}^n$. Show that the $k$th lift-and-project closure of $P$ is equal to

$$\{x \in \mathbb{R}^n_+ : \sum_{j \in J} x_j \leq 1 \text{ for all } J \text{ such that } |J| = k + 2\}.$$

# Lecture 3

# Intersection cuts and corner polyhedra

In this lecture, we present two classical points of view for approximating a mixed integer linear set: Gomory's corner polyhedron and Balas' intersection cuts. It turns out that they are equivalent: the nontrivial valid inequalities for the corner polyhedron are exactly the intersection cuts. We show that the best possible intersection cuts are generated from maximal lattice-free convex sets.

## 3.1 Corner polyhedron

We consider a mixed integer linear set defined by the following constraints

$$\begin{array}{ll} Ax = b & \\ x_j \in \mathbb{Z} & \text{for } j = 1, \dots, p \\ x_j \geq 0 & \text{for } j = 1, \dots, n \end{array} \tag{3.1}$$

where $p \leq n$, $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ is a column vector. We assume that the matrix $A$ has full row rank $m$. Given a feasible basis $B$, let $N := \{1, \dots, n\} \setminus B$ index the nonbasic variables. We rewrite the system $Ax = b$ as

$$x_i \;=\; \bar{b}_i - \textstyle\sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for } i \in B \tag{3.2}$$

where $\bar{b}_i \geq 0$, $i \in B$. The corresponding basic solution is $\bar{x}_i = \bar{b}_i, i \in B$, $\bar{x}_j = 0, j \in N$. If $\bar{b}_i \in \mathbb{Z}$ for all $i \in B \cap \{1, \dots, p\}$, then $\bar{x}$ is a feasible solution to (3.1).

If this is not the case, we address the problem of finding valid inequalities for the set (3.1) that are violated by the point $\bar{x}$. Typically, $\bar{x}$ is an optimal solution of the LP relaxation of a mixed integer linear program having (3.1) as feasible set.

The key idea is to work with the corner polyhedron introduced by Gomory [39], which is obtained from (3.1) by dropping the nonnegativity restriction on all the basic variables $x_i$, $i \in B$. Note that in this relaxation we can drop the constraints $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for all $i \in B \cap \{p+1, \dots, n\}$ because these variables $x_i$ are continuous and they only appear in one equation and no other constraint (recall that we dropped the nonnegativity constraint on these variables). Therefore from now on we assume that all basic variables in (3.2) are integer variables, i.e. $B \subseteq \{1, \ldots, p\}$.

Under this assumption, the relaxation of (3.1) introduced by Gomory is

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1, \ldots, p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array} \tag{3.3}$$

The convex hull of the feasible solutions to (3.3) is called the *corner polyhedron* relative to the basis $B$ and it is denoted by corner$(B)$. Any valid inequality for the corner polyhedron is valid for the set (3.1).

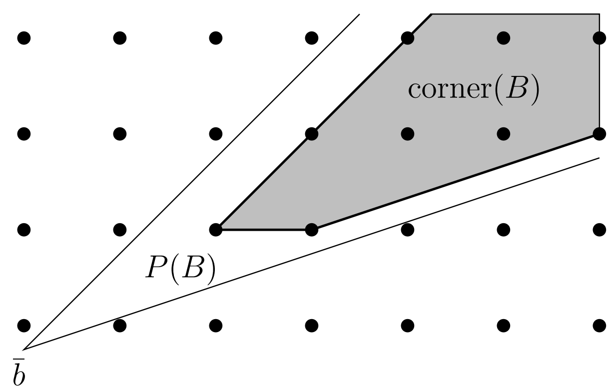


Figure 3.1: Corner polyhedron viewed in the space of the basic variables

Let $P(B)$ be the linear relaxation of (3.3). $P(B)$ is a polyhedron whose vertices and extreme rays are simple to describe, a property that will be useful in generating valid inequalities for corner$(B)$. Indeed, the point $\bar{x}$ defined by $\bar{x}_i = \bar{b}_i$, for $i \in B$, $\bar{x}_j = 0$, for $j \in N$, is the unique vertex of $P(B)$. In particular $P(B)$ is a translate of its recession cone, that is $P(B) = \{\bar{x}\} + \text{rec}(P(B))$. The recession cone of $P(B)$ is defined by the following linear system.

$$\begin{array}{rcll} x_i & = & -\sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Since the projection of this cone onto $\mathbb{R}^N$ is defined by the inequalities $x_j \geq 0$, $j \in N$, and variables $x_i$, $i \in B$, are defined by the above equations, its extreme rays are the vectors satisfying at equality all but one of the nonnegativity constraints. Thus there are $|N|$ extreme rays, $\bar{r}^j$ for $j \in N$, defined by

$$\bar{r}^j_h = \begin{cases} -\bar{a}_{hj} & \text{if } h \in B, \\ 1 & \text{if } h = j, \\ 0 & \text{if } h \in N \setminus \{j\}. \end{cases} \tag{3.4}$$

**Remark 3.1.** *The vectors $\bar{r}^j$, $j \in N$, are linearly independent. Hence $P(B)$ is an $|N|$-dimensional polyhedron whose affine hull is defined by the equations $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i \in B$.*

The rationality assumption of the matrix $A$ will be used in the proof of the next lemma.

**Lemma 3.2.** *If the affine hull of $P(B)$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then* corner$(B)$ *is an $|N|$-dimensional polyhedron. Otherwise* corner$(B)$ *is empty.*

*Proof.* Since corner$(B)$ is contained in the affine hull of $P(B)$, corner$(B)$ is empty when the affine hull of $P(B)$ contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

Next we assume that the affine hull of $P(B)$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and we show that corner$(B)$ is an $|N|$-dimensional polyhedron. We first show that corner$(B)$ is nonempty.

Let $x' \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ belong to the affine hull of $P(B)$. By Remark 3.1 $x'_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x'_j$ for $i \in B$.

Let $N^- := \{j \in N : x'_j < 0\}$. If $N^-$ is empty, then $x' \in$ corner$(B)$. Let $D \in \mathbb{Z}_+$ be such that $D\bar{a}_{ij} \in \mathbb{Z}$ for all $i \in B$ and $j \in N^-$. Define the point $x''$ as follows

$$x''_j := x'_j,\ j \in N \setminus N^-; \quad x''_j := x'_j - D\lfloor \frac{x'_j}{D} \rfloor,\ j \in N^-; \quad x''_i := \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x''_j,\ i \in B.$$

By construction, $x''_j \geq 0$ for all $j \in N$ and $x''_i$ is integer for $i = 1, \ldots, p$. Since $x''$ satisfies $x''_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x''_j$, $x''$ belongs to corner$(B)$. This shows that corner$(B)$ is nonempty.

The recession cones of $P(B)$ and corner$(B)$ coincide by Meyer's theorem, because $P(B)$ is a rational polyhedron. By Remark 3.1, this implies that the dimension of corner$(B)$ is $|N|$. □

**Example 3.3.** Consider the pure integer program

$$\begin{array}{rcl} \max \frac{1}{2}x_2 + x_3 & & \\ x_1 + x_2 + x_3 & \leq & 2 \\ x_1 - \frac{1}{2}x_3 & \geq & 0 \\ x_2 - \frac{1}{2}x_3 & \geq & 0 \\ x_1 + \frac{1}{2}x_3 & \leq & 1 \\ -x_1 + x_2 + x_3 & \leq & 1 \\ x_1, x_2, x_3 \in \mathbb{Z} & & \\ x_1, x_2, x_3 \geq 0. & & \end{array} \tag{3.5}$$

This problem has four feasible solutions $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$, all satisfying $x_3 = 0$. These four points are shown in the $(x_1, x_2)$-space in Figure 3.2.

We first write the problem in standard form (3.1) by introducing continuous slack or surplus variables $x_4, \ldots, x_8$. Solving the LP relaxation, we obtain

$$\begin{array}{rclllll} x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\ x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\ x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\ x_4 & = & 0 & -\frac{1}{2}x_6 & +\frac{3}{2}x_7 & +\frac{1}{2}x_8 \\ x_5 & = & 0 & +\frac{1}{2}x_6 & -\frac{1}{2}x_7 & +\frac{1}{2}x_8 \end{array}$$

The optimal basic solution is $x_1 = x_2 = \frac{1}{2}$, $x_3 = 1$, $x_4 = \ldots = x_8 = 0$.

Relaxing the nonnegativity of the basic variables and dropping the two constraints relative to the continuous basic variables $x_4$ and $x_5$, we obtain the following realization of formulation (3.3) for this example:

$$\begin{array}{rcllll} x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\ x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\ x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\ & & \multicolumn{4}{l}{x_1, x_2, x_3 \in \mathbb{Z}} \\ & & \multicolumn{4}{l}{x_6, x_7, x_8 \geq 0.} \end{array} \tag{3.6}$$

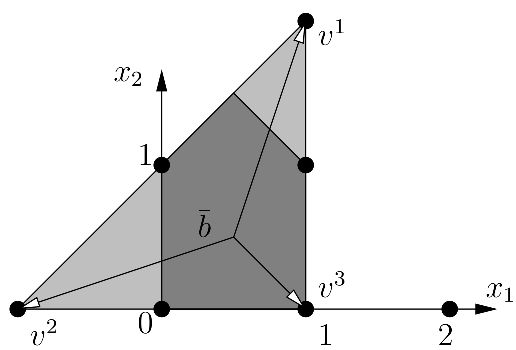


Figure 3.2: Intersection of the corner polyhedron with the plane $x_3 = 0$

Let $P(B)$ be the linear relaxation of (3.6). The projection of $P(B)$ in the space of original variables $x_1, x_2, x_3$ is a polyhedron with unique vertex $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. The extreme rays of its recession cone are $v^1 = (\frac{1}{2}, \frac{3}{2}, -1)$, $v^2 = (-\frac{3}{2}, -\frac{1}{2}, -1)$ and $v^3 = (\frac{1}{2}, -\frac{1}{2}, -1)$. In Figure 3.2, the shaded region (both light and dark) is the intersection of $P(B)$ with the plane $x_3 = 0$.

The last equation in (3.6) and the facts that $x_6+x_7+x_8 > 0$ and $x_3 \in \mathbb{Z}$ in every solution of (3.6) imply that $x_3 \leq 0$ is a valid inequality for corner$(B)$. In fact, corner$(B)$ is exactly the intersection of $P(B)$ with $x_3 \leq 0$ since this later polyhedron has integral vertices and the same recession cone as $P(B)$. Therefore corner$(B)$ is entirely defined by the inequalities $x_3 \leq 0$ and $x_6, x_7, x_8 \geq 0$. Equivalently, in the original $(x_1, x_2, x_3)$-space, corner$(B)$ is entirely defined by $x_3 \leq 0$, $x_2 - \frac{1}{2}x_3 \geq 0$, $x_1 + \frac{1}{2}x_3 \leq 1$, $-x_1 + x_2 + x_3 \leq 1$. In Figure 3.2, the shaded region (both light and dark) is therefore also the intersection of corner$(B)$ with the plane $x_3 = 0$.

Let $P$ be the polyhedron defined by the inequalities of (3.5) that are satisfied at equality by the point $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. The intersection of $P$ with the plane $x_3 = 0$ is the dark shaded region in Figure 3.2. Thus $P$ is strictly contained in $P(B)$. ■

Using the fact that every basic variable is a linear combination of nonbasic ones, note that every valid linear inequality for corner$(B)$ can be written in terms of the nonbasic variables $x_j$ for $j \in N$ only, as $\sum_{j\in N} \gamma_j x_j \geq \delta$. We say that a valid inequality $\sum_{j\in N} \gamma_j x_j \geq \delta$ for corner$(B)$ is *trivial* if it is implied by the nonnegativity constraints $x_j \geq 0$, $j \in N$. This is the case if and only if $\gamma_j \geq 0$ for all $j \in N$ and $\delta \leq 0$. A valid inequality is said to be *nontrivial* otherwise.

**Lemma 3.4.** *Assume* corner$(B)$ *is nonempty. Every nontrivial valid inequality for* corner$(B)$ *can be written in the form* $\sum_{j\in N} \gamma_j x_j \geq 1$ *where* $\gamma_j \geq 0$ *for all* $j \in N$.

*Proof.* We already observed that every valid linear inequality for corner($B$) can be written as $\sum_{j\in N}\gamma_j x_j \geq \delta$. We argue next that $\gamma_j \geq 0$ for all $j \in N$. Indeed, if $\gamma_k < 0$ for some $k \in N$, then consider $\bar{r}^k$ defined in (3.4). We have $\sum_{j\in N}\gamma_j \bar{r}^k_j = \gamma_k < 0$, hence $\min\{\sum_{j\in N}\gamma_j x_j \,:\, x \in \text{corner}(B)\}$ is unbounded, because $\bar{r}^k$ is in the recession cone of corner($B$), contradicting the fact that $\sum_{j\in N}\gamma_j x_j \geq \delta$ is valid for corner($B$).

If $\delta \leq 0$, the inequality $\sum_{j\in N}\gamma_j x_j \geq \delta$ is trivial since it is implied by the nonnegativity constraints $x_j \geq 0,\ j \in N$. Hence $\delta > 0$ and, up to multiplying by $\delta^{-1}$, we may assume that $\delta = 1$. □

Since the variables $x_i,\ i \in B$, are free integer variables, (3.3) can be reformulated as follows

$$\begin{array}{rcll} \sum_{j\in N}\bar{a}_{ij}x_j & \equiv & \bar{b}_i \mod 1 & \text{for } i \in B \\ x_j & \in & \mathbb{Z} & \text{for } j \in \{1,\ldots,p\}\cap N \\ x_j & \geq & 0 & \text{for } j \in N. \end{array} \tag{3.7}$$

This point of view was introduced by Gomory and extensively studied by Gomory and Johnson. We will discuss it in Lecture 4.

## 3.2 Intersection cuts

We describe a paradigm introduced by Balas [3] for constructing valid inequalities for the corner polyhedron cutting off the basic solution $\bar{x}$.

Consider a closed convex set $C \subseteq \mathbb{R}^n$ such that the interior of $C$ contains the point $\bar{x}$. (Recall that $\bar{x}$ belongs to the interior of $C$ if $C$ contains an $n$-dimensional ball centered at $\bar{x}$. This implies that $C$ is full-dimensional). Assume that the interior of $C$ contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. In particular $C$ does not contain any feasible point of (3.3) in its interior. For each of the $|N|$ extreme rays of corner($B$), define

$$\alpha_j := \max\{\alpha \geq 0 \,:\, \bar{x} + \alpha\bar{r}^j \in C\}. \tag{3.8}$$

Since $\bar{x}$ is in the interior of $C$, $\alpha_j > 0$. When the half-line $\{\bar{x} + \alpha\bar{r}^j \,:\, \alpha \geq 0\}$ intersects the boundary of $C$, then $\alpha_j$ is finite, the point $\bar{x} + \alpha_j\bar{r}^j$ belongs to the boundary of $C$ and the semi-open segment $\{\bar{x} + \alpha\bar{r}^j,\ 0 \leq \alpha < \alpha_j\}$ is contained in the interior of $C$. When $\bar{r}_j$ belongs the recession cone of $C$, we have $\alpha_j = +\infty$. Define $\frac{1}{+\infty} := 0$. The inequality

$$\sum_{j\in N}\frac{x_j}{\alpha_j} \geq 1 \tag{3.9}$$

is the *intersection cut* defined by $C$ for corner($B$).

**Theorem 3.5.** *Let $C \subset \mathbb{R}^n$ be a closed convex set whose interior contains the point $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. The intersection cut (3.9) defined by $C$ is a valid inequality for* corner($B$).

*Proof.* The set of points of the linear relaxation $P(B)$ of corner$(B)$ that are cut off by (3.9) is $S := \{x \in P(B) : \sum_{j\in N} \frac{x_j}{\alpha_j} < 1\}$. We will show that $S$ is contained in the interior of $C$. Since the interior of $C$ does not contain a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, the result will follow.

Consider polyhedron $\bar{S} := \{x \in P(B) : \sum_{j\in N} \frac{x_j}{\alpha_j} \leq 1\}$. By Remark 3.1, $\bar{S}$ is a $|N|$-dimensional polyhedron with vertices $\bar{x}$ and $\bar{x} + \alpha_j \bar{r}^j$ for $\alpha_j$ finite, and extreme rays $\bar{r}^j$ for $\alpha_j = +\infty$. Since the vertices of $\bar{S}$ that lie on the hyperplane $\{x \in \mathbb{R}^n : \sum_{j\in N} \frac{x_j}{\alpha_j} = 1\}$ are the points $\bar{x} + \alpha_j \bar{r}^j$ for $\alpha_j$ finite, every point in $S$ can be expressed as a convex combination of points in the segments $\{\bar{x} + \alpha \bar{r}^j,\ 0 \leq \alpha < \alpha_j\}$ for $\alpha_j$ finite, plus a conic combination of extreme rays $\bar{r}^j$, for $\alpha_j = +\infty$. By the definition of $\alpha_j$, the interior of $C$ contains the segments $\{\bar{x} + \alpha \bar{r}^j,\ 0 \leq \alpha < 1\}$ when $\alpha_j$ is finite, and the rays $\bar{r}^j$ belong to the recession cone of $C$ when $\alpha_j = +\infty$. Therefore, the set $S$ is contained in the interior of $C$. □

The intersection cut has a simple geometric interpretation. Denoting the interior of $C$ by int$(C)$, it follows that $P(B) \setminus \text{int}(C)$ contains all points of $P(B) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. If we define $Q$ to be the closed convex hull of $P(B) \setminus \text{int}(C)$, then corner$(B) \subseteq Q$. One can show that $Q$ is also a polyhedron, and indeed

$$Q = \{x \in P(B) : \sum_{j\in N} \frac{x_j}{\alpha_j} \geq 1\},$$

where $\alpha_j$, $j \in N$, are defined as in (3.8) (see Exercise 3.4). In other words, the intersection cut is the only inequality one needs to add to the description of $P(B)$ in order to obtain $Q$.

Note that Corollary 2.7 and Remark 2.8 imply that split inequalities are a special case of intersection cuts, where the convex set $C$ is of the form $C = \{x : \pi_0 \leq \pi x \leq \pi_0 + 1\}$ for some split $(\pi, \pi_0)$.

Consider two valid inequalities $\sum_{j\in N} \gamma_j x_j \geq 1$ and $\sum_{j\in N} \gamma'_j x_j \geq 1$ for corner$(B)$. We say that the first inequality *dominates* the second if every point $x \in \mathbb{R}^n_+$ satisfying the second inequality also satisfies the first. Note that $\sum_{j\in N} \gamma_j x_j \geq 1$ dominates $\sum_{j\in N} \gamma'_j x_j \geq 1$ if and only if $\gamma_j \leq \gamma'_j$ for all $j \in N$.

**Remark 3.6.** *Let $C_1$, $C_2$ be two closed convex sets whose interiors contain $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. If $C_1$ is contained in $C_2$, then the intersection cut defined by $C_2$ dominates the intersection cut defined by $C_1$.*

A closed convex set $C$ whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is *maximal* if $C$ is not strictly contained in a closed convex set with the same properties. Any closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is contained in a maximal such set [12]. This property and Remark 3.6 imply that it is enough to consider intersection cuts defined by maximal closed convex sets whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

A set $K \subset \mathbb{R}^p$ that contains no point of $\mathbb{Z}^p$ in its interior is called *$\mathbb{Z}^p$-free* or *lattice-free*.

**Remark 3.7.** *One way of constructing a closed convex set $C$ whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is as follows. In the space $\mathbb{R}^p$, construct a $\mathbb{Z}^p$-free closed convex set $K$ whose interior contains the orthogonal projection of $\bar{x}$ onto $\mathbb{R}^p$. The cylinder $C = K \times \mathbb{R}^{n-p}$ is a closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.*

**Example 3.8.** Consider the following 4-variable mixed integer linear set

$$\begin{aligned} x_1 &= b_1 + a_{11}y_1 + a_{12}y_2 \\ x_2 &= b_2 + a_{21}y_1 + a_{22}y_2 \\ x &\in \mathbb{Z}^2 \\ y &\geq 0 \end{aligned} \tag{3.10}$$

where the rays $r^1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, r^2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \in \mathbb{R}^2$ are not collinear and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \notin \mathbb{Z}^2$.

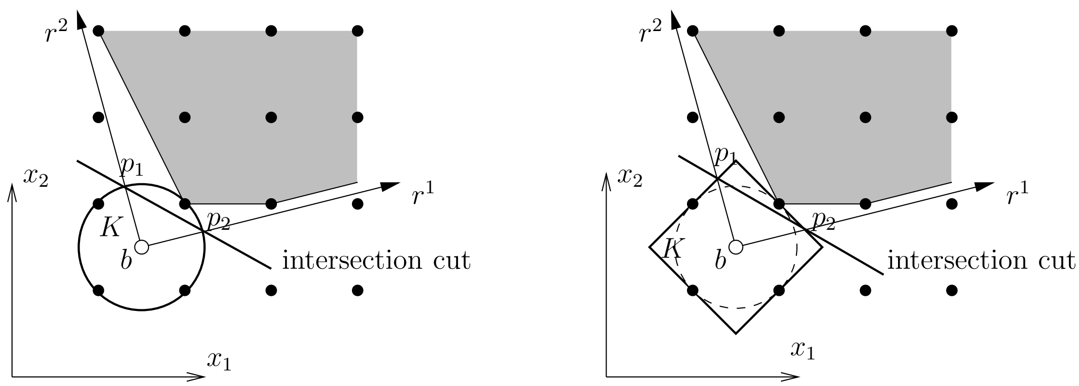


Figure 3.3: Intersection cuts determined by lattice-free convex sets.

Figure 3.3 represents the projection of the feasible region of (3.10) in the space of the variables $x_1$, $x_2$. The set of feasible points $x \in \mathbb{R}^2$ for the linear relaxation of (3.10) is the cone with apex $b$ and extreme rays $r^1$, $r^2$. The feasible points $x \in \mathbb{Z}^2$ for (3.10) are represented by the black dots in this cone. The shaded region represents the projection of the corner polyhedron in the $(x_1, x_2)$-space. The figure depicts two examples of lattice-free convex sets $K \subset \mathbb{R}^2$ containing $b$ in their interior, a disk in the left example and a square that contains this disk on the right.

Because there are two nonbasic variables in this example, the intersection cut can be represented by a line in the space of the basic variables, namely the line passing through the intersection points $p^1$, $p^2$ of the boundary of $K$ with the half lines $\{b + \alpha r^1 \, : \, \alpha \geq 0\}$, $\{b + \alpha r^2 \, : \, \alpha \geq 0\}$.

The coefficients $\alpha_1$, $\alpha_2$ defining the intersection cut $\frac{y_1}{\alpha_1} + \frac{y_2}{\alpha_2} \geq 1$ are $\alpha_j = \frac{\|p^j - b\|}{\|r^j\|}$, $j = 1, 2$, using the definition of $\alpha_j$ in (3.8). Note that the intersection cut on the right dominates the one on the left, as observed in Remark 3.6, because the lattice-free set on the right contains the one on the left. ■

**Example 3.9.** (Intersection cut defined by a split)

Given $\pi \in \mathbb{Z}^p$ and $\pi_0 \in \mathbb{Z}$, let $K := \{x \in \mathbb{R}^p \, : \, \pi_0 \leq \pi x \leq \pi_0 + 1\}$. The set $K$ is a $\mathbb{Z}^p$-free convex set since either $\pi\bar{x} \leq \pi_0$ or $\pi\bar{x} \geq \pi_0 + 1$, for any $\bar{x} \in \mathbb{Z}^p$. Furthermore it is easy to verify that if the entries of $\pi$ are relatively prime, both hyperplanes $\{x \in \mathbb{R}^p \, : \, \pi x = \pi_0\}$ and $\{x \in \mathbb{R}^p \, : \, \pi x = \pi_0 + 1\}$ contain points in $\mathbb{Z}^p$. Therefore $K$ is a maximal $\mathbb{Z}^p$-free convex set in this case. Consider the cylinder $C := K \times \mathbb{R}^{n-p} = \{x \in \mathbb{R}^n \, : \, \pi_0 \leq \sum_{j=1}^p \pi_j x_j \leq \pi_0 + 1\}$.

Such a set $C$ is called a *split set*. By Remark 3.7, $C$ is a convex set whose interior contains no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

Given a corner polyhedron corner$(B)$, let $\bar{x}$ be the unique vertex of its linear relaxation $P(B)$. If $\bar{x}_j \notin \mathbb{Z}$ for some $j = 1, \dots, p$, there exist $\pi \in \mathbb{Z}^p$, $\pi_0 \in \mathbb{Z}$ such that $\pi_0 < \sum_{j=1}^p \pi_j \bar{x}_j < \pi_0 + 1$. Let $\pi_j := 0$ for $j = p+1, \dots, n$. Then the split set $C$ defined above contains $\bar{x}$ in its interior. We apply formula (3.8) to $C$. Define $\epsilon := \pi\bar{x} - \pi_0$. Since $\pi_0 < \pi\bar{x} < \pi_0 + 1$, we have $0 < \epsilon < 1$. Also, for $j \in N$, define scalars:

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi\bar{r}^j} & \text{if } \pi\bar{r}^j < 0, \\ \frac{1-\epsilon}{\pi\bar{r}^j} & \text{if } \pi\bar{r}^j > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.11}$$

where $\bar{r}^j$ is defined in (3.4).

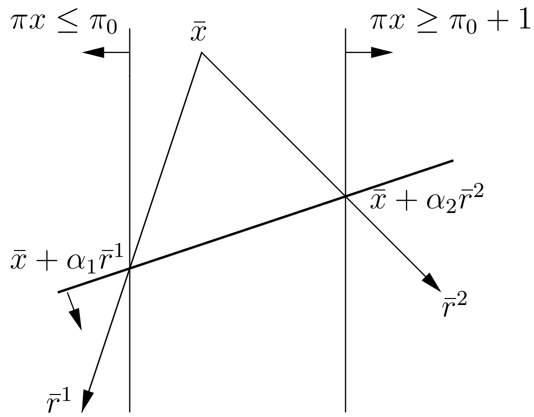


Figure 3.4: Intersection cut defined by a split set.

As observed earlier, the interpretation of $\alpha_j$ is the following. Consider the half-line $\bar{x} + \alpha\bar{r}^j$, where $\alpha \geq 0$, starting from $\bar{x}$ in the direction $\bar{r}^j$. The value $\alpha_j$ is the largest $\alpha \geq 0$ such that $\bar{x} + \alpha\bar{r}^j$ belongs to $C$. In other words, when the above half-line intersects one of the hyperplanes $\pi x = \pi_0$ or $\pi x = \pi_0 + 1$, this intersection point $\bar{x} + \alpha_j \bar{r}^j$ defines $\alpha_j$ (see Figure 3.4) and when the direction $\bar{r}^j$ is parallel to the hyperplane $\pi x = \pi_0$, $\alpha_j = +\infty$. The intersection cut defined by the split set $C$ is given by:

$$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1. \tag{3.12}$$

■

**Example 3.10.** (Gomory's mixed integer cuts from the tableau)

We already mentioned that split cuts are intersection cuts. We can interpret the formula of a Gomory mixed integer cut derived from a row of the simplex tableau (3.2) in the context of an intersection cut defined by a split set. The argument is as follows. Consider a simplex tableau (3.2), the corresponding basic solution $\bar{x}$, and the corner polyhedron corner$(B)$ described by the system (3.3). Let $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ be an equation where $\bar{b}_i$ is fractional.

Let $f := \bar{b}_i - \lfloor \bar{b}_i \rfloor$ and $f_j := \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$ for $j \in N \cap \{1, \ldots, p\}$. Define $\pi_0 := \lfloor \bar{b}_i \rfloor$, and for $j = 1, \ldots, p$, define

$$\pi_j := \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in N \text{ and } f_j \le f, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in N \text{ and } f_j > f, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

For $j = p+1, \ldots, n$, define $\pi_j := 0$. Note that $\pi_0 < \pi \bar{x} < \pi_0 + 1$.

Next we derive the intersection cut from the split set $C := \{x \in \mathbb{R}^n : \pi_0 \le \pi x \le \pi_0 + 1\}$ following Example 3.9. We will compute $\alpha_j$ for $j \in N$ using formula (3.11). To do this, we need to compute $\epsilon$ and $\pi \bar{r}^j$.

$$\epsilon = \pi \bar{x} - \pi_0 = \sum_{i \in B} \pi_i \bar{x}_i - \pi_0 = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f.$$

Let $j \in N$. Using (3.4) and (3.13), we get $\pi \bar{r}^j = \sum_{h \in N} \pi_h \bar{r}^j_h + \sum_{h \in B} \pi_h \bar{r}^j_h = \pi_j \bar{r}^j_j + \pi_i \bar{r}^j_i$ because $\bar{r}^j_h = 0$ for all $h \in N \setminus \{j\}$ and $\pi_h = 0$ for all $h \in B \setminus \{i\}$. This gives us

$$\pi \bar{r}^j = \begin{cases} \lfloor \bar{a}_{ij} \rfloor - \bar{a}_{ij} & = -f_j & \text{if } 1 \le j \le p \text{ and } f_j \le f, \\ \lceil \bar{a}_{ij} \rceil - \bar{a}_{ij} & = 1 - f_j & \text{if } 1 \le j \le p \text{ and } f_j > f, \\ -\bar{a}_{ij} & & \text{if } j \ge p+1. \end{cases} \tag{3.14}$$

Now $\alpha_j$ follows from formula (3.11). Therefore the intersection cut (3.12) defined by the split set $C$ is

$$\sum_{\substack{j \in N,\, j \le p \\ f_j \le f}} \frac{f_j}{f} x_j + \sum_{\substack{j \in N,\, j \le p \\ f_j > f}} \frac{1 - f_j}{1 - f} x_j + \sum_{\substack{p+1 \le j \le n \\ \bar{a}_{ij} > 0}} \frac{\bar{a}_{ij}}{f} x_j - \sum_{\substack{p+1 \le j \le n \\ \bar{a}_{ij} < 0}} \frac{\bar{a}_{ij}}{1 - f} x_j \ge 1. \tag{3.15}$$

This is exactly the Gomory mixed integer cut (2.18).

The Gomory formula looks complicated, and it may help to think of it as an inequality of the form

$$\sum_{j=1}^{p} \pi(-\bar{a}_{ij}) x_j + \sum_{j=p+1}^{n} \psi(-\bar{a}_{ij}) x_j \ge 1$$

where the functions $\pi$ and $\psi$, associated with the integer and continuous variables, respectively, are defined by

$$\pi(r) := \min\left\{ \frac{r - \lfloor r \rfloor}{1 - f}, \frac{1 + \lfloor r \rfloor - r)}{f} \right\}, \quad \psi(r) := \max\left\{ \frac{r}{1 - f}, \frac{-r}{f} \right\}. \tag{3.16}$$

These two functions produce the Gomory mixed integer cut. Section 4.3 studies properties that general functions $\pi$ and $\psi$ must satisfy in order to produce valid inequalities for corner$(B)$. ■

The next example shows that intersection cuts can be much stronger than split inequalities.

**Example 3.11.** (Intersection cuts can have an arbitrarily large split rank)

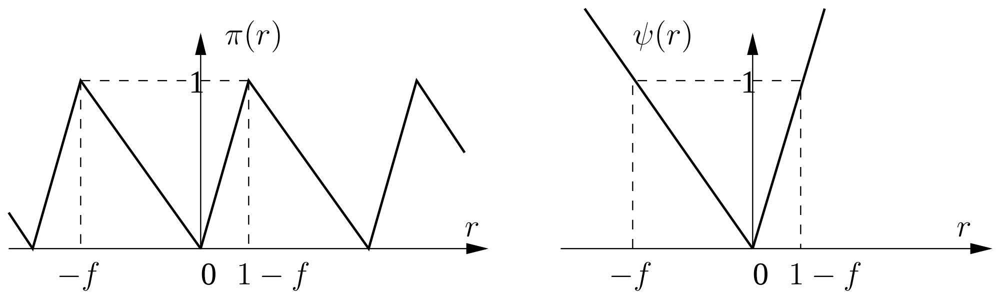


Figure 3.5: Gomory functions

We refer the reader to Section 2.1.2 for the definition of split rank of a valid inequality. Consider the polytope $P := \{(x_1, x_2, y) \in \mathbb{R}^3_+ \,:\, x_1 \geq y,\ x_2 \geq y,\ x_1 + x_2 + 2y \leq 2\}$, and let $S := \{(x_1, x_2, y) \in P \,:\, x_1, x_2 \in \mathbb{Z}\}$. Example 2.9 shows that the inequality $y \leq 0$ does not have a finite split rank. We show next that $y \leq 0$ can be obtained as an intersection cut. By adding slack or surplus variables, the system defining $P$ is equivalent to

$$\begin{array}{rcl} -x_1 + y + s_1 & = & 0 \\ -x_2 + y + s_2 & = & 0 \\ x_1 + x_2 + 2y + s_3 & = & 2 \\ x_1, x_2, y, s_1, s_2, s_3 & \geq & 0. \end{array}$$

The tableau relative to the basis $B$ defining the vertex $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $y = \frac{1}{2}$, $s_1 = s_2 = s_3 = 0$ is

$$\begin{array}{rcl} x_1 & = & \frac{1}{2} + \frac{3}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ x_2 & = & \frac{1}{2} - \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{4}s_3 \\ y & = & \frac{1}{2} - \frac{1}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ & & x_1, x_2, y, s_1, s_2, s_3 \geq 0. \end{array}$$

Since $y$ is a continuous basic variable, we drop the corresponding tableau row. The corner polyhedron corner$(B)$ is the convex hull of the points satisfying

$$\begin{array}{rcl} x_1 & = & \frac{1}{2} + \frac{3}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ x_2 & = & \frac{1}{2} - \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{4}s_3 \\ & & s_1, s_2, s_3 \geq 0 \\ & & x_1, x_2 \in \mathbb{Z}. \end{array}$$

The extreme rays of corner$(B)$ are the vectors $(\frac{3}{4}, -\frac{1}{4}, 1, 0, 0)$, $(-\frac{1}{4}, \frac{3}{4}, 0, 1, 0)$ and $(-\frac{1}{4}, -\frac{1}{4}, 0, 0, 1)$. Let $K$ be the triangle conv$\{(0,0), (2,0), (0,2)\}$, and $C := K \times \mathbb{R}^3$. One can readily observe that $K$ is lattice-free. We may therefore consider the intersection cut defined by $C$. The largest $\alpha$ such that $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0) + \alpha(\frac{3}{4}, -\frac{1}{4}, 1, 0, 0)$ belongs to $C$ is $\alpha_1 = 2$, the largest $\alpha$ such that $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0) + \alpha(-\frac{1}{4}, \frac{3}{4}, 0, 1, 0)$ belongs to $C$ is $\alpha_2 = 2$ and the largest $\alpha$ such that $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0) + \alpha(-\frac{1}{4}, -\frac{1}{4}, 0, 0, 1)$ belongs to $C$ is $\alpha_3 = 2$. The intersection cut defined by $C$ is therefore $\frac{1}{2}s_1 + \frac{1}{2}s_2 + \frac{1}{2}s_3 \geq 1$. Since $y = \frac{1}{2} - \frac{1}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3$, the intersection cut is equivalent to $y \leq 0$. Adding this single inequality to the initial formulation, we obtain conv$(S)$. But, as mentioned above, the intersection cut $y \leq 0$ does not have finite split rank.

Dey and Louveaux [32] study the split rank of intersection cuts for problems with two integer variables. Surprisingly, they show that all intersection cuts have finite split rank except for the ones defined by lattice-free triangles with integral vertices and an integral point in the middle of each side. These triangles are all unimodular transformations of the triangle $K$ defined above. ■

Theorem 3.5 shows that intersection cuts are valid for corner($B$). The following theorem provides a converse statement, namely that corner($B$) is completely defined by the trivial inequalities and intersection cuts. We assume here that corner($B$) is nonempty.

**Theorem 3.12.** *Every nontrivial facet-defining inequality for* corner($B$) *is an intersection cut.*

*Proof.* We prove the theorem in the pure integer case, that is, when $p = n$ (see [25] for the general case). Consider a nontrivial valid inequality for corner($B$). By Lemma 3.4 it is of the form $\sum_{j\in N} \gamma_j x_j \geq 1$. We show that it is an intersection cut.

Consider the polyhedron $S := P(B) \cap \{x \in \mathbb{R}^n : \sum_{j\in N} \gamma_j x_j \leq 1\}$. Since $\sum_{j\in N} \gamma_j x_j \geq 1$ is a valid inequality for corner($B$), all points of $\mathbb{Z}^n \cap S$ satisfy $\sum_{j\in N} \gamma_j x_j = 1$.

Since $P(B)$ is a rational polyhedron, $P(B) = \{x \in \mathbb{R}^n : Cx \leq d\}$ for some integral matrix $C$ and vector $d$. Let

$$T := \{x \in \mathbb{R}^n : Cx \leq d + 1, \sum_{j\in N} \gamma_j x_j \leq 1\}.$$

We first show that $T$ is a $\mathbb{Z}^n$-free convex set. Assume that the interior of $T$ contains an integral point $\tilde{x}$. That is, $\tilde{x}$ satisfies all inequalities defining $T$ strictly. Since $Cx \leq d+1$ is an integral system, then $C\tilde{x} \leq d$ and $\sum_{j\in N} \gamma_j \tilde{x}_j < 1$. This contradicts the fact that all points of $\mathbb{Z}^n \cap S$ satisfy $\sum_{j\in N} \gamma_j x_j = 1$.

Since the basic solution $\bar{x}$ belongs to $S$ and $\sum_{j\in N} \gamma_j \bar{x}_j = 0$, $T$ is a $\mathbb{Z}^n$-free convex set containing $\bar{x}$ in its interior. Note that the intersection cut defined by $T$ is $\sum_{j\in N} \gamma_j \bar{x}_j \geq 1$. □

### 3.2.1 The gauge function

Intersection cuts have a nice description in the language of convex analysis. Let $K \subseteq \mathbb{R}^n$ be a closed, convex set with the origin in its interior. A standard concept in convex analysis [43, 53] is that of *gauge* (also known as Minkowski function), which is the function $\gamma_K$ defined by

$$\gamma_K(r) := \inf\{t > 0 : \frac{r}{t} \in K\}, \quad \text{for all } r \in \mathbb{R}^n.$$

Since the origin is in the interior of $K$, $\gamma_K(r) < +\infty$ for all $r \in \mathbb{R}^n$. Furthermore $\gamma_K(r) \leq 1$ if and only if $r \in K$ (Exercise 3.5).

The coefficients $\alpha_j$ of the intersection cut defined in (3.8) can be expressed in terms of the gauge of $K := C - \bar{x}$, namely $\frac{1}{\alpha_j} = \gamma_K(\bar{r}^j)$.

**Remark 3.13.** *The intersection cut defined by a $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set $C$ is precisely $\sum_{j\in N} \gamma_K(\bar{r}^j) x_j \geq 1$, where $K := C - \bar{x}$.*

Next we discuss some important properties of the gauge function. A function $g : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $g(r^1) + g(r^2) \geq g(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$. The function $g$ is *positively homogeneous* if $g(\lambda r) = \lambda g(r)$ for every $r \in \mathbb{R}^n$ and every $\lambda > 0$. The function $g$ is *sublinear* if it is both subadditive and positively homogeneous.

Note that if $g : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous, then $g(0) = 0$. Indeed, for any $\lambda > 0$, we have that $g(0) = g(\lambda 0) = \lambda g(0)$, which implies that $g(0) = 0$.

**Lemma 3.14.** *Given a closed convex set $K$ with the origin in its interior, the gauge $\gamma_K$ is a nonnegative sublinear function.*

*Proof.* It follows from the definition of gauge that $\gamma_K$ is positively homogeneous and nonnegative. Since $K$ is a closed convex set, $\gamma_K$ is a convex function. We now show that $\gamma_K$ is subadditive. We have that $\gamma_K(r^1)+\gamma_K(r^2) \geq 2\gamma_K(\frac{r^1+r^2}{2}) = \gamma_K(r^1+r^2)$, where the inequality follows by convexity and the equality follows by positive homogeneity. □

**Lemma 3.15.** *Every sublinear function $g : \mathbb{R}^n \to \mathbb{R}$ is convex, and therefore continuous.*

*Proof.* Let $g$ be a sublinear function. The convexity of $g$ follows from $\frac{1}{2}(g(r^1) + g(r^2)) = g(\frac{r^1}{2}) + g(\frac{r^2}{2}) \geq g(\frac{r^1+r^2}{2})$ for every $r^1, r^2 \in \mathbb{R}^n$, where the equality follows by positive homogeneity and the inequality by subadditivity. Every convex function is continuous, see e.g. Rockafellar [53]. □

**Theorem 3.16.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative sublinear function and let $K := \{r \in \mathbb{R}^n : g(r) \leq 1\}$. Then $K$ is a closed convex set with the origin in its interior and $g$ is the gauge of $K$.*

*Proof.* By Lemma 3.15, $g$ is continuous and convex, therefore $K$ is a closed convex set. Since the interior of $K$ is $\{r \in \mathbb{R}^n : g(r) < 1\}$ and $g(0) = 0$, the origin is in the interior of $K$.

Let $r \in \mathbb{R}^n$. We need to show that $g(r) = \gamma_K(r)$. If the half-line $\{\alpha r : \alpha \geq 0\}$ intersects the boundary of $K$, let $\alpha^* > 0$ be such that $g(\alpha^* r) = 1$. Since $g$ is positively homogeneous, $g(r) = \frac{1}{\alpha^*} = \inf\{t > 0 : \frac{r}{t} \in K\} = \gamma_K(r)$. If $\{\alpha r : \alpha \geq 0\}$ does not intersect the boundary of $K$, then, since $g$ is positively homogeneous, $g(\alpha r) = \alpha g(r) \leq 1$ for all $\alpha > 0$, therefore $g(r) = 0$ because $g$ is nonnegative. Hence $g(r) = 0 = \inf\{t > 0 : \frac{r}{t} \in K\} = \gamma_K(r)$. □

### 3.2.2 Maximal lattice-free convex sets

For a good reference on lattices and convexity, we recommend Barvinok [10]. Here we will only work with the integer lattice $\mathbb{Z}^p$. By Remark 3.6, the undominated intersection cuts are the ones defined by full-dimensional *maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex sets* in $\mathbb{R}^n$, that is, full-dimensional subsets of $\mathbb{R}^n$ that are convex, have no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in their interior, and are inclusion maximal with respect to these two properties.

**Lemma 3.17.** *Let $C$ be a full-dimensional maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set and let $K$ be its orthogonal projection onto $\mathbb{R}^p$. Then $K$ is a maximal $\mathbb{Z}^p$-free convex set and $C = K \times \mathbb{R}^{n-p}$.*

*Proof.* A classical result in convex analysis implies that the interior of $K$ is the orthogonal projection onto $\mathbb{R}^p$ of the interior of $C$ (see Theorem 6.6 in Rockafellar [53]). Since $C$ is a $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set, it follows that $K$ is a $\mathbb{Z}^p$-free convex set. Let $K'$ be a maximal

$\mathbb{Z}^p$-free convex set containing $K$. Then the set $K' \times \mathbb{R}^{n-p}$ is a $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set and $C \subseteq K \times \mathbb{R}^{n-p} \subseteq K' \times \mathbb{R}^{n-p}$. Since $C$ is maximal, these three sets coincide and the result follows. □

The above lemma shows that it suffices to study $\mathbb{Z}^p$-free convex sets. Next we state a characterization of lattice-free sets due to Lovász [49]. We recall that the *relative interior* of a set $S \subseteq \mathbb{R}^n$ is the set of all points $x \in S$ for which there exists a ball $B \subseteq \mathbb{R}^n$ centered at $x$ such that $B \cap \text{aff}(S)$ is contained in $S$.

**Theorem 3.18** (Lovász [49])**.** *Let $K \subset \mathbb{R}^p$ be a full-dimensional set. Then $K$ is a maximal lattice-free convex set if and only if $K$ is a polyhedron that does not contain any point of $\mathbb{Z}^p$ in its interior and there is at least one point of $\mathbb{Z}^p$ in the relative interior of each facet of $K$.*

*Furthermore, if $K$ is a maximal lattice-free convex set, then* $\text{rec}(K) = \text{lin}(K)$.

We prove the theorem under the assumption that $K$ is a bounded set. A complete proof of the above theorem appears in [12].

*Proof of Theorem 3.18 in the bounded case.* Let $K$ be a maximal lattice-free convex set and assume that $K$ is bounded. Then there exist vectors $l, u$ in $\mathbb{Z}^p$ such that $K$ is contained in the box $B = \{x \in \mathbb{R}^p : l \le x \le u\}$. Since $K$ is a lattice-free convex set, for each $v \in B \cap \mathbb{Z}^p$ there exists a half-space $\{x \in \mathbb{R}^p : \alpha^v x \le \beta^v\}$ containing $K$ such that $\alpha^v v = \beta^v$ (see the separation theorem for convex sets [10]). Since $B$ is a bounded set, $B \cap \mathbb{Z}^p$ is a finite set. Therefore the set

$$P := \{x \in \mathbb{R}^p : l \le x \le u,\ \alpha^v x \le \beta^v \ \text{ for all } v \in B \cap \mathbb{Z}^p\}$$

is a polytope. By construction, $P$ is lattice-free and $K \subseteq P$, thus $K = P$ by maximality of $K$.

We now show that each facet of $K$ contains a lattice point in its relative interior. Assume $K = \{x : a^i x \le b_i,\ i \in M\}$, where $a^i x \le b_i$, $i \in M$, are all distinct facet-defining inequalities for $K$. Assume by contradiction that the facet $F_t := \{x \in K : a^t x = b_t\}$ does not contain a point of $\mathbb{Z}^p$ in its relative interior. Given $\varepsilon > 0$, let $K' := \{x : a^i x \le b_i,\ i \in M \setminus \{t\},\ a^t x \le b_t + \varepsilon\}$. Since the recession cones of $K$ and $K'$ coincide, $K'$ is a polytope. Since $K$ is a maximal lattice-free convex set and $K \subset K'$, $K'$ contains points of $\mathbb{Z}^p$ in its interior. Since $K'$ is a polytope, the number of points in $\text{int}(K') \cap \mathbb{Z}^p$ is finite, hence there exists one such point minimizing $a^t x$, say $z$. Note that $a^t z > b_t$. By construction, the polytope $K'' := \{x : a^i x \le b_i,\ i \in M \setminus \{t\},\ a^t x \le a^t z\}$ does not contain any point of $\mathbb{Z}^p$ in its interior, and the inclusion $K'' \supset K$ is strict. This contradicts the maximality of $K$. □

Doignon [35], Bell [14] and Scarf [55] show the following.

**Theorem 3.19.** *Any full-dimensional maximal lattice-free convex set $K \subseteq \mathbb{R}^p$ has at most $2^p$ facets.*

*Proof.* By Theorem 3.18, each facet $F$ contains an integral point $x^F$ in its relative interior. If there are more than $2^p$ facets, then there exist two distinct facets $F, F'$ such that $x^F$ and $x^{F'}$ are congruent modulo 2. Now their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral and it is in the interior of $K$, contradicting the fact that $K$ is lattice-free. □

In $\mathbb{R}^2$, Theorem 3.19 implies that full-dimensional maximal lattice-free convex sets have at most 4 facets. Using Theorem 3.18, one can show that they are either:

1. Splits, namely sets of the form $\{x \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$, where $\pi_0, \pi_1, \pi_2 \in \mathbb{Z}$ and $\pi_1, \pi_2$ are relatively prime;

2. Triangles with an integral point in the relative interior of each facet and no integral point in the interior of the triangle;

3. Quadrilaterals with an integral point in the relative interior of each facet and no integral point in the interior of the quadrilateral.

A sharpening of the above classification is given in Exercises 3.6 and 3.7.

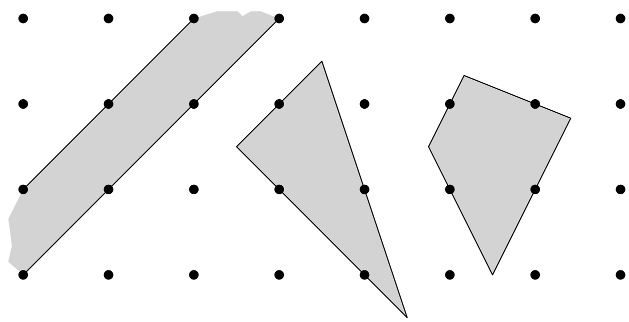

Figure 3.6: Maximal lattice-free convex sets with nonempty interior in $\mathbb{R}^2$

Consider the corner polyhedron corner$(B)$ and the linear relaxation $P(B)$ of (3.3). As in Section 3.1, we denote by $\bar{x}$ the apex of $P(B)$ and by $\bar{r}^j$, $j \in N$ its extreme rays (recall (3.4)). By Remark 3.6, undominated intersection cuts for corner$(B)$ are defined by maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex sets containing $\bar{x}$ in their interior. By Lemma 3.17 and Theorem 3.18, these sets are polyhedra of the form $K \times \mathbb{R}^{n-p}$, where $K$ is a maximal lattice-free polyhedron in $\mathbb{R}^p$. The next theorem shows how to compute the coefficients of the intersection cut from a facet description of $K$.

**Theorem 3.20.** *Let $K$ be a $\mathbb{Z}^p$-free polyhedron containing $(\bar{x}_1, \ldots, \bar{x}_p)$ in its interior. Then $K$ can be uniquely written in the form $K = \{x \in \mathbb{R}^p : \sum_{h=1}^p d_h^i (x_h - \bar{x}_h) \leq 1,\ i = 1, \ldots, t\}$, where $t$ is the number of facets of $K$ and $d^1, \ldots, d^t \in \mathbb{R}^p$. The coefficients in the intersection cut (3.9) defined by $C := K \times \mathbb{R}^{n-p}$ are*

$$\frac{1}{\alpha_j} = \max_{i=1,\ldots,t} \sum_{h=1}^p d_h^i \bar{r}_h^j \qquad j \in N. \tag{3.17}$$

*Proof.* Every facet-defining inequality for $K$ can be written in the form $\sum_{h=1}^p d_h (x_h - \bar{x}_h) \leq \delta$. Since $(\bar{x}_1, \ldots, \bar{x}_p)$ is in the interior of $K$, it follows that $\sum_{h=1}^p d_h (\bar{x}_h - \bar{x}_h) < \delta$, thus $\delta > 0$. Possibly by multiplying by $\delta^{-1}$, every facet-defining inequality for $K$ can be written in the form $\sum_{h=1}^p d_h (x_h - \bar{x}_h) \leq 1$.

We next show (3.17). Since $\alpha_j := \max\{\alpha \geq 0 : \bar{x} + \alpha\bar{r}^j \in C\}$ and $C = \{x \in \mathbb{R}^n : \sum_{h=1}^p d^i_h(x_h - \bar{x}_h) \leq 1,\ i = 1,\dots,t\}$, it follows that $\frac{1}{\alpha_j} = \max\{0,\ \sum_{h=1}^p d^i_h \bar{r}^j_h \ \ i = 1,\dots,t\}$. We only need to show that there exists $i \in \{1,\dots,t\}$ such that $\sum_{h=1}^p d^i_h \bar{r}^j_h \geq 0$.

Since $K$ is contained in a maximal $\mathbb{Z}^p$-free convex set, it follows from the last part of Theorem 3.18 that the recession cone of $K$ has dimension less than $p$, hence it has empty interior. Thus, the system of strict inequalities $\sum_{h=1}^p d^i_h r_h < 0$, $i = 1,\dots,t$ admits no solution. This shows that there exists $i \in \{1,\dots,t\}$ such that $\sum_{h=1}^p d^i_h \bar{r}^j_h \geq 0$. □

Let $\rho^j \in \mathbb{R}^p$ denote the restriction of $\bar{r}^j \in \mathbb{R}^n$ to the first $p$ components. Theorem 3.20 states that intersection cuts are of the form $\sum_{j\in N} \psi(\rho^j)x_j \geq 1$, where $\psi : \mathbb{R}^p \to \mathbb{R}_+$ is defined by

$$\psi(\rho) := \max_{i=1,\dots,t} d^i\rho. \tag{3.18}$$

Note that $\psi$ is the gauge of the set $K - (\bar{x}_1,\dots,\bar{x}_p)$. The definition of $\psi$ depends only on the number $p$ of integer variables and the values $\bar{b}_i \in \mathbb{Q}$, $i \in B$ in (3.3). If these are fixed, then $\sum_{j\in N} \psi(\rho^j)x_j \geq 1$ is valid for corner($B$) regardless of the number of continuous variables or of the values of the coefficients $\bar{a}_{ij}$, $i \in B, j \in N$. So $\psi$ gives a formula for generating valid inequalities that is independent of the specific data of the problem.

In the next section we will establish a framework to study functions with such a property, even when the number of integer variables is not fixed.

**Example 3.21.** Consider the following instance of (3.3) with no integer nonbasic variable.

$$\begin{array}{rcllllll} x_1 & = & \frac{1}{2} & +\frac{1}{4}x_3 & -\frac{3}{4}x_4 & -\frac{1}{4}x_5 & +x_6 \\ x_2 & = & \frac{1}{2} & +\frac{3}{4}x_3 & -\frac{1}{4}x_4 & +\frac{3}{4}x_5 & -\frac{3}{4}x_6 \\ & & x_1, x_2 \in \mathbb{Z} \\ & & x_3, x_4, x_5, x_6 \geq 0. \end{array}$$

Let $B$ be the triangle with vertices $(0,0), (2,0), (0,2)$. $B$ is a maximal $\mathbb{Z}^2$-free convex set, since it contains no integral point in its interior and all three sides have integral middle points, namely, $(1,0), (0,1), (1,1)$ (Figure 3.7). Note that $\bar{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ is in the interior of $B$, and $B$ can be written in the form

$$B = \{x \in \mathbb{R}^2 : -2(x_1 - \frac{1}{2}) \leq 1,\ -2(x_2 - \frac{1}{2}) \leq 1,\ (x_1 - \frac{1}{2}) + (x_2 - \frac{1}{2}) \leq 1\}.$$

The gauge of the set $K := B - \bar{b}$ is the function defined by

$$\psi(\rho) = \max\{-2\rho_1,\ -2\rho_2,\ \rho_1 + \rho_2\}, \qquad \rho \in \mathbb{R}^2$$

Because $\rho^3 = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$, $\rho^4 = \begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{4} \end{pmatrix}$, $\rho^5 = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$, $\rho^6 = \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix}$, the intersection cut defined by $B$ is therefore

$$x_3 + \frac{3}{2}x_4 + \frac{1}{2}x_5 + \frac{3}{2}x_6 \geq 1.$$

■

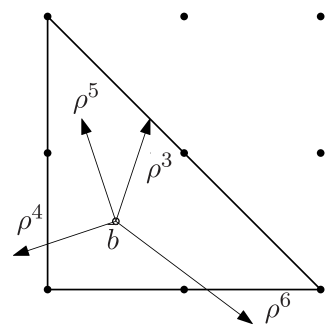


Figure 3.7: Maximal $\mathbb{Z}^2$-free triangle $B$ and vectors $\rho^3 \ldots, \rho^6$.

## 3.3 Exercises

**Exercise 3.1.** Show that the Gomory mixed integer inequality generated from a tableau row (3.2) is valid for corner($B$).

**Exercise 3.2.** Give a minimal system of linear inequalities describing the corner polyhedron for the integer program of Example 3.3 when the choice of basic variables is $x_1, x_2, x_3, x_5, x_7$.

**Exercise 3.3.** Reformulate (3.6) in the modular form (3.7). Show that the solution set of this modular problem is contained in the union of the simplices $S_k := \{(x_6, x_7, x_8) \in \mathbb{R}^3_+ : x_6 + x_7 + x_8 = 2k\}$ where $k$ is a positive integer. Show that the modular problem admits solutions $(x_6, x_7, x_8)$ of the form $(2k, 0, 0)$, $(0, 2k, 0)$ and $(0, 0, 2k)$ for every positive integer $k$. Describe the corner polyhedron corner($B$) in the space of the variables $x_6, x_7, x_8$. Deduce a description of the corner polyhedron corner($B$) in the space of the variables $x_1, x_2, x_3$.

**Exercise 3.4.** Consider a closed convex set $C \subseteq \mathbb{R}^n$ whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. Let $Q$ be the closed convex hull of $P(B) \setminus \text{int}(C)$, where $\text{int}(C)$ denotes the interior of $C$. Show that $Q$ is the set of points in $P(B)$ satisfying the intersection cut defined by C.

**Exercise 3.5.** Let $K \subseteq \mathbb{R}^n$ be a closed convex set with the origin in its interior. Prove that the gauge function satisfies $\gamma_K(r) \leq 1$ if and only if $r \in K$.

**Exercise 3.6.** Let $T$ be a maximal lattice-free convex set in $\mathbb{R}^2$ which is a triangle. Show that $T$ satisfies one of the following.

a) All vertices of $T$ are integral points and $T$ contains exactly one integral point in the relative interior of each facet.

b) At least one vertex of $T$ is not an integral point and the opposite facet contains at least two integral points in its relative interior.

c) $T$ contains exactly three integral points, one in the relative interior of each facet.

**Exercise 3.7.** Let $Q$ be a maximal lattice-free convex set in $\mathbb{R}^2$ which is a quadrilateral.

a) Show that $Q$ contains *exactly* four integral points on its boundary.

b) Show that these four integral points are the vertices of a parallelogram of area one.

**Exercise 3.8.** A convex set in $\mathbb{R}^2$ is $\mathbb{Z}^2_+$-free if it contains no point of $\mathbb{Z}^2_+$ in its interior. Characterize the maximal $\mathbb{Z}^2_+$-free convex sets that are not $\mathbb{Z}^2$-free convex sets.

**Exercise 3.9.** Let $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$, such that the system $Ax \leq b$ has no integral solution but any system obtained from it by removing one of the $m$ inequalities has an integral solution. Show that $m \leq 2^n$.

**Exercise 3.10.** In $\mathbb{R}^p$, let $f_i = \frac{1}{2}$ for $i = 1, \ldots, p$. Define the octahedron $\Omega_f$ centered at $f$ with vertices $f \pm \frac{p}{2}e^i$, where $e^i$ denotes the $i$th unit vector. $\Omega_f$ has $2^p$ facets, each of which contains a 0,1 point in its center.

a) Show that the intersection cut $\sum \pi(r)y_r \geq 1$ obtained from the octahedron $\Omega_f$ is obtained from the function $\pi(r) := \frac{2}{p}(|r_1| + \ldots + |r_p|)$.

b) Show that the above intersection cut from the octahedron is implied by $p$ split inequalities.

# Lecture 4

# Infinite Relaxation

Theorem 3.20 gives a formula for computing the coefficients of an intersection cut, namely $\frac{1}{\alpha_j} = \psi(\rho^j)$, where $\psi$ is the function defined in (3.18). As we pointed out, the definition of $\psi$ does not depend on the number of continuous variables nor on the vectors $\rho^j$s. Any function with such properties can therefore be used as a "black box" to generate cuts from the tableau of any integer program. Similarly, the functions $\pi$ and $\psi$ defined in (3.16) provide a formula to generate valid inequalities from one equation of the simplex tableau $x_i + \sum_{j\in N} \bar{a}_{ij}x_j = \bar{b}_i$, namely the inequality $\sum_{j\in N,\, j\le p} \pi(-\bar{a}_{ij})x_j + \sum_{p+1\le j\le n} \psi(-\bar{a}_{ij}) \ge 1$.

Gomory and Johnson [40] introduced a convenient setting to formalize and study these functions. In this framework, one works with a model with a fixed number of basic variables, but an infinite number of nonbasic ones, namely one for every possible choice of variables coefficients in (3.3).

Consider the constraints (3.7). We rename $f_i := \bar{b}_i$ and $r_i^j := -\bar{a}_{ij}$ for $i \in B$ and $j \in N$. In other words, defining $q := |B|$, the vector $r^j \in \mathbb{R}^q$ is the restriction of the vector $\bar{r}^j \in \mathbb{R}^n$ to the components $i \in B$. Renaming the variables so that the nonbasic integer variables are $x_j$, $j \in I$, and the nonbasic continuous variables are $y_j$, $j \in C$, (3.7) is written in the form

$$\begin{array}{rcll} f_i + \sum_{j\in I} r_i^j x_j + \sum_{j\in C} r_i^j y_j & \in & \mathbb{Z} & i = 1,\dots,q \\ x_j \in \mathbb{Z}_+ & & & \text{for all } j \in I \\ y_j \ge 0 & & & \text{for all } j \in C. \end{array} \tag{4.1}$$

Gomory and Johnson [40] suggested relaxing the space of variables $x_j$, $j \in I$, $y_j$, $j \in C$, to an infinite-dimensional space, where an integer variable $x_r$ and a continuous variable $y_r$ are introduced for every $r \in \mathbb{R}^q$. We obtain the following *infinite relaxation*

$$\begin{array}{ll} f + \sum_{r\in\mathbb{R}^q} r x_r + \sum_{r\in\mathbb{R}^q} r y_r \in \mathbb{Z}^q & \\ x_r \in \mathbb{Z}_+ & \text{for } r \in \mathbb{R}^q \\ y_r \ge 0 & \text{for } r \in \mathbb{R}^q \\ x, y \text{ have a finite support.} & \end{array} \tag{4.2}$$

The infinite dimensional vectors $x, y$ having *finite support* means that the sets $\{r \in \mathbb{R}^q : x_r > 0\}$ and $\{r \in \mathbb{R}^q : y_r > 0\}$ are finite.

Every problem of the type (4.1) can be obtained from (4.2) by setting to 0 all but a finite number of variables. This is why $x$ and $y$ are restricted to have finite support in the above model. Furthermore, the study of model (4.2) yields information on (4.1) that is independent of the data in (4.1), but depends only on the vector $f \in \mathbb{R}^q$.

We denote by $M_f \subset \mathbb{Z}^{\mathbb{R}^q} \times \mathbb{R}^{\mathbb{R}^q}$ the set of feasible solutions to (4.2). Note that $M_f \neq \emptyset$ since defining $x_r = 1$ for $r = -f$, $x_r = 0$ otherwise, and setting $y = 0$, yields a feasible solution to (4.2).

A function $(\pi, \psi) : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ is *valid* for $M_f$ if $\pi \geq 0$ and the linear inequality

$$\sum_{r \in \mathbb{R}^q} \pi(r) x_r + \sum_{r \in \mathbb{R}^q} \psi(r) y_r \geq 1 \tag{4.3}$$

is satisfied by all vectors in $M_f$.

The relevance of the above definition rests on the fact that any valid function $(\pi, \psi)$ yields a valid inequality for the original set defined in (4.1), namely

$$\sum_{j \in I} \pi(r^j) x_j + \sum_{j \in C} \psi(r^j) y_j \geq 1.$$

Observe that, if we are given valid functions $(\pi', \psi')$ and $(\pi'', \psi'')$ for $M_f$, such that $\psi' \leq \psi''$ and $\pi' \leq \pi''$, then the inequality (4.3) defined by $(\pi, \psi) := (\pi', \psi')$ is stronger than that defined by $(\pi, \psi) := (\pi'', \psi'')$. This observation naturally leads to the following definition: a valid function $(\pi, \psi)$ for $M_f$ is *minimal* if there is no valid function $(\pi', \psi')$, distinct from $(\pi, \psi)$, where $\pi' \leq \pi$ and $\psi' \leq \psi$.

We remark, omitting the proof, that for every valid function $(\pi, \psi)$ there exists a minimal valid function $(\pi', \psi')$ such that $\pi' \leq \pi$ and $\psi' \leq \psi$. It follows that we only need to focus our attention on minimal valid functions.

While the concept of valid function is natural, the assumption that $\pi \geq 0$ in the definition might, however, seem artificial. Indeed, if we omitted this assumption in the definition, then there would be valid functions for which $\pi$ takes negative values. However, we next show that any valid function should be nonnegative over the rational vectors. Thus, since data in integer programming problems are usually rational and valid functions should be nonnegative over rational vectors, it makes sense to assume that $\pi \geq 0$.

To show that $\pi$ should be nonnegative over the rational vectors, consider a function $(\pi, \psi)$ such that (4.3) holds for every $(x, y) \in M_f$, and suppose $\pi(\tilde{r}) < 0$ for some $\tilde{r} \in \mathbb{Q}^q$. Let $D \in \mathbb{Z}_+$ be such that $D\tilde{r}$ is an integral vector, and let $(\bar{x}, \bar{y}) \in M_f$. Define $\tilde{x}$ by $\tilde{x}_{\tilde{r}} := \bar{x}_{\tilde{r}} + MD$ where $M$ is a positive integer, and $\tilde{x}_r := \bar{x}_r$ for $r \neq \tilde{r}$. It follows that also $(\tilde{x}, \bar{y})$ is an element of $M_f$. We have $\sum \pi(r)\tilde{x}_r + \sum \psi(r)\bar{y}_r = \sum \pi(r)\bar{x}_r + \pi(\tilde{r})MD + \sum \psi(r)\bar{y}_r$. If we choose $M > (\sum \pi(r)\bar{x}_r + \sum \psi(r)\bar{y}_r \; - 1)/(D|\pi(\tilde{r})|)$, then $\sum \pi(r)\tilde{x}_r + \sum \psi(r)\bar{y}_r < 1$, contradicting the fact that $(\pi, \psi)$ is valid.

In the next section we start by considering the pure integer case, namely the case where $y_r = 0$ for all $r \in \mathbb{R}^q$. We will then focus on the "continuous case", where $x_r = 0$ for all $r \in \mathbb{R}^q$, and finally we will give a characterization of minimal valid functions for the set $M_f$.

## 4.1 Pure integer infinite relaxation

If in (4.2) we disregard the continuous variables, we obtain the following *pure integer infinite relaxation*.

$$\begin{array}{rcll} f+\sum_{r\in\mathbb{R}^q} rx_r & \in & \mathbb{Z}^q & \\ x_r & \in & \mathbb{Z}_+ & \text{for all } r\in\mathbb{R}^q \\ x & & & \text{has a finite support.} \end{array} \tag{4.4}$$

Denote by $G_f$ the set of feasible solutions to (4.4). Note that $G_f \neq \emptyset$ since the vector $x$ defined by $x_r = 1$ for $r = -f$ and $x_r = 0$ otherwise is a feasible solution to (4.4).

A function $\pi : \mathbb{R}^q \to \mathbb{R}$ is *valid* for $G_f$ if $\pi \geq 0$ and the linear inequality

$$\sum_{r\in\mathbb{R}^q} \pi(r)x_r \geq 1 \tag{4.5}$$

is satisfied by all feasible solutions of (4.4).

A valid function for $G_f$, $\pi:\ \mathbb{R}^q \to \mathbb{R}_+$, is *minimal* if there is no valid function $\pi' \neq \pi$ such that $\pi'(r) \leq \pi(r)$ for all $r \in \mathbb{R}^q$.

Note that any minimal valid function $\pi$ must satisfy $\pi(r) \leq 1$ for all $r \in \mathbb{R}^q$ because every $x \in G_f$ has integral components, and therefore for all $r \in \mathbb{R}^q$ either $x_r = 0$ or $x_r \geq 1$. Furthermore, $\pi$ must satisfy $\pi(-f) = 1$, since the vector defined by $x_{-f} = 1$, $x_r = 0$ for all $r \neq -f$ is in $G_f$.

Observe that, given $\bar{r} \in \mathbb{R}^q$, the vector $x$ defined by $x_{\bar{r}} = x_{-f-\bar{r}} = 1$, $x_r = 0$ for all $r \neq \bar{r}, -f-\bar{r}$, is an element of $G_f$, therefore $\pi(\bar{r}) + \pi(-f-\bar{r}) \geq 1$. A function $\pi : \mathbb{R}^q \to \mathbb{R}$ is said to satisfy the *symmetry condition* if $\pi(r) + \pi(-f-r) = 1$ for all $r \in \mathbb{R}^q$.

A function $\pi:\ \mathbb{R}^q \to \mathbb{R}$ is *periodic* if $\pi(r) = \pi(r+w)$, for every $w \in \mathbb{Z}^q$. Therefore a periodic function is entirely defined by its values in $[0,1[^q$. The next theorem shows that minimal valid functions are completely characterized by subadditivity, symmetry, and periodicity.

**Theorem 4.1** (Gomory and Johnson [40]). *A function $\pi : \mathbb{R}^q \to \mathbb{R}_+$ is a minimal valid function for $G_f$ if and only if $\pi(0) = 0$, $\pi$ is subadditive, periodic and satisfies the symmetry condition.*

*Proof.* We first prove the "only if" part of the statement. Assume that $\pi$ is a minimal valid function for $G_f$. We need to show the following four facts.

a) $\pi(0) = 0$. If $\bar{x}$ is a feasible solution of $G_f$, then so is $\tilde{x}$ defined by $\tilde{x}_r := \bar{x}_r$ for $r \neq 0$, and $\tilde{x}_0 = 0$. Therefore the function $\pi'$ defined by $\pi'(r) = \pi(r)$ for $r \neq 0$ and $\pi'(0) = 0$ is also valid. Since $\pi$ is minimal and nonnegative, it follows that $\pi(0) = 0$.

b) $\pi$ *is subadditive.* Let $r^1, r^2 \in \mathbb{R}^q$. We need to show $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$. This inequality holds when $r^1 = 0$ or $r^2 = 0$ because $\pi(0) = 0$. Assume now that $r^1 \neq 0$ and $r^2 \neq 0$. Define the function $\pi'$ as follows.

$$\pi'(r) := \begin{cases} \pi(r^1) + \pi(r^2) & \text{if } r = r^1 + r^2 \\ \pi(r) & \text{if } r \neq r^1 + r^2. \end{cases}$$

We show that $\pi'$ is valid. Consider any $\bar{x} \in G_f$. We need to show that $\sum_r \pi'(r)\bar{x}_r \geq 1$. Define $\tilde{x}$ as follows

$$\tilde{x}_r := \begin{cases} \bar{x}_{r^1} + \bar{x}_{r^1+r^2} & \text{if } r = r^1 \\ \bar{x}_{r^2} + \bar{x}_{r^1+r^2} & \text{if } r = r^2 \\ 0 & \text{if } r = r^1 + r^2 \\ \bar{x}_r & \text{otherwise.} \end{cases}$$

Note that $\tilde{x} \geq 0$ and $f + \sum r\tilde{x}_r = f + \sum r\bar{x}_r \in \mathbb{Z}^q$, thus $\tilde{x} \in G_f$. Using the definitions of $\pi'$ and $\tilde{x}$, it is easy to verify that $\sum_r \pi'(r)\bar{x}_r = \sum_r \pi(r)\tilde{x}_r \geq 1$, where the last inequality follows from the facts that $\pi$ is valid and $\tilde{x} \in G_f$. This shows that $\pi'$ is valid. Since $\pi$ is minimal, we get $\pi(r^1 + r^2) \leq \pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$.

c) $\pi$ *is periodic.* Suppose not. Then $\pi(\tilde{r}) > \pi(\tilde{r} + w)$ for some $\tilde{r} \in \mathbb{R}^q$ and $w \in \mathbb{Z}^q \setminus \{0\}$. Define the function $\pi'$ by $\pi'(\tilde{r}) := \pi(\tilde{r} + w)$ and $\pi'(r) = \pi(r)$ for $r \neq \tilde{r}$. We show that $\pi'$ is valid. Consider any $\bar{x} \in G_f$. Let $\tilde{x}$ be defined by

$$\tilde{x}_r := \begin{cases} \bar{x}_r & \text{if } r \neq \tilde{r},\ \tilde{r} + w \\ 0 & \text{if } r = \tilde{r} \\ \bar{x}_{\tilde{r}} + \bar{x}_{\tilde{r}+w} & \text{if } r = \tilde{r} + w. \end{cases}$$

Since $\bar{x} \in G_f$ and $w\bar{x}_{\tilde{r}} \in \mathbb{Z}^q$, we have that $\tilde{x} \in G_f$. By the definition of $\pi'$ and $\tilde{x}$, $\sum \pi'(r)\bar{x}_r = \sum \pi(r)\tilde{x}_r \geq 1$, where the last inequality follows from the facts that $\pi$ is valid and $\tilde{x} \in G_f$. This contradicts the fact that $\pi$ is minimal, since $\pi' \leq \pi$ and $\pi'(\tilde{r}) < \pi(\tilde{r})$.

d) $\pi$ *satisfies the symmetry condition.* Suppose there exists $\tilde{r} \in \mathbb{R}^q$ such that $\pi(\tilde{r}) + \pi(-f - \tilde{r}) \neq 1$. Since $\pi$ is valid, $\pi(\tilde{r}) + \pi(-f - \tilde{r}) = 1 + \delta$ where $\delta > 0$. Note that, since $\pi(r) \leq 1$ for all $r \in \mathbb{R}^q$, it follows that $\pi(\tilde{r}) > 0$. Define the function $\pi'$ by

$$\pi'(r) := \begin{cases} \frac{1}{1+\delta}\pi(\tilde{r}) & \text{if } r = \tilde{r}, \\ \pi(r) & \text{if } r \neq \tilde{r}, \end{cases} \qquad r \in \mathbb{R}^q.$$

We show that $\pi'$ is valid. Consider any $\bar{x} \in G_f$. Note that

$$\sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r = \sum_{\substack{r \in \mathbb{R}^q \\ r \neq \tilde{r}}} \pi(r)\bar{x}_r + \frac{1}{1+\delta}\pi(\tilde{r})\bar{x}_{\tilde{r}}$$

If $\bar{x}_{\tilde{r}} = 0$ then $\sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r = \sum_{r \in \mathbb{R}^q} \pi(r)\bar{x}_r \geq 1$ because $\pi$ is valid. If $\bar{x}_{\tilde{r}} \geq (1+\delta)/\pi(\tilde{r})$ then $\sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r \geq 1$. Thus we can assume that $1 \leq \bar{x}_{\tilde{r}} < (1+\delta)/\pi(\tilde{r})$.

Observe that $\sum_{r \in \mathbb{R}^q r \neq \tilde{r}} \pi(r)\bar{x}_r + \pi(\tilde{r})(\bar{x}_{\tilde{r}} - 1) \geq \sum_{r \in \mathbb{R}^q r \neq \tilde{r}} \pi(r\bar{x}_r) + \pi(\tilde{r}(\bar{x}_{\tilde{r}} - 1)) \geq \pi(\sum_{r \in \mathbb{R}^q r \neq \tilde{r}} r\bar{x}_r + \tilde{r}(\bar{x}_{\tilde{r}} - 1)) = \pi(-f - \tilde{r})$, where the inequalities follow by the subadditivity of $\pi$ and the equality follows by the periodicity of $\pi$. Therefore

$$\begin{aligned} \sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r &= \sum_{\substack{r \in \mathbb{R}^q \\ r \neq \tilde{r}}} \pi(r)\bar{x}_r + \pi(\tilde{r})(\bar{x}_{\tilde{r}} - 1) + \pi(\tilde{r}) - \frac{\delta}{1+\delta}\pi(\tilde{r})\bar{x}_{\tilde{r}} \\ &\geq \pi(-f - \tilde{r}) + \pi(\tilde{r}) - \delta \\ &= 1 + \delta - \delta = 1, \end{aligned}$$

This shows that $\pi'$ is valid, contradicting the minimality of $\pi$.

We now prove the "if" part of the statement. Assume that $\pi(0) = 0$, $\pi$ is subadditive, periodic and satisfies the symmetry condition.

We first show that $\pi$ is valid. The symmetry condition implies $\pi(0) + \pi(-f) = 1$. Since $\pi(0) = 0$, we have $\pi(-f) = 1$. Any $\bar{x} \in G_f$ satisfies $\sum r\bar{x}_r = -f + w$ for some $w \in \mathbb{Z}^q$. We have that $\sum \pi(r)\bar{x}_r \geq \pi(\sum r\bar{x}_r) = \pi(-f + w) = \pi(-f) = 1$, where the inequality comes from subadditivity and the second to last equality comes from periodicity. Thus $\pi$ is valid.

To show that $\pi$ is minimal, suppose by contradiction that there exists a valid function $\pi' \leq \pi$ such that $\pi'(\tilde{r}) < \pi(\tilde{r})$ for some $\tilde{r} \in \mathbb{R}^q$. Then $\pi(\tilde{r}) + \pi(-f - \tilde{r}) = 1$ implies $\pi'(\tilde{r}) + \pi'(-f - \tilde{r}) < 1$, contradicting the validity of $\pi'$. □

### 4.1.1 One-dimensional examples

We give examples of minimal valid functions for the case $q = 1$ in (4.4). By periodicity it suffices to describe them in $[0, 1]$. These examples share the following property.

A function $\pi : [0, 1] \to \mathbb{R}$ is *piecewise-linear* if there are finitely many values $0 = r_0 < r_1 < \ldots < r_k = 1$ such that the function is of the form $\pi(r) = a_j r + b_j$ in interval $]r_{j-1}, r_j[$, for $j = 1, \ldots, k$. The $r_j$s for $j = 0, \ldots, k$ are the *breakpoints*. The *slopes* of a piecewise-linear function are the different values of $a_j$ for $j = 1, \ldots, k$. Note that a piecewise-linear function $\pi : [0, 1] \to \mathbb{R}$ is continuous if and only if, for $j = 1, \ldots, k - 1$, $a_j r_j + b_j = a_{j+1} r_j + b_{j+1}$.

**Example 4.2.** Let $q = 1$ and $0 < t < f < 1$. Consider the Gomory function $\pi$ defined in Example 3.10, and the functions $\pi_1$, $\pi_2$ defined in $[0, 1]$ as follows

$$\pi_1(r) := \begin{cases} \frac{r}{1-f} & \text{if } 0 \leq r \leq 1-f \\ \frac{1-r+t-f}{t} & \text{if } 1-f \leq r \leq 1-f+\frac{t/2}{1-f+t} \\ \frac{r-1/2}{1-f} & \text{if } 1-f+\frac{t/2}{1-f+t} \leq r \leq 1-\frac{t/2}{1-f+t} \\ \frac{1-r}{t} & \text{if } 1-\frac{t/2}{1-f+t} \leq r \leq 1 \end{cases}$$

$$\pi_2(r) := \begin{cases} \frac{r}{1-f} & \text{if } 0 \leq r \leq 1-f \\ \frac{1-r+t-f}{t} & \text{if } 1-f \leq r \leq 1-f+\frac{t}{2-f+t} \\ \frac{r}{2-f} & \text{if } 1-f+\frac{t}{2-f+t} \leq r \leq 1-\frac{t}{2-f+t} \\ \frac{1-r}{t} & \text{if } 1-\frac{t}{2-f+t} \leq r \leq 1 \end{cases}$$

and elsewhere by periodicity. The three functions are illustrated in Figure 4.1. Note the symmetry relative to the points $(\frac{1-f}{2}, \frac{1}{2})$ and $(1 - \frac{f}{2}, \frac{1}{2})$.

Consider a continuous nonnegative periodic function $\pi : \mathbb{R} \to \mathbb{R}_+$ that is piecewise-linear in the interval $[0, 1]$ and satisfies $\pi(0) = 0$. By Theorem 4.1, such a function $\pi$ is minimal if it is subadditive and satisfies the symmetry condition. Checking whether the symmetry condition $\pi(r) + \pi(-f - r) = 1$ holds for all $r \in \mathbb{R}$ is easy: It suffices to check it at the breakpoints of the function in the interval $[0, 1]$. Checking subadditivity of a function is a nontrivial task in general. Gomory and Johnson [42] showed that, for a nonnegative continuous periodic piecewise-linear function $\pi$ that is symmetric, it is enough to check that $\pi(a) + \pi(b) \geq \pi(a + b)$ for all pairs of breakpoints $a, b$ (possibly $a = b$) in the interval $[0, 1]$

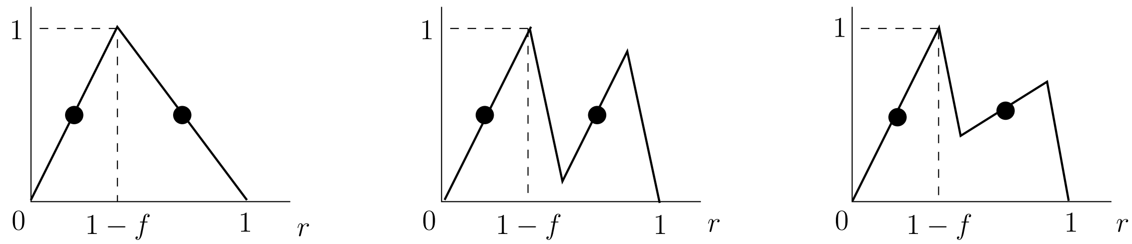


Figure 4.1: Minimal valid functions $\pi$, $\pi_1$, $\pi_2$.

where the function is locally convex. Using this, the reader can check that all three functions given above are minimal. ■

### 4.1.2 Extreme valid functions and the two-slope theorem

In order to describe a full-dimensional polyhedron in $\mathbb{R}^n$, the facet-defining inequalities suffice and they cannot be written as nonnegative combinations of inequalities defining distinct faces. This concept can be generalized to the infinite-dimensional set $G_f$.

A valid function $\pi$ for $G_f$ is *extreme* if it cannot be expressed as a convex combination of two distinct valid functions. That is, if $\pi$ is extreme and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ where $\pi_1$ and $\pi_2$ are valid functions, then $\pi = \pi_1 = \pi_2$.

It follows from the definition that one is interested only in extreme valid functions for $G_f$, since the inequality (4.5) defined by a valid function $\pi$ that is not extreme is implied by the two inequalities defined by $\pi_1$ and $\pi_2$, where $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, $\pi_1, \pi_2 \neq \pi$. It also follows easily from the definition that extreme valid inequalities are minimal.

**Example 4.3.** The first two functions of Figure 4.1 are extreme; this will follow from the two-slope theorem (see below). The third function is extreme when $f \geq t + 1/2$; the proof is left as an exercise (Exercise 4.5). We remark that extreme valid functions are not always continuous. Indeed, Dey, Richard, Li and Miller [33] show that, for $0 < 1 - f < .5$, the following discontinuous valid function is extreme (see Figure 4.2).

$$\pi(r) := \begin{cases} \frac{r}{1-f} & \text{for } 0 \leq r \leq 1-f \\ \frac{r}{2-f} & \text{for } 1-f < r < 1 \end{cases}$$

■

Given a minimal valid function $\pi$, we recall that $\pi$ must be subadditive (by Theorem 4.1). We denote by $E(\pi)$ the (possibly infinite) set of all possible inequalities $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ that are satisfied as an equality.

**Lemma 4.4.** *Let $\pi$ be a minimal valid function. Assume $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, where $\pi_1$ and $\pi_2$ are valid functions. Then $\pi_1$ and $\pi_2$ are minimal functions and $E(\pi) \subseteq E(\pi_1) \cap E(\pi_2)$.*

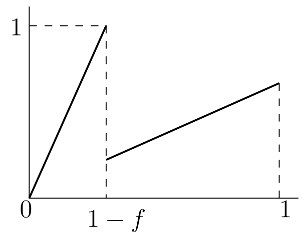


Figure 4.2: A discontinuous extreme valid function

*Proof.* Suppose $\pi_1$ is not minimal. Let $\pi_1' \neq \pi$ be a valid function, such that $\pi_1' \leq \pi_1$. Then $\pi' = \frac{1}{2}\pi_1' + \frac{1}{2}\pi_2$ is a valid function, distinct from $\pi$, and $\pi' \leq \pi$. This contradicts the minimality of $\pi$.

Suppose $E(\pi) \not\subseteq E(\pi_1) \cap E(\pi_2)$. We may assume $E(\pi) \not\subseteq E(\pi_1)$. That is, there exist $r^1$, $r^2$ such that $\pi(r^1) + \pi(r^2) = \pi(r^1 + r^2)$ and $\pi_1(r^1) + \pi_1(r^2) > \pi_1(r^1 + r^2)$. Since $\pi_2$ is minimal, it is subadditive by Theorem 4.1 and therefore $\pi_2(r^1) + \pi_2(r^2) \geq \pi_2(r^1 + r^2)$. This contradicts the assumption that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. □

No general characterization of the extreme valid functions is known. In fact, checking that a valid function is extreme, or proving that a certain class of valid functions are extreme, can be challenging. Gomory and Johnson [42] give an interesting class of extreme valid functions for the case of a single row problem ($q = 1$ in model (4.4)). We present their result in Theorem 4.6, the "two-slope theorem". A useful tool for showing that a given valid function is extreme is the so-called *interval lemma*, which we prove next.

**Lemma 4.5** (Interval lemma)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is bounded on every bounded interval. Let $a_1 < a_2$ and $b_1 < b_2$. Consider the intervals $A := [a_1, a_2]$, $B := [b_1, b_2]$ and $A + B := [a_1 + b_1, a_2 + b_2]$. If $f(a) + f(b) = f(a + b)$ for all $a \in A$ and $b \in B$, then $f$ is an affine function in each of the sets $A$, $B$ and $A + B$, and it has the same slope in each of these sets.*

*Proof.* We first show the following.

*Claim 1. Let $a \in A$, and let $\varepsilon > 0$ such that $b_1 + \varepsilon \in B$. For every nonnegative integer $p$ such that $a + p\varepsilon \in A$, we have $f(a + p\varepsilon) - f(a) = p(f(b_1 + \varepsilon) - f(b_1))$.*

For $h = 1, \ldots, p$, by hypothesis $f(a+h\varepsilon)+f(b_1) = f(a+h\varepsilon+b_1) = f(a+(h-1)\varepsilon)+f(b_1+\varepsilon)$. Thus $f(a + h\varepsilon) - f(a + (h - 1)\varepsilon) = f(b_1 + \varepsilon) - f(b_1)$, for $h = 1, \ldots, p$. By summing these $p$ equations, we obtain $f(a + p\varepsilon) - f(a) = p(f(b_1 + \varepsilon) - f(b_1))$. This concludes the proof of Claim 1.

Let $\bar{a}, \bar{a}' \in A$ such that $\bar{a} - \bar{a}' \in \mathbb{Q}$ and $\bar{a} > \bar{a}'$. Define $c := \frac{f(\bar{a}) - f(\bar{a}')}{\bar{a} - \bar{a}'}$.

*Claim 2. For every $a, a' \in A$ such that $a - a' \in \mathbb{Q}$, we have $f(a) - f(a') = c(a - a')$.*

We may assume $a > a'$. Choose a rational $\varepsilon > 0$ such that $b_1 + \varepsilon \in B$ and the numbers $\bar{p} := \frac{\bar{a} - \bar{a}'}{\varepsilon}$ and $p = \frac{a - a'}{\varepsilon}$ are both integer. By Claim 1,

$$f(\bar{a}) - f(\bar{a}') = \bar{p}(f(b_1 + \varepsilon) - f(b_1)) \quad \text{and} \quad f(a) - f(a') = p(f(b_1 + \varepsilon) - f(b_1)).$$

Dividing the last equality by $a - a' = p\varepsilon$ and the second to last by $\bar{a} - \bar{a}' = \bar{p}\varepsilon$, we obtain

$$\frac{f(b_1 + \varepsilon) - f(b_1)}{\varepsilon} = \frac{f(\bar{a}) - f(\bar{a}')}{\bar{a} - \bar{a}'} = \frac{f(a) - f(a')}{a - a'} = c.$$

Thus $f(a) - f(a') = c(a - a')$. This concludes the proof of Claim 2.

*Claim 3. For every $a \in A$, $f(a) = f(a_1) + c(a - a_1)$.*

Let $\delta(x) := f(x) - cx$. Since $f$ is bounded on every bounded interval, $\delta$ is bounded over $A, B$ and $A + B$. Let $M$ be a number such that $|\delta(x)| \leq M$ for all $x \in A \cup B \cup (A + B)$.

We will show that $\delta(a) = \delta(a_1)$ for all $a \in A$, which proves the claim. Suppose by contradiction that, for some $a^* \in A$, $\delta(a^*) \neq \delta(a_1)$. Let $N$ be a positive integer such that $N|\delta(a^*) - \delta(a_1)| > 2M$.

By Claim 2, $\delta(a^*) = \delta(a)$ for every $a \in A$ such that $a^* - a$ is rational. Thus there exists $\bar{a}$ such that $\delta(\bar{a}) = \delta(a^*)$, $a_1 + N(\bar{a} - a_1) \in A$ and $b_1 + \bar{a} - a_1 \in B$. Let $\varepsilon := \bar{a} - a_1$. By Claim 1,

$$\delta(a_1 + N\varepsilon) - \delta(a_1) = N(\delta(b_1 + \varepsilon) - \delta(b_1)) = N(\delta(a_1 + \varepsilon) - \delta(a_1)) = N(\delta(\bar{a}) - \delta(a_1))$$

Thus $|\delta(a_1 + N\varepsilon) - \delta(a_1)| = N|\delta(\bar{a}) - \delta(a_1)| = N|\delta(a^*) - \delta(a_1)| > 2M$, which implies $|\delta(a_1 + N\varepsilon)| + |\delta(a_1)| > 2M$, a contradiction. This concludes the proof of Claim 3.

By symmetry between $A$ and $B$, Claim 3 implies that there exists some constant $c'$ such that, for every $b \in B$, $f(b) = f(b_1) + c'(b - b_1)$. We show $c' = c$. Indeed, given $\varepsilon > 0$ such that $a_1 + \varepsilon \in A$ and $b_1 + \varepsilon \in B$, $c\varepsilon = f(a_1 + \varepsilon) - f(a_1) = f(b_1 + \varepsilon) - f(b_1) = c'\varepsilon$, where the second equality follows from Claim 1.
Therefore, for every $b \in B$, $f(b) = f(b_1) + cf(b - b_1)$. Finally, since $f(a) + f(b) = f(a + b)$ for every $a \in A$ and $b \in B$, it follows that for every $w \in A + B$, $f(w) = f(a_1 + b_1) + c(w - a_1 - b_1)$. □

**Theorem 4.6** (Two-slope theorem)**.** *Let $\pi : \mathbb{R} \to \mathbb{R}$ be a minimal valid function. If the restriction of $\pi$ to the interval $[0, 1]$ is a continuous piecewise-linear function with only two slopes, then $\pi$ is extreme.*

*Proof.* Consider valid functions $\pi_1$, $\pi_2$ such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. By Lemma 4.4, $\pi_1$ and $\pi_2$ are minimal valid functions. Since $\pi$, $\pi_1$, $\pi_2$ are minimal, by Theorem 4.1 they are nonnegative and $\pi(0) = \pi_1(0) = \pi_2(0) = 0$, $\pi(1) = \pi_1(1) = \pi_2(1) = 0$, $\pi(1-f) = \pi_1(1-f) = \pi_2(1-f) = 1$. We will prove $\pi = \pi_1 = \pi_2$. We recall that minimal valid functions can only take values between 0 and 1, thus $\pi, \pi_1, \pi_2$ are bounded everywhere.

Consider $0 = r_0 < r_1 < \cdots < r_{k-1} < r_k = 1$, where $r_1, \ldots, r_{k-1}$ are the points in $[0, 1]$ where the slope of $\pi$ changes. Since $\pi$ is continuous and $\pi(0) = \pi(1) = 0$, one of the slopes must be positive and the other negative. Let $s^+$ and $s^-$ be the positive and negative slopes of $\pi$. Therefore $\pi(r) = s^+ r$ for $0 \leq r \leq r_1$ and $\pi(r) = \pi(r_{k-1}) + s^-(r - r_{k-1})$ for $r_{k-1} \leq r \leq r_k = 1$. Furthermore $\pi$ has slope $s^+$ in interval $[r_i, r_{i+1}]$ if $i$ is even and slope $s^-$ if $i$ is odd, $i = 0, \ldots, k-1$.

We next show the following. *$\pi_1$, $\pi_2$ are continuous piecewise-linear functions with two slopes. In intervals $[r_i, r_{i+1}]$, $i$ even, $\pi_1, \pi_2$ have positive slopes $s_1^+$, $s_2^+$. In intervals $[r_i, r_{i+1}]$, $i$ odd, $\pi_1, \pi_2$ have negative slopes $s_1^-$, $s_2^-$.*

Let $i \in \{0, \dots, k\}$. Assume first $i$ even. Let $\epsilon$ be a sufficiently small positive number and define $A = [0, \epsilon]$, $B = [r_i, r_{i+1} - \epsilon]$. Then $A + B = [r_i, r_{i+1}]$ and $\pi$ has slope $s^+$ in all three intervals. Since $\pi(0) = 0$, then $\pi(a) + \pi(b) = \pi(a + b)$ for every $a \in A$ and $b \in B$. By Lemma 4.4, $\pi_1(a) + \pi_1(b) = \pi_1(a + b)$ and $\pi_2(a) + \pi_2(b) = \pi_2(a + b)$ for every $a \in A$ and $b \in B$. Thus, by the Interval lemma (Lemma 4.5), $\pi_1$ and $\pi_2$ are affine functions in each of the closed intervals $A$, $B$ and $A + B$, where $\pi_1$ has positive slope $s_1^+$ and $\pi_2$ has positive slope $s_2^+$ in each of these sets. The proof for the case $i$ odd is identical, only one needs to choose intervals $A = [r_i + \epsilon, r_{i+1}]$, $B = [1 - \epsilon, 1]$ and use the fact that $\pi(1) = 0$. This shows that, for $i$ even, $\pi_1(r) = \pi_1(r_j) + s_1^+(r - r_j)$ and $\pi_2(r) = \pi_2(r_j) + s_2^+(r - r_j)$, while, for $i$ odd, $\pi_1(r) = \pi_1(r_j) + s_1^-(r - r_j)$ and $\pi_2(r) = \pi_2(r_j) + s_2^-(r - r_j)$. In particular $\pi_1$ and $\pi_2$ are continuous piecewise-linear functions.

Define $L_\ell^+$ and $L_r^+$ as the sum of the lengths of the intervals of positive slope included in $[0, 1 - f]$ and $[1 - f, 1]$, respectively. Define $L_\ell^-$ and $L_r^-$ as the sum of the lengths of the intervals of negative slope included in $[0, 1 - f]$ and $[1 - f, 1]$, respectively. Note that $L_\ell^+ > 0$ and $L_r^- > 0$.

By the above claim, since $\pi(0) = \pi_1(0) = \pi_2(0) = 0$, $\pi(1) = \pi_1(1) = \pi_2(1) = 0$ and $\pi(1 - f) = \pi_1(1 - f) = \pi_2(1 - f) = 1$, it follows that the vectors $(s^+, s^-)$, $(s_1^+, s_1^-)$, $(s_2^+, s_2^-)$ all satisfy the system

$$\begin{aligned} L_\ell^+ \sigma^+ + L_\ell^- \sigma^- &= 1 \\ L_r^+ \sigma^+ + L_r^- \sigma^- &= -1. \end{aligned}$$

Suppose the constraint matrix of the above system is singular. Then the vector $(L_r^+, L_r^-)$ is a multiple of $(L_\ell^+, L_\ell^-)$, so it must be a nonnegative multiple, but this is impossible since the right-hand-side of the two equations are one positive and one negative. Thus the constraint matrix is nonsingular, so the system has a unique solution. This implies that $\sigma^+ = s^+ = s_1^+ = s_2^+$ and $\sigma^- = s^- = s_1^- = s_2^-$, and therefore $\pi = \pi_1 = \pi_2$. □

## 4.2 Continuous infinite relaxation

If in (4.2) we disregard the integer variables, we obtain the following *continuous infinite relaxation*

$$\begin{aligned} f + \sum_{r \in \mathbb{R}^q} r y_r &\in \mathbb{Z}^q \\ y_r \geq 0 & \qquad \text{for all } r \in \mathbb{R}^q \\ y & \qquad \text{has a finite support.} \end{aligned} \tag{4.6}$$

Denote by $R_f$ the set of feasible solutions to (4.6). A function $\psi : \mathbb{R}^q \to \mathbb{R}$ is *valid* for $R_f$ if the linear inequality

$$\sum_{r \in \mathbb{R}^q} \psi(r) y_r \geq 1 \tag{4.7}$$

is satisfied by all vectors in $R_f$.

A valid function $\psi : \mathbb{R}^q \to \mathbb{R}$ for $R_f$ is *minimal* if there is no valid function $\psi' \neq \psi$ such that $\psi'(r) \leq \psi(r)$ for all $r \in \mathbb{R}^q$.

Note that the notions of valid function and minimal valid function defined above are closely related to the notions introduced at the end of Section 3.2. In particular, we will show a one-to-one correspondence between minimal valid functions for $R_f$ and maximal $\mathbb{Z}^q$-free convex sets containing $f$ in their interior.

The next lemma establishes how $\mathbb{Z}^q$-free convex sets with $f$ in their interior naturally yield valid functions for $R_f$.

**Lemma 4.7.** *Let $B$ be a $\mathbb{Z}^q$-free closed convex set with $f$ in its interior. Let $\psi$ be the gauge of $B - f$. Then $\psi$ is a valid function.*

*Proof.* By Lemma 3.14, $\psi$ is sublinear. Consider $\bar{y} \in R_f$. Then $\sum r\bar{y}_r = \bar{x} - f$, for some $\bar{x} \in \mathbb{Z}^q$.

$$\sum \psi(r)\bar{y}_r = \sum \psi(r\bar{y}_r) \geq \psi(\sum r\bar{y}_r) = \psi(\bar{x} - f) \geq 1$$

where the first equality follows by positive homogeneity, the first inequality by subadditivity, and the last from the fact that $B$ is a $\mathbb{Z}^q$-free convex set and that $\psi$ is the gauge of $B - f$. □

On the other hand, we will prove that all minimal valid functions are gauges of maximal $\mathbb{Z}^q$-free convex sets containing $f$ in their interior. First, we need to prove the following.

**Lemma 4.8.** *If $\psi : \mathbb{R}^q \to \mathbb{R}$ is a minimal valid function for $R_f$, then $\psi$ is nonnegative and sublinear.*

*Proof.* We first note that $\psi(0) \geq 0$. Indeed, consider any $\bar{y} \in R_f$. Let $\tilde{y} = \bar{y}$ except for the component $\tilde{y}_0$ which is set to an arbitrarily large value $M$. We have $\tilde{y} \in R_f$. Therefore $\sum \psi(r)\tilde{y}_r \geq 1$. For this inequality to hold for all $M > 0$, we must have $\psi(0) \geq 0$.

a) *$\psi$ is sublinear.* We first prove that $\psi$ is subadditive. When $r^1 = 0$ or $r^2 = 0$, $\psi(r^1)+\psi(r^2) \geq \psi(r^1 + r^2)$ follows from $\psi(0) \geq 0$. The proof for the case that $r^1, r^2 \neq 0$ is identical to part b) of the proof of Theorem 4.1.

We next show that $\psi$ is positively homogeneous. Suppose there exists $\tilde{r} \in \mathbb{R}^q$ and $\lambda > 0$ such that $\psi(\lambda\tilde{r}) \neq \lambda\psi(\tilde{r})$. Without loss of generality we may assume that $\psi(\lambda\tilde{r}) < \lambda\psi(\tilde{r})$, else we can consider $\lambda\tilde{r}$ instead of $\tilde{r}$ and $\lambda^{-1}$ instead of $\lambda$. Define a function $\psi'$ by $\psi'(\tilde{r}) := \lambda^{-1}\psi(\lambda\tilde{r})$, $\psi'(r) := \psi(r)$ for all $r \neq \tilde{r}$. We will show that $\psi'$ is valid. Consider any $\bar{y} \in R_f$. We need to show that $\sum_r \psi'(r)\bar{y}_r \geq 1$. Define $\tilde{y}$ as follows

$$\tilde{y}_r := \begin{cases} 0 & \text{if } r = \tilde{r} \\ \bar{y}_{\lambda\tilde{r}} + \lambda^{-1}\bar{y}_{\tilde{r}} & \text{if } r = \lambda\tilde{r} \\ \bar{y}_r & \text{otherwise.} \end{cases}$$

Note that $f + \sum r\bar{y}_r = f + \sum r\tilde{y}_r \in \mathbb{Z}^q$ and $\tilde{y} \geq 0$, thus $\tilde{y} \in R_f$. Using the definitions of $\psi'$ and $\tilde{y}$, it follows that $\sum_r \psi'(r)\bar{y}_r = \sum_r \psi(r)\tilde{y}_r \geq 1$, where the latter inequality follows from the facts that $\psi$ is valid and $\tilde{y} \in R_f$. This shows that $\psi'$ is valid, contradicting the fact that $\psi$ is minimal. Therefore $\psi$ is positively homogeneous.

b) *$\psi$ is nonnegative.* Suppose $\psi(\tilde{r}) < 0$ for some $\tilde{r} \in \mathbb{Q}^q$. Let $D \in \mathbb{Z}_+$ such that $D\tilde{r}$ is an integral vector, and let $\bar{y}$ be a feasible solution of $R_f$ (for example $\bar{y}_r = 1$ for $r = -f$, $\bar{y}_r = 0$ otherwise). Let $\tilde{y}$ be defined by $\tilde{y}_{\tilde{r}} := \bar{y}_{\tilde{r}} + MD$ where $M$ is a positive integer, and $\tilde{y}_r := \bar{y}_r$ for $r \neq \tilde{r}$. It follows that $\tilde{y}$ is a feasible solution of $R_f$.

We have $\sum \psi(r)\tilde{y}_r = \sum \psi(r)\bar{y}_r + \psi(\tilde{r})MD$. Choose the integer $M$ large enough, namely $M > \frac{\sum \psi(r)\bar{y}_r - 1}{D|\psi(\tilde{r})|}$. Then $\sum \psi(r)\tilde{y}_r < 1$, contradicting the fact that $\tilde{y}$ is feasible.

Since $\psi$ is sublinear, by Lemma 3.15 it is continuous. Thus, since $\psi$ is nonnegative over $\mathbb{Q}^q$ and $\mathbb{Q}^q$ is dense in $\mathbb{R}^q$, $\psi$ is nonnegative over $\mathbb{R}^q$. □

**Theorem 4.9.** *A function $\psi$ is a minimal valid function for $R_f$ if and only if there exists some maximal $\mathbb{Z}^q$-free convex set $B$ such that $\psi$ is the gauge of $B - f$.*

*Proof.* For the "only if" part, let $\psi$ be a minimal valid function. Define $B := \{x \in \mathbb{R}^q : \psi(x - f) \leq 1\}$. Since $\psi$ is a minimal valid function, by Lemma 4.8, $\psi$ is a nonnegative sublinear function. Thus, by Theorem 3.16, $B$ is a closed convex set with $f$ in its interior and $\psi$ is the gauge of $B - f$. Furthermore, $B$ is a $\mathbb{Z}^q$-free convex set because, given that $\psi$ is valid, $\psi(\bar{x} - f) \geq 1$ for every $\bar{x} \in \mathbb{Z}^q$. We only need to prove that $B$ is a maximal $\mathbb{Z}^q$-free convex set. Suppose not, and let $B'$ be a $\mathbb{Z}^q$-free convex set properly containing $B$. Let $\psi'$ be the gauge of $B' - f$. Then by definition of gauge $\psi' \leq \psi$, and $\psi' \neq \psi$ since $B' \neq B$. By Lemma 4.7, $\psi'$ is a valid function, a contradiction to the minimality of $\psi$.

To prove the "if" part, assume that $\psi$ is the gauge of $B - f$ for some maximal $\mathbb{Z}^q$-free convex set $B$. By Lemma 4.7 $\psi$ is valid for $R_f$. Suppose there exists a valid function $\psi'$ such that $\psi' \leq \psi$ and $\psi' \neq \psi$. Then $B' := \{x : \psi'(x - f) \leq 1\}$ is a $\mathbb{Z}^q$-free convex set and $B' \supset B$, contradicting the maximality of $B$. □

A function $\psi : \mathbb{R}^q \to \mathbb{R}$ is *piecewise-linear* if $\mathbb{R}^q$ can be covered by a finite number of polyhedra $P_1, \ldots, P_t$ whose interiors are pairwise disjoint, so that the restriction of $\psi$ to the interior of $P_i$, $i = 1, \ldots, t$ is an affine function. The restrictions of $\psi$ to $P_i$, $i = 1, \ldots, t$ are the *pieces* of $\psi$.

Given a maximal $\mathbb{Z}^q$-free convex set $B$ containing $f$ in its interior, if follows from Theorem 3.18 that $B$ is a polyhedron, thus by Theorem 3.20 $B$ can be written in the form

$$B = \{x \in \mathbb{R}^q : d^i(x - f) \leq 1, ; i = 1, \ldots, t\}$$

for some $d^1, \ldots, d^t \in \mathbb{R}^q$, and that the gauge of $B - f$ is the function defined by

$$\psi(r) = \max_{i=1,\ldots,t} d^i r. \tag{4.8}$$

Note that the function $\psi$ defined by (4.8) is piecewise-linear, and it has as many pieces as the number of facets of $B$. This discussion and Theorems 3.19, 4.9 imply the following.

**Corollary 4.10.** *Every minimal valid function for $R_f$ is a nonnegative sublinear piecewise-linear function with at most $2^q$ pieces.*

**Example 4.11.** Consider the maximal $\mathbb{Z}^2$-free set $B$ defined in Example 3.21, and let $f = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. The corresponding function $\psi$ has three pieces, corresponding to the three polyhedral cones $P_1$, $P_2$, $P_3$ shown in Figure 4.3, and we have

$$\psi(r) = \begin{cases} -2r_1 & \text{for } r \in P_1 \\ -2r_2 & \text{for } r \in P_2 \\ r_1 + r_2 & \text{for } r \in P_3 \end{cases} \qquad r \in \mathbb{R}^2.$$

■

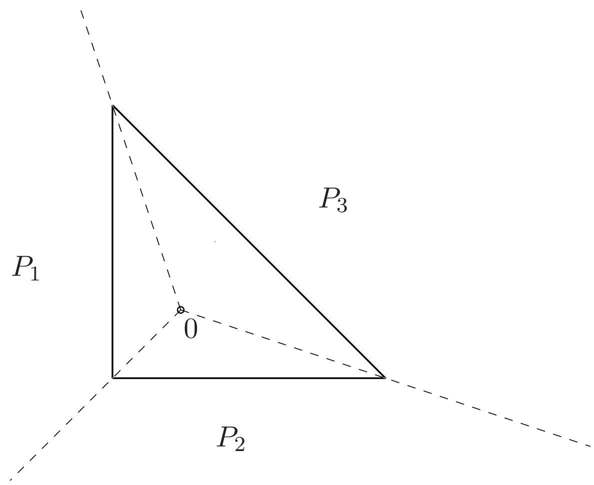


Figure 4.3: Set $B - f$, and the three pieces $P_1, P_2, P_3$ of the associated gauge $\psi$.

## 4.3 The mixed integer infinite relaxation

We finally return to the infinite relaxation (4.2) defined at the beginning of Section 5:

$$\begin{array}{rcl} f + \sum_{r\in\mathbb{R}^q} rx_r + \sum_{r\in\mathbb{R}^q} ry_r & \in & \mathbb{Z}^q \\ x_r \in \mathbb{Z}_+ & & \text{for all } r \in \mathbb{R}^q \\ y_r \geq 0 & & \text{for all } r \in \mathbb{R}^q \\ x, y & & \text{have a finite support.} \end{array}$$

Recall that $M_f$ denotes the set of feasible solutions to (4.2). The purpose of this section is to provide a characterization of the minimal valid inequalities for $M_f$.

**Lemma 4.12.** *Let $(\pi, \psi)$ be a minimal valid function for $M_f$. Then $\pi \leq \psi$ and $\psi$ is a nonnegative sublinear function.*

*Proof.* The same proof as that in Lemma 4.8 shows that $\psi$ is nonnegative and sublinear. We next show that $\pi \leq \psi$. Suppose not, and let $\tilde{r} \in \mathbb{R}^q$ such that $\pi(\tilde{r}) > \psi(\tilde{r})$. Let $\pi'$ be the function defined by $\pi'(\tilde{r}) := \psi(\tilde{r})$, $\pi'(r) := \pi(r)$ for $r \neq \tilde{r}$. We will show that $(\pi', \psi)$ is valid for $M_f$. Given $(\bar{x}, \bar{y}) \in M_f$, define

$$\tilde{x}_r := \begin{cases} 0 & \text{for } r = \tilde{r} \\ \bar{x}_r & \text{for } r \neq \tilde{r} \end{cases} \qquad \tilde{y}_r := \begin{cases} \bar{x}_r + y_r & \text{for } r = \tilde{r} \\ \bar{y}_r & \text{for } r \neq \tilde{r}. \end{cases}$$

It is immediate to check that $(\tilde{x}, \tilde{y}) \in M_f$. It follows that $\sum_{r\in\mathbb{R}^q} \pi'(r)\bar{x}_r + \sum_{r\in\mathbb{R}^q} \psi(r)\bar{y}_r = \sum_{r\in\mathbb{R}^q} \pi(r)\tilde{x}_r + \sum_{r\in\mathbb{R}^q} \psi(r)\tilde{y}_r \geq 1$. This shows that $(\pi', \psi)$ is a valid function, contradicting the minimality of $(\pi, \psi)$. □

The next theorem, due to Johnson [46], provides a characterization of minimal valid functions, and it shows that in a minimal valid function $(\pi, \psi)$ for $M_f$, the function $\psi$ is uniquely determined by $\pi$.

**Theorem 4.13.** *Let $(\pi, \psi)$ be a valid function for $M_f$. The function $(\pi, \psi)$ is minimal for $M_f$ if and only if $\pi$ is a minimal valid function for $G_f$ and $\psi$ is defined by*

$$\psi(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \quad \textit{for every } r \in \mathbb{R}^q. \tag{4.9}$$

*Proof.* Using the same arguments as in points a)-d) of the proof of Theorem 4.1, one can show that, if $(\pi, \psi)$ is minimal, then the function $\pi : \mathbb{R}^q \to \mathbb{R}$ is subadditive, periodic and satisfies the symmetry condition, and $\pi(0) = 0$. Thus, by Theorem 4.1, if $(\pi, \psi)$ is a minimal valid function for $M_f$ then $\pi$ is a minimal valid function for $G_f$.

Therefore, we only need to show that, given a function $(\pi, \psi)$ such that $\pi$ is minimal for $G_f$, $(\pi, \psi)$ is a minimal valid function for $M_f$ if and only if $\psi$ is defined by (4.9).

Let us define the function $\psi'$ by

$$\psi'(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \quad \text{for every } r \in \mathbb{R}^q.$$

We will show that $\psi'$ is well defined, $(\pi, \psi')$ is valid for $M_f$, and that $\psi' \leq \psi$. This will imply that $(\pi, \psi)$ is minimal if and only if $\psi = \psi'$, and the statement will follow.

We now show that $\psi'$ is well defined. This amounts to showing that the $\limsup$ in (4.9) is always finite. We recall that

$$\limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} := \lim_{\alpha \to 0^+} \sup\{\frac{\pi(\epsilon r)}{\epsilon} \,:\, 0 < \epsilon \leq \alpha\} = \inf_{\alpha > 0} \sup\{\frac{\pi(\epsilon r)}{\epsilon} \,:\, 0 < \epsilon \leq \alpha\}.$$

Let $\psi''$ be a function such that $\psi'' \leq \psi$ and $(\pi, \psi'')$ is a minimal valid function for $M_f$ (as mentioned at the beginning of Section 5, such a function exists).

By Lemma 4.12, $\pi \leq \psi''$ and $\psi''$ is a sublinear function. Thus, for every $\epsilon > 0$ and every $r \in \mathbb{R}^q$, it follows that

$$\frac{\pi(\epsilon r)}{\epsilon} \leq \frac{\psi''(\epsilon r)}{\epsilon} = \psi''(r)$$

thus

$$\limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \leq \psi''(r).$$

This shows that $\psi'$ is well defined and $\psi' \leq \psi'' \leq \psi$. Furthermore, it follows from the definition of $\psi'$ and the definition of $\limsup$ that $\psi'$ is sublinear.

We conclude the proof by showing that $(\pi, \psi')$ is valid for $M_f$. Let $(\bar{x}, \bar{y}) \in M_f$. Suppose by contradiction that

$$\sum_{r \in \mathbb{R}^q} \pi(r)\bar{x}_r + \sum_{r \in \mathbb{R}^q} \psi'(r)\bar{y}_r = 1 - \delta$$

where $\delta > 0$. Define $\bar{r} := \sum_{r \in \mathbb{R}^q} r\bar{y}_r$. By definition of $\psi'$, it follows that, for some $\bar{\alpha} > 0$ sufficiently small,

$$\frac{\pi(\epsilon \bar{r})}{\epsilon} < \psi'(\bar{r}) + \delta \quad \text{for all } 0 < \epsilon \leq \bar{\alpha}. \tag{4.10}$$

Choose $D \in \mathbb{Z}$ such that $1/D \leq \bar{\alpha}$, and define, for all $r \in \mathbb{R}^q$,

$$\tilde{x}_r = \begin{cases} \bar{x}_r & r \neq \frac{\bar{r}}{D} \\ \bar{x}_r + D & r = \frac{\bar{r}}{D} \end{cases}$$

Note that all entries of $\tilde{x}$ are nonnegative integers and that $\sum_{r\in\mathbb{R}^q} r\tilde{x}_r = \sum_{r\in\mathbb{R}^q} r\bar{x}_r + \sum_{r\in\mathbb{R}^q} r\bar{y}_r$, thus $\tilde{x}$ is in $G_f$. Now

$$\begin{array}{rcll} \sum_{r\in\mathbb{R}^q} \pi(r)\tilde{x}_r & = & \sum_{r\in\mathbb{R}^q} \pi(r)\bar{x}_r + \frac{\pi(\bar{r}/D)}{1/D} & \\ & < & \sum_{r\in\mathbb{R}^q} \pi(r)\bar{x}_r + \psi'(\bar{r}) + \delta & \text{(by (4.10) because } 1/D \leq \bar{\alpha}) \\ & \leq & \sum_{r\in\mathbb{R}^q} \pi(r)\bar{x}_r + \sum_{r\in\mathbb{R}^q} \psi'(r)\bar{y}_r + \delta = 1, & \text{(by sublinearity of } \psi') \end{array}$$

contradicting the fact that $\pi$ is valid for $G_f$. □

If $(\pi, \psi)$ is a minimal valid function, then, by Theorem 4.13, $\pi$ is a minimal valid function for $G_f$. However, the next example illustrates that in general it is not the case that $\psi$ is a minimal valid function for $R_f$.

**Example 4.14.** Consider the three functions $\pi, \pi_1, \pi_2$ of Figure 4.1, where $t > 0$ and $t + 1/2 < f < 1$. As discussed in Example 4.3, these functions are extreme for $G_f$. For ease of notation, let $\pi_0 := \pi$. For $i = 0, 1, 2$, let $s_i^+$ be the positive slope of $\pi_i$ at 0 and $s_i^-$ be the negative slope at 1 (or at 0, since the function is periodic). By Theorem 4.13, for each $\pi_i$, the function $\psi_i$ for which $(\pi_i, \psi_i)$ is minimal for $M_f$ is the function defined by

$$\psi_i(r) := \begin{cases} s_i^+ r & \text{if } r \geq 0 \\ s_i^- r & \text{if } r < 0 \end{cases} \qquad r \in \mathbb{R}.$$

The positive slopes are identical ($s_i^+ = (1-f)^{-1}$ for $i = 0, 1, 2$), while the most negative slope is $s_0^- = f^{-1}$, thus $\psi_0$ is pointwise smaller than the other two functions. In particular $\psi_i$ is not minimal for $R_f$ for $i = 1, 2$. Note that $\psi_0$ is minimal for $R_f$, since it is the gauge of the set $B - f$, where $B := [0, 1]$ is a maximal $\mathbb{Z}$-free set. ■

## 4.4 Trivial and unique liftings

While Theorem 4.13 implies that minimal valid functions $(\pi, \psi)$ are entirely determined by the function $\pi$, and that they are in one-to-one correspondence with the minimal valid functions for $G_f$, verifying that a function $\pi$ is valid for $G_f$, let alone minimal, is a difficult task in general.

On the other hand, the function $\psi$ has a nice geometric characterization. Indeed, for any minimal valid function $(\pi, \psi)$ for $M_f$, Lemma 4.12 and Theorem 3.16 imply that the set $B := \{x \in \mathbb{R}^q : \psi(x - f) \leq 1\}$ is a $\mathbb{Z}^q$-free convex set and $\psi$ is the gauge of $B - f$. Conversely, Lemma 4.7 show that $\mathbb{Z}^q$-free convex sets define valid functions for $R_f$. Therefore, it may be desirable to start from a valid sublinear function $\psi$ for $R_f$, and construct a function $\pi$ such that $(\pi, \psi)$ is valid for $M_f$. We say that any such function $\pi$ is a *lifting* for $\psi$, and that $\pi$ is a *minimal lifting* for $\psi$ if there is no lifting $\pi'$ for $\psi$ such that $\pi' \leq \pi$, $\pi' \neq \pi$.

Note that, if we start from a minimal valid function $\psi$ for $R_f$, then, for every minimal lifting $\pi$ of $\psi$, the function $(\pi, \psi)$ is a minimal valid function for $M_f$. Also, it follows from the definition that, given a valid function $\psi$ for $R_f$ and a minimal lifting $\pi$ for $\psi$, there exists some function $\psi' \leq \psi$ such that $(\pi, \psi')$ is a minimal valid function for $M_f$. In particular, by Theorem 4.13, every minimal lifting $\pi$ for $\psi$ is a minimal valid function for $G_f$. It follows from Lemma 4.12 that $\pi \leq \psi$ for every minimal lifting $\pi$ of $\psi$. Moreover, since by Theorem 4.1 $\pi$ is periodic over the unit hypercube, it must be the case that $\pi(r) \leq \psi(r+w)$ for all $r \in \mathbb{R}^q$ and every $w \in \mathbb{Z}^q$.

**Remark 4.15.** *Let $\psi$ be a valid function for $R_f$. Define the function $\bar{\pi}$ by*

$$\bar{\pi}(r) = \inf_{w \in \mathbb{Z}^q} \psi(r+w) \qquad r \in \mathbb{R}^q. \tag{4.11}$$

*Then $\pi \leq \bar{\pi}$ for every minimal lifting $\pi$ of $\psi$. In particular, $\bar{\pi}$ is a lifting for $\psi$.*

The function $\bar{\pi}$ defined in (4.11) is called the *trivial lifting* of $\psi$ [8, 40].

**Example 4.16.** Let us consider the case $q = 1$. Assume that $0 < f < 1$, and Let $B = [0, 1]$. Let $\psi$ be the gauge of $B - f$. As one can easily check,

$$\psi(r) = \max\left\{\frac{r}{1-f}, -\frac{r}{f}\right\}.$$

One can verify that the trivial lifting $\bar{\pi}$ for $\psi$ is the following

$$\bar{\pi}(r) = \begin{cases} \frac{r - \lfloor r \rfloor}{1-f} & \text{if } r - \lfloor r \rfloor \leq 1 - f \\ \frac{\lceil r \rceil - r}{f} & \text{if } r - \lfloor r \rfloor > 1 - f \end{cases}$$

Observe that $\psi$ and $\bar{\pi}$ are the functions, given in (3.16), that define the Gomory mixed-integer inequalities.

It follows from the discussion in Example 4.2 that $\bar{\pi}$, in this case, is a minimal valid function for $G_f$, therefore $\bar{\pi}$ is in this case a minimal lifting. In particular, it follows from Remark 4.15 that, in this example, $\bar{\pi}$ is the unique minimal lifting of $\psi$. ■

In general, the trivial lifting is not minimal. However, we can argue that, if we start from a minimal valid function $\psi$, there always exists an infinite region of $\mathbb{R}^q$ within which all minimal liftings of $\psi$ coincide with the trivial lifting $\bar{\pi}$.

**Lemma 4.17.** *Let $(\pi, \psi)$ be a minimal valid function for $M_f$. Given $r^* \in \mathbb{R}^q$, if*

$$\psi(r^*) + \psi(z - f - r^*) = \psi(z - f) = 1 \quad \textit{for some } z \in \mathbb{Z}^q, \tag{4.12}$$

*then $\pi(r^*) = \psi(r^*) = \inf_{w \in \mathbb{Z}^q} \psi(r^* + w)$.*

*Proof.* Given $z \in \mathbb{Z}^q$, define

$$x_r := \begin{cases} 1 & \text{for } r = r^* \\ 0 & \text{for } r \neq r^* \end{cases} \qquad y_r := \begin{cases} 1 & \text{for } r = z - f - r^* \\ 0 & \text{for } r \neq z - f - r^* \end{cases}$$

It is straightforward to check that $(x, y) \in M_f$. Therefore we have

$$1 \leq \pi(r^*) + \psi(z - f - r^*) \leq \psi(r^*) + \psi(z - f - r^*) = \psi(z - f) = 1 \qquad (4.13)$$

where the first inequality follows from the fact that $(x, y) \in M_f$ and that $(\pi, \psi)$ is a valid function for $M_f$, the second inequality follows because $\pi(r^*) \leq \psi(r^*)$ by Lemma 4.12. Now (4.13) implies $\pi(r^*) = \psi(r^*)$. Finally, by Remark 4.15, $\pi(r^*) \leq \inf_{w \in \mathbb{Z}^q} \psi(r^* + w) \leq \psi(r^*)$, thus we have inequality throughout. □

Given a minimal valid function $\psi$ for $R_f$, if we let $R(\psi) := \{r \in \mathbb{R}^q : \psi(r^*) + \psi(\bar{z} - f - r) = \psi(\bar{z} - f) = 1 \text{ for some } \bar{z} \in \mathbb{Z}^q\}$, it follows from the above lemma that all minimal valid liftings coincide with the trivial lifting over the region $R(\psi) + \mathbb{Z}^q = \{r + w : r \in R^f, w \in \mathbb{Z}^q\}$. In particular, whenever $R(\psi) + \mathbb{Z}^q = \mathbb{R}^q$, the trivial lifting is the unique minimal lifting for $\psi$.

In [11] a converse of the above statement is proven. Namely, if $R(\psi) + \mathbb{Z}^q \subset \mathbb{R}^q$, then there exist more than one minimal lifting, so in particular the trivial lifting is not minimal.

**Example 4.18.** (Dey and Wolsey [34]) Let $q = 2$. Consider the maximal lattice-free triangle $B = \text{conv}(\binom{0}{0}, \binom{2}{0}, \binom{0}{2})$, and let $f$ be a point in the interior of $B$ (see Figure 4.4). Let $\psi$ be the gauge of $B - f$. By Theorem 4.9, $\psi$ is a minimal valid function for $R_f$.

For each of the three points $z_1 = \binom{1}{0}$, $z_2 = \binom{0}{1}$, $z_3 = \binom{1}{1}$ on the boundary of $B$, we have that $\psi(z_i - f) = 1$. For $i = 1, 2, 3$, define $R(z_i) = \{r \in \mathbb{R}^2 : \psi(r) + \psi(z_i - f - r) = \psi(z_i - f) = 1\}$, and let $R = R(z_1) \cup R(z_2) \cup R(z_3)$. Observe that $R \subseteq R(\psi)$. We will show that $R + \mathbb{Z}^q = \mathbb{R}^q$, so in this case the trivial lifting $\bar{\pi}$ is the unique minimal lifting of $\psi$.

Since $B$ is a maximal lattice-free convex set, the function $\psi$ is given by (4.8). Therefore the regions $R(z_1)$, $R(z_2)$, $R(z_3)$ are three quadrilaterals, namely they are obtained by translating the grey quadrilaterals depicted in Figure 4.4 by $-f$.

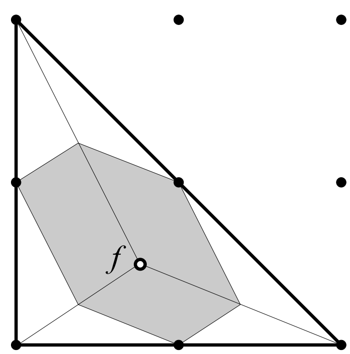


Figure 4.4: Lattice free triangle giving an inequality with a unique minimal lifting. The shaded region depicts $f + R$.

For $r \in \mathbb{R}^2$, $r - \lfloor r \rfloor$ is the unit box $[0, 1] \times [0, 1]$. Thus it suffices to show that every point in $[0, 1] \times [0, 1]$ can be translated by an integral vector into $f + R$. Note that $[0, 1] \times [0, 1] \setminus (f + R)$ is the union of the two triangles $\text{conv}(\binom{0}{0}, \binom{1}{0}, \frac{f}{2})$ and $\text{conv}(\binom{0}{0}, \binom{0}{1}, \frac{f}{2})$. The first one can be translated into $f + R$ by adding the vector $\binom{0}{1}$ and the second can be translated into $f + R$ by adding the vector $\binom{1}{0}$. The above argument shows that integral translations of $R$ cover $\mathbb{R}^2$. Since the area of $R$ is equal to 1, integral translations of $R$ actually define a tiling of

$\mathbb{R}^2$. This discussion implies that the trivial lifting can be computed efficiently. Indeed, for for any $r \in \mathbb{R}^2$, it gives a construction for an integral vector $\bar{w}$ such that $r + \bar{w} \in R$ and, by Lemma 4.17, $\inf_{w \in \mathbb{Z}^2} \psi(r + w) = \psi(r + \bar{w})$. ■

## 4.5 Exercises

**Exercise 4.1.** Prove that the following functions $g : \mathbb{R} \to \mathbb{R}$ are subadditive.

a) $g(x) := \lceil x \rceil$

b) $g(x) := x \mod t$, where $t$ is a given positive integer.

**Exercise 4.2.**

a) Show that, if $g : \mathbb{R}^n \to \mathbb{R}$ is a subadditive function, $g(0) \geq 0$.

b) Show that, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are two subadditive functions, $\max(f, g)$ is also a subadditive function.

**Exercise 4.3.** Let $g : \mathbb{R}^m \to \mathbb{R}$ be a nondecreasing subadditive function such that $g(0) = 0$. (Function $g$ is *nondecreasing* if for any $a, a' \in \mathbb{R}^m$ such that $a \leq a'$, we have $g(a) \leq g(a')$.)

Let $S := P \cap \mathbb{Z}^n$ where $P := \{x \in \mathbb{R}^n_+ : Ax \geq b\}$. We denote by $a^j$ the $j$th column of the $m \times n$ matrix $A$. Prove that $\sum_{j=1}^n g(a^j)x_j \geq g(b)$ is a valid inequality for $\text{conv}(S)$.

**Exercise 4.4.** Consider a continuous nonnegative periodic function $\pi : \mathbb{R} \to \mathbb{R}_+$ that is piecewise-linear in the interval $[0, 1]$ and satisfies $\pi(0) = 0$.
a) Show that, in order to check whether the symmetry condition $\pi(r) + \pi(-f - r) = 1$ holds for all $r \in \mathbb{R}$, it suffices to check it at the breakpoints of the function in the interval $[0, 1]$.
b) Assume that, in addition to the above properties, $\pi$ satisfies the symmetry condition. Show that, in order to check whether subadditivity $\pi(a) + \pi(b) \geq \pi(a + b)$ holds for all $a, b \in \mathbb{R}$, it is enough to check the inequality $\pi(a) + \pi(b) \geq \pi(a + b)$ at all the breakpoints $a, b$ in the interval $[0, 1]$ where the function is locally convex.

**Exercise 4.5.** Assume $t > 0$ and $1/2 + t \leq f < 1$. Show that the function $\pi_2$ in Example 4.2 is extreme.

**Exercise 4.6.** Consider the model

$$\begin{aligned} f + \sum_{r \in \mathbb{R}^q} r y_r \quad &\in \quad \mathbb{Z}^q_+ \\ y_r \geq 0 \quad & \quad \text{for all } r \in \mathbb{R}^q \\ y \quad & \quad \text{has a finite support} \end{aligned}$$

A convex set in $\mathbb{R}^q$ is $\mathbb{Z}^q_+$-free if it contains no point of $\mathbb{Z}^q_+$ in its interior. A function $\psi : \mathbb{R}^q \to \mathbb{R}$ is *valid* for the above model if the inequality $\sum_{r \in \mathbb{R}^q} \psi(r) y_r \geq 1$ is satisfied by every feasible vector $y$.

a) Show that if $\psi : \mathbb{R}^q \to \mathbb{R}$ is sublinear and the set $B_\psi := \{x \in \mathbb{R}^q : \psi(x - f) \leq 1\}$ is $\mathbb{Z}^q_+$-free, then $\psi$ is valid.

b) Let $q = 2$ and $f = \binom{1/4}{1/2}$. Show that the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\psi(r) = \max\{4r_1 + 4r_2, 4r_1 - 4r_2\}$ ($r \in \mathbb{R}^2$) is valid, by showing that $\psi$ is sublinear and $B_\psi$ $\mathbb{Z}^q_+$-free.

**Exercise 4.7.** Let $q = 2$. Consider the triangle $K$ with vertices $(-\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$, $(\frac{1}{2}, 2)$ and the point $f = (\frac{1}{2}, \frac{1}{2})$.
a) Show that $K$ is a maximal lattice-free convex set;
b) Compute the function $\psi_K$ given by (4.8);
c) Let $\pi_K$ be any minimal lifting of $\psi_K$. Determine the region $R := \{r \in \mathbb{R}^2 : \pi_K(r) = \psi_K(r)\}$;
d) Show that $\psi_K$ has a unique minimal lifting $\pi_K$ (Hint: Show that $R + \mathbb{Z}^2$ covers the plane).

**Exercise 4.8.** Let $\psi : \mathbb{R}^q \to \mathbb{R}$ be a minimal valid function for (4.6). Given $d \in \mathbb{R}^q$, consider the model

$$\begin{array}{rcl} f + \sum_{r \in \mathbb{R}^q} r y_r + dz & \in & \mathbb{Z}^q \\ y_r \geq 0 & & \text{for all } r \in \mathbb{R}^q \\ y & & \text{has a finite support} \\ z \in \mathbb{Z}_+. & & \end{array} \tag{4.14}$$

Let $\pi_\ell(d)$ be the minimum scalar $\lambda$ such that the inequality $\sum_{r \in \mathbb{R}^q} \psi(r) y_r + \lambda z \geq 1$ is valid for (4.14). Prove that when $(\pi_\ell, \psi)$ is valid for (4.2), then $\pi_\ell$ is the unique minimal lifting of $\psi$.

# Lecture 5

# Cut-Generating Functions for Integer Variables

Consider a pure integer linear program and the optimal simplex tableau of its linear programming relaxation. We select $n$ rows of the tableau, corresponding to $n$ basic variables $\{x_i\}_{i=1}^n$. Let $\{y_j\}_{j=1}^m$ denote the nonbasic variables. The tableau restricted to these $n$ rows is of the form

$$\begin{aligned} x =& f + \sum_{j=1}^{m} r^j y_j, \\ & x \in \mathbb{Z}^n_+, \\ & y_j \in \mathbb{Z}_+, \forall j \in \{1, \dots, m\}, \end{aligned} \tag{5.1}$$

where $f \in \mathbb{R}^n_+$ and $r^j \in \mathbb{R}^n$ for all $j \in \{1, \dots, m\}$. We assume $f \notin \mathbb{Z}^n$; therefore, the basic solution $x = f$, $y = 0$ is not feasible. We would like to generate cutting-planes that cut off this infeasible solution.

A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (5.1) if the inequality $\sum_{j=1}^m \pi(r^j) y_j \geq 1$ holds for all feasible solutions $(x, y)$ to (5.1) for any possible number $m$ of nonbasic variables and any choice of nonbasic columns $r^j$. In Lecture 4, we saw how Gomory and Johnson [40, 41] and Johnson [46] characterized such functions for the corner relaxation of (5.1) obtained by relaxing $x \in \mathbb{Z}^n_+$ to $x \in \mathbb{Z}^n$. They also introduced the infinite group relaxation

$$\begin{aligned} x =& f + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in \mathbb{Z}^n, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & y \text{ has finite support}, \end{aligned} \tag{5.2}$$

where an infinite-dimensional vector is said to have *finite support* if it has a finite number of nonzero entries. This model was analysed in detail in Section 4.1.

Here we consider the following generalization of the Gomory-Johnson model:

$$\begin{aligned} x =& f + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in S, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & y \text{ has finite support,} \end{aligned} \tag{5.3}$$

where $S$ can be any nonempty subset of the Euclidean space. This flexibility in the choice of $S$ makes (5.3) a relevant model for 1) integer convex and conic programs and 2) integer programs with complementarity constraints, as well as integer linear programs; see [23]. The Gomory-Johnson model (5.2) is the special case of (5.3) where $S = \mathbb{Z}^n$. The models studied in [45, 16, 2] are closely related to the case $S = \{0\}$. The case where $S = \mathbb{Z}^n_+$, or more generally where $S$ is the set of integer points in a full-dimensional rational polyhedron, is of particular interest in integer linear programming due to its connection to (5.1) above. It is a main focus of this lecture.

Note that (5.3) is nonempty since for any $\bar{x} \in S$, the solution $x = \bar{x}$, $y_{\bar{x}-f} = 1$, and $y_r = 0$ for all $r \neq \bar{x} - f$ is feasible. In the remainder of the paper, we assume that $f \in \mathbb{R}^n \setminus S$. Therefore, the basic solution $x = f$, $y = 0$ is not a feasible solution of (5.3). We are interested in valid inequalities for (5.3) that cut off the above infeasible basic solution.

We can generalize the notion of cut-generating function as follows. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (5.3) if the inequality $\sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ holds for all feasible solutions $(x, y)$ to (5.3). For example, the function that takes the value 1 for all $r \in \mathbb{R}^n$ is a cut-generating function because every feasible solution of (5.3) satisfies $y_r \geq 1$ for at least one $r \in \mathbb{R}^n$. When $S = \mathbb{Z}^n_+$, we recover the earlier definition of cut-generating function for (5.1).

A key feature that distinguishes the cut-generating functions for model (5.3) from those that were studied by Gomory and Johnson for model (5.2) is that they need not be nonnegative even if we assume continuity. In fact, they can take any real value, positive and negative, as the following examples illustrate.

**Example 5.1.** Consider the model (5.3) where $n = 1$, $0 < f < 1$, and $S = \mathbb{Z}_+$. Cornuéjols, Kis, and Molinaro [24] showed that for $0 < \alpha \leq 1$, the following family of functions are cut-generating functions:

$$\pi^1_\alpha(r) \;=\; \min\left\{ \frac{r - \lfloor \alpha r \rfloor}{1 - f},\; \frac{-r}{f} + \frac{\lceil \alpha r \rceil (1 - \alpha f)}{\alpha f (1 - f)} \right\}.$$

Note that when $\alpha = 1$, the function $\pi^1_1(r) = \min\{\frac{r - \lfloor r \rfloor}{1-f}, \frac{\lceil r \rceil - r}{f}\}$ is the well-known Gomory function. This function is periodic and takes its values in the interval $[0, 1]$. However, when $\alpha < 1$, this is not the case any more: The function $\pi^1_\alpha$ takes all real values between $-\infty$ and $+\infty$, and it is not periodic in the usual sense. See Figure 5.1. ■

The next example is mostly of theoretical interest. It illustrates another property of model (5.3) that does not arise in the Gomory-Johnson model (5.2).

**Example 5.2.** Consider the model (5.3) where $n = 1$, $f > 0$, and $S = \{0\}$. In this case, the model (5.3) reduces to $\sum_{r \in \mathbb{R}} r y_r = -f$, $y_r \in \mathbb{Z}_+$ for $r \in \mathbb{R}$, and $y$ has finite support. For

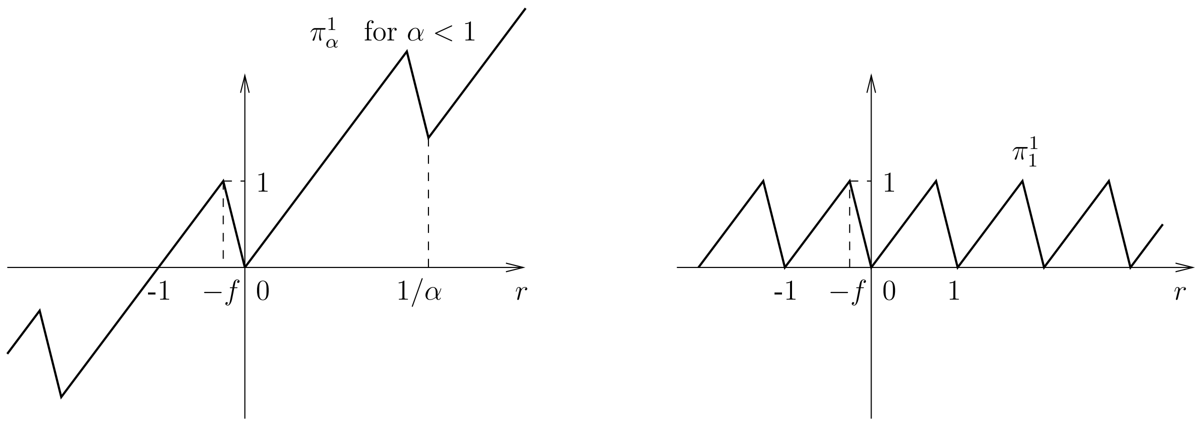


Figure 5.1: Two cut-generating functions: $\pi^1_\alpha$ for some $\alpha < 1$ and $\pi^1_1$.

any $\alpha \leq -\frac{1}{f} < 0$, the linear function $\pi^2_\alpha(r) = \alpha r$ is a cut-generating function. This can be seen by observing that for any $y$ feasible to (5.3), we have $\sum_{r\in\mathbb{R}} \pi^2_\alpha(r) y_r = \sum_{r\in\mathbb{R}} (\alpha r) y_r = \alpha \sum_{r\in\mathbb{R}} r y_r = -\alpha f \geq 1$. ■

## 5.1 Minimal Cut-Generating Functions

We say that a cut-generating function $\pi'$ for (5.3) *dominates* another cut-generating function $\pi$ if $\pi \geq \pi'$, that is, $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. A cut-generating function $\pi$ is *minimal* if there is no cut-generating function $\pi'$ distinct from $\pi$ that dominates $\pi$. When $n = 1$, $S = \mathbb{Z}_+$, and $0 < f < 1$, the cut-generating functions $\pi^1_\alpha$ of Example 5.1 are minimal [24]. Later in Section 5.2, we will show that the linear cut-generating functions $\pi^2_\alpha$ of Example 5.2 are also minimal. The following theorem shows that minimal cut-generating functions indeed always exist when $S \neq \emptyset$ in (5.3).

**Theorem 5.3** (Basu and Paat [13], Yıldız and Cornuéjols [57]). *Every cut-generating function for* (5.3) *is dominated by a minimal cut-generating function.*

*Proof.* Let $\pi$ be a cut-generating function for (5.3). Denote by $\Pi$ the set of cut-generating functions $\pi'$ that dominate $\pi$. Let $\{\pi_\ell\}_{\ell\in L} \subset \Pi$ be a nonempty family of cut-generating functions such that for any pair $\ell', \ell'' \in L$, we have $\pi_{\ell'} \leq \pi_{\ell''}$ or $\pi_{\ell'} \geq \pi_{\ell''}$. To prove the claim, it is enough to show by Zorn's Lemma (see, e.g., [22]) that there exists a cut-generating function that is a lower bound on $\{\pi_\ell\}_{\ell\in L}$.

Define the function $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}\cup\{-\infty\}$ as $\bar{\pi}(r) = \inf_\ell\{\pi_\ell(r) : \ell \in L\}$. Clearly, the function $\bar{\pi}$ is a lower bound on $\{\pi_\ell\}_{\ell\in L}$. We show that it is a cut-generating function for (5.3). First we prove that $\bar{\pi}$ is finite everywhere. Choose $\bar{x} \in S$. For any $\bar{r} \in \mathbb{R}^n$, let $\bar{y}$ be defined as $\bar{y}_{\bar{r}} = 1$, $\bar{y}_{\bar{x}-f-\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise. The solution $(\bar{x}, \bar{y})$ is feasible to (5.3). Then for any $\ell \in L$, the cut-generating function $\pi_\ell$ satisfies $\sum_{r\in\mathbb{R}^n} \pi_\ell(r)\bar{y}_r = \pi_\ell(\bar{r}) + \pi_\ell(\bar{x} - f - \bar{r}) \geq 1$. Moreover, we have $\pi_\ell \leq \pi$ because $\pi_\ell \in \Pi$; hence,

$$\pi_\ell(\bar{r}) \geq 1 - \pi_\ell(\bar{x} - f - \bar{r}) \geq 1 - \pi(\bar{x} - f - \bar{r}).$$

Therefore, $\bar{\pi}(\bar{r}) \geq 1 - \pi(\bar{x} - f - \bar{r})$. This shows that $\bar{\pi}(r)$ is finite for all $r \in \mathbb{R}^n$. That is, $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}$. Now consider any feasible solution $(x, y)$ of (5.3). Note that $\{\pi_\ell\}_{\ell \in L}$ is a totally ordered set, $\bar{\pi}$ is finite everywhere, and only a finite number of the terms $y_r$ are nonzero. Combining these facts, we can write

$$\sum_{r \in \mathbb{R}^n} \bar{\pi}(r) y_r = \sum_{r \in \mathbb{R}^n} \inf_{\ell} \{\pi_\ell(r) : \ell \in L\} \, y_r = \inf_{\ell} \left\{ \sum_{r \in \mathbb{R}^n} \pi_\ell(r) y_r : \ell \in L \right\} \geq 1.$$

This proves that $\bar{\pi}$ is a cut-generating function. □

Theorem 5.3 shows that one can focus on minimal cut-generating functions since non-minimal ones are not needed in the description of the convex hull of feasible solutions to (5.3).

Recall that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$; it is *symmetric* or satisfies the *symmetry condition* if $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^n$; and it is *periodic with respect to* $\mathbb{Z}^n$ if $\pi(r) = \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n$. When $S = \mathbb{Z}^n$, cut-generating functions are traditionally assumed to be nonnegative. Theorem 4.1, due to Gomory and Johnson, shows that subadditivity, symmetry, and periodicity with respect to $\mathbb{Z}^n$ are necessary and sufficient for a nonnegative cut-generating function to be minimal. However, for general $S$, Examples 5.1 and 5.2 show that minimal cut-generating functions do not necessarily satisfy periodicity with respect to $\mathbb{Z}^n$, nor symmetry. We define a new condition, which we call the generalized symmetry condition, to replace symmetry and periodicity in the characterization of minimal cut-generating functions for (5.3). Let $\mathbb{Z}_{++}$ be the set of strictly positive integers. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the *generalized symmetry condition* if

$$\pi(r) = \sup_{x,k} \left\{ \frac{1}{k} \left(1 - \pi(x - f - kr)\right) : x \in S, k \in \mathbb{Z}_{++} \right\} \text{ for all } r \in \mathbb{R}^n. \tag{5.4}$$

The functions $\pi^1_\alpha$ and $\pi^2_\alpha$ of Examples 5.1 and 5.2 satisfy the generalized symmetry condition. We briefly outline the proof in each case.

Let $S = \mathbb{Z}_+$ and $0 < f < 1$ in (5.3), and consider the function $\pi^1_\alpha$ defined in Example 5.1. The inequality $\bar{k}\pi^1_\alpha(\bar{r}) + \pi^1_\alpha(\bar{x} - f - \bar{k}\bar{r}) \geq 1$ holds for any $\bar{r} \in \mathbb{R}$, $\bar{k} \in \mathbb{Z}_{++}$, and $\bar{x} \in \mathbb{Z}_+$ because $\pi^1_\alpha$ is a cut-generating function [24] and the solution $x = \bar{x}$, $y_{\bar{r}} = \bar{k}$, $y_{\bar{x} - f - \bar{k}\bar{r}} = 1$, and $y_r = 0$ otherwise is feasible to (5.3). Hence, $\pi^1_\alpha(r) \geq \frac{1}{k}(1 - \pi^1_\alpha(x - f - kr))$ for all $r \in \mathbb{R}$, $k \in \mathbb{Z}_{++}$, and $x \in \mathbb{Z}_+$. Furthermore, the graph of $\pi^1_\alpha$ is symmetric relative to the point $(-f/2, 1/2)$. In other words, the symmetry condition holds: $\pi^1_\alpha(r) = 1 - \pi^1_\alpha(-f - r)$ for all $r \in \mathbb{R}$. Therefore, for all $r \in \mathbb{R}$, we have

$$\pi^1_\alpha(r) = 1 - \pi^1_\alpha(-f - r) \leq \sup_{x,k} \left\{ \frac{1}{k} \left(1 - \pi^1_\alpha(x - f - kr)\right) : x \in \mathbb{Z}_+, k \in \mathbb{Z}_{++} \right\} \leq \pi^1_\alpha(r).$$

This shows that $\pi^1_\alpha$ satisfies the generalized symmetry condition.

Now let $f > 0$ and $S = \{0\}$ in (5.3), and consider the function $\pi^2_\alpha$ of Example 5.2. Because $S = \{0\}$, the term $x$ disappears in (5.4). Using $\alpha \leq \frac{-1}{f}$, for any $r \in \mathbb{R}$, we can write

$$\sup_{k \in \mathbb{Z}_{++}} \left\{ \frac{1}{k} \left(1 - \pi^2_\alpha(-f - kr)\right) \right\} = \alpha r + \sup_{k \in \mathbb{Z}_{++}} \left\{ \frac{1 + \alpha f}{k} \right\} = \alpha r = \pi^2_\alpha(r).$$

This shows that $\pi^2_\alpha$ satisfies the generalized symmetry condition.

The main result about minimal cut-generating functions for (5.3) is the following theorem which holds for any choice of $S \neq \emptyset$.

**Theorem 5.4.** *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function $\pi$ is a minimal cut-generating function for* (5.3) *if and only if $\pi(0) = 0$, $\pi$ is subadditive and satisfies the generalized symmetry condition.*

Refer to [57] for the proof of this theorem.

## 5.2 Strengthening the Notion of Minimality

The notion of minimality that we defined above can be unsatisfactory for certain choices of $S$. We illustrate this in the next proposition and remark.

**Proposition 5.5.** *If a cut-generating function for* (5.3) *is linear, then it is minimal.*

*Proof.* Let $\pi$ be a linear cut-generating function for (5.3). By Theorem 5.3, there exists a minimal cut-generating function $\pi'$ such that $\pi' \leq \pi$. By Theorem 5.4, $\pi'$ is subadditive and $\pi'(0) = 0$. For any $r \in \mathbb{R}^n$, the inequality $\pi' \leq \pi$ implies $\pi(r) + \pi(-r) \geq \pi'(r) + \pi'(-r) \geq \pi'(0) = 0 = \pi(r) + \pi(-r)$ where the last equality follows from the linearity of $\pi$. Hence, $\pi' = \pi$. ☐

**Remark 5.6.** *For a minimal cut-generating function $\pi$, it is possible that the inequality $\sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ is implied by an inequality $\sum_{r \in \mathbb{R}^n} \pi'(r) y_r \geq 1$ arising from some other cut-generating function $\pi'$. Indeed, for $n = 1$, $f > 0$, and $S = \{0\}$, consider again the cut-generating functions $\pi^2_\alpha$ of Example 5.2 with $\alpha \leq -\frac{1}{f}$. These are minimal by Proposition 5.5. However, the inequalities $|\alpha| f \sum_{r \in \mathbb{R}} \frac{-r}{f} y_r \geq 1$ generated from $\pi^2_\alpha$ for $\alpha < -\frac{1}{f}$ are implied by the inequality $\sum_{r \in \mathbb{R}} \frac{-r}{f} y_r \geq 1$ generated for $\alpha = -\frac{1}{f}$.*

Therefore, it makes sense to define a stronger notion of minimality as follows. A cut-generating function $\pi'$ for (5.3) *implies* another cut-generating function $\pi$ *via scaling* if there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$. Note that when the function $\pi'$ is nonnegative, this notion is identical to the notion of domination introduced earlier; however, the two notions are distinct when $\pi'$ can take negative values. A cut-generating function $\pi$ is *restricted minimal* if there is no cut-generating function $\pi'$ distinct from $\pi$ that implies $\pi$ via scaling. This notion was the one used by Jeroslow [45], Blair [16], and Bachem, Johnson, and Schrader [2]; they just called it minimality. In this paper, we call it restricted minimality to distinguish it from the notion of minimality introduced in Section 5.1. The next proposition shows that restricted minimal cut-generating functions are the minimal cut-generating functions which enjoy an additional "tightness" property.

**Proposition 5.7.** *A cut-generating function $\pi$ for* (5.3) *is restricted minimal if and only if it is minimal and* $\inf_x \{\pi(x - f) : x \in S\} = 1$.

Refer to [57] for the proof of this proposition.

The next proposition shows that restricted minimal cut-generating functions exist and they are always sufficient to separate the infeasible basic solution $x = f$, $y = 0$ from the closed convex hull of feasible solutions to (5.3).

**Proposition 5.8.** *Every cut-generating function for* (5.3) *is implied via scaling by a restricted minimal cut-generating function.*

*Proof.* Let $\pi$ be a cut-generating function. Let $\mu = \inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi(r)y_r : (x,y) \text{ satisfies } (5.3)\}$; note that $\mu \geq 1$. Define $\pi' = \frac{\pi}{\mu}$. The function $\pi'$ is also a cut-generating function, and it satisfies $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi'(r)y_r : (x,y) \text{ satisfies } (5.3)\} = 1$. By Theorem 5.3, there exists a minimal cut-generating function $\pi^*$ that dominates $\pi'$. The function $\pi^*$ implies $\pi$ via scaling since $\mu\pi^* \leq \mu\pi' = \pi$. We claim that $\pi^*$ is restricted minimal. First note that $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi^*(r)y_r : (x,y) \text{ satisfies } (5.3)\} = 1$. Now consider $\beta \geq 1$ and a cut-generating function $\pi^{**}$ such that $\pi^* \geq \beta\pi^{**}$. We must have $\beta = 1$ since $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi^{**}(r)y_r : (x,y) \text{ satisfies } (5.3)\} \geq 1$. Then because $\pi^*$ is minimal, we get $\pi^{**} = \pi^*$. This proves the claim. ☐

For the case $S = \{0\}$, Bachem, Johnson, and Schrader [2] showed that restricted minimal cut-generating functions satisfy the symmetry condition. This can be generalized as in the next theorem. Let us say that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *nondecreasing with respect to a set* $S \subset \mathbb{R}^n$ if $\pi(r) \leq \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in S$.

**Theorem 5.9.** *Let* $K \subset \mathbb{R}^n$ *be a closed convex cone and* $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. *Let* $\pi : \mathbb{R}^n \to \mathbb{R}$. *The function* $\pi$ *is a restricted minimal cut-generating function for* (5.3) *if and only if* $\pi(0) = 0$, $\pi$ *is subadditive, nondecreasing with respect to* $S$, *and satisfies the symmetry condition.*

Refer to [57] for the proof of this theorem.

The notion of minimality can be strengthened even further by taking into consideration the linear inequalities that are valid for $S$. Let $\alpha^\top(x-f) \geq \alpha_0$ be a valid inequality for $S \subset \mathbb{R}^n$. Because $f + \sum_{r\in\mathbb{R}^n} ry_r = x \in S$ for any $(x,y)$ feasible to (5.3), such a valid inequality can be translated to the space of the nonbasic variables $y$ as $\sum_{r\in\mathbb{R}^n} \alpha^\top ry_r \geq \alpha_0$. We say that a cut-generating function $\pi'$ for (5.3) *implies* another cut-generating function $\pi$ for (5.3) if there exists a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $S$ and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \alpha^\top r + \beta\pi'(r)$ for all $r \in \mathbb{R}^n$. This definition makes sense because if $\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq 1$ is a valid inequality for (5.3), then $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq \sum_{r\in\mathbb{R}^n} \alpha^\top ry_r + \beta\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq \alpha_0 + \beta \geq 1$ is also valid for (5.3). When the closed convex hull of $S$, $\overline{\text{conv}}(S)$, is equal to the whole of $\mathbb{R}^n$, the only inequalities that are valid for $S$ are those that have $\alpha = 0$ and $\alpha_0 \leq 0$; in this case, a cut-generating function may imply another only via scaling. However, the two notions may be different when $\overline{\text{conv}}(S) \subsetneq \mathbb{R}^n$. We say that a cut-generating function $\pi$ is *strongly minimal* if there does not exist a cut-generating function $\pi'$ distinct from $\pi$ that implies $\pi$. Note that strongly minimal cut-generating functions are restricted minimal. Indeed, if $\pi$ is a cut-generating function that is not restricted minimal, there exists a cut-generating function $\pi' \neq \pi$ and $\beta \geq 1$ such that $\pi \geq \beta\pi'$; but then $\pi'$ implies $\pi$ by taking $\alpha = 0$ and $\alpha_0 = 0$ which shows that $\pi$ is not strongly minimal.

For a set which is contained in the nonnegative orthant, minimality of a valid inequality is usually defined with respect to the nonnegative orthant. In a model for disjunctive conic programs, Kılınç-Karzan [47] generalized this notion broadly by defining and studying the minimality of a valid inequality with respect to an arbitrary regular cone which contains the feasible solution set. The various definitions of the minimality of a cut-generating function that we explore in this lecture could be viewed from a similar perspective. The next result demonstrates how strengthening the notion of minimality imposes additional structure on cut-generating functions for (5.3).

Let $[k] = \{1, \ldots, k\}$ for $k \in \mathbb{Z}_{++}$; let $e^i$ denote the $i^{\text{th}}$ standard unit vector in $\mathbb{R}^n$ for $i \in [n]$. [57] proves the following theorem about strongly minimal cut-generating functions for (5.3) when $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$.

**Theorem 5.10.** *Let $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ and $f \in \mathbb{R}^n_+ \setminus S$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function $\pi$ is a strongly minimal cut-generating function for* (5.3) *if and only if $\pi(0) = 0$, $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$, $\pi$ is subadditive and satisfies the symmetry condition.*

[57] gives an example showing that strongly minimal cut-generating functions do not always exist. On the other hand, we can show that they always exist when the closed convex hull of $S$ is a full-dimensional polyhedron.

**Theorem 5.11.** *Suppose $\overline{\text{conv}}(S)$ is a full-dimensional polyhedron. Let $f \in \overline{\text{conv}}(S)$. Then every cut-generating function for* (5.3) *is implied by a strongly minimal cut-generating function.*

Refer to [57] for the proof of this theorem.

A yet stronger notion is that of extreme cut-generating function. A cut-generating function $\pi$ is *extreme* if whenever cut-generating functions $\pi_1, \pi_2$ satisfy $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, we have $\pi = \pi_1 = \pi_2$. It follows from this definition that extreme cut-generating functions are minimal. It is shown in [57] that extreme cut-generating functions must in fact be strongly minimal. Furthermore [57] proves a 2-Slope Theorem for extreme cut-generating functions for (5.3) when $S = \mathbb{Z}_+$, in the spirit of the Gomory-Johnson 2-Slope Theorem for $S = \mathbb{Z}$.

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