Algorithmic Game Theory

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Today's Outline

- Nash equilibria of Games
- Price of Anarchy and smooth games

Next:

- Learning in Games, and learning outcomes, quality of learning outcomes
- Auction as a game, games with incomplete information

Games and Solution Quality

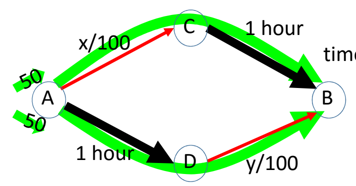
Tragedy of the Commons

- Rational selfish action can lead to outcome bad for everyone

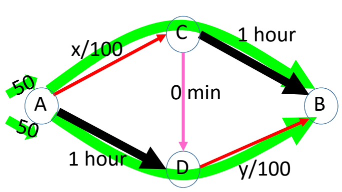
Model:

- Value for each cow decreasing function of # of cows
- Too many cows: no value left

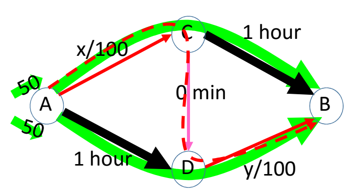
Example: flow equilibrium with 100 travelers



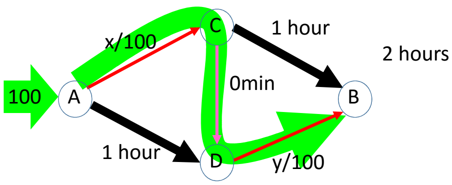
Add a new edge



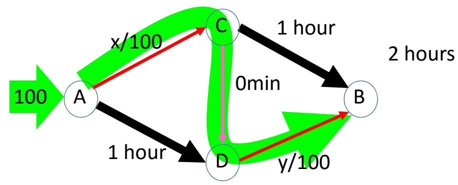
Not equilibrium!



Equilibrium



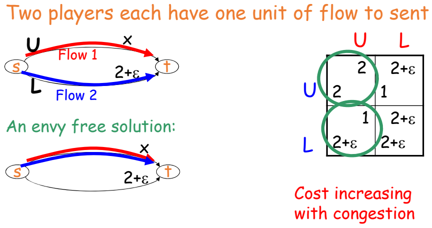
Braess' Paradox



Paradox: players optimize their own flow, yet total not optimal?

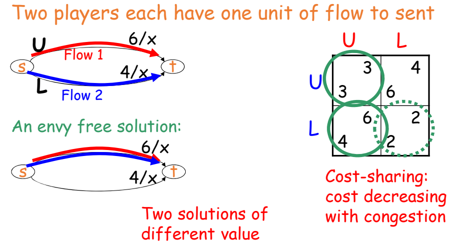
Examples on two links: load balancing

Two players each have one unit of flow to sent



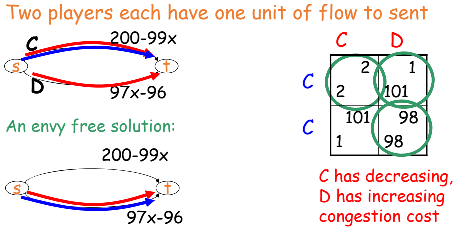
Examples on two links: coordination game

Two players each have one unit of flow to sent



Examples on two links: Prisoner's Dilemma

Two players each have one unit of flow to sent



What is Selfish Outcome?

Classical: Nash equilibrium

- Current strategy "best response" for all players (no incentive to deviate)

Theorem [Nash 1952]:

- Always exists if we allow randomized strategies

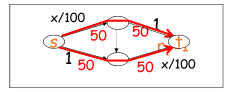
Price of Anarchy: $\dfrac{\text{cost of worst (pure) Nash}}{\text{"socially optimum" cost}}$

Games of minimizing cost

- Finite set of players 1,…,n
- strategy sets S_i for player i:
- Resulting in strategy vector: s=$(s_1, \dots, s_n)$ for each $s_i \in S_i$
- Cost of player i: $c_i(s)$ or $c_i(s_i, s_{-i})$
  Pure Nash equilibrium if $c_i(s) \le c_i(s_i', s_{-i})$ for all players and all alternate strategies $s'_i \in S_i$
- Social welfare: $\sum_i c_i(s)$
  Optimum: $\min_s \sum_i c_i(s)$

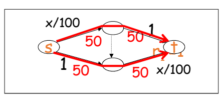
Model of Routing Game

- A directed graph G = (V,E)
- source–sink pairs s_i,t_i for i=1,..,k



•Goal minimum delay:
delay adds along path
edge-cost/delay is a function $c_e(\cdot)$ of the load on the edge e

Delay Functions

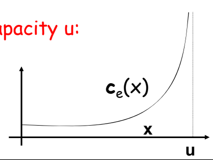


Assume $\ell_e(x)$ continuous and monotone increasing in load x on edge

No capacity of edges for now

Example to model capacity u:

$c_e(x)$= a/(u-x)



Goal's of the Game: min delay

Personal objective: minimize

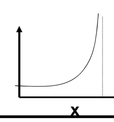
c_P(f) = sum of delays of edges along **P**
(wrt. flow **f**)

Overall objective:

C(f) = total delay of a flow **f**: = Σ_P $f_P \cdot c_P(f)$

= - social welfare
or total/average delay

Also:
C(f) = Σ_e $f_e \cdot c_P(f_e)$



Goal's of the Game: min cost

Personal objective: minimize

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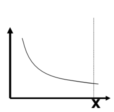


Price of Anarchy: proof technique

- What can work with:
  Optimum $s^* = (s_1^*, s_2^*, \dots, s_n^*)$
  Nash: $s = (s_1, s_2, \dots, s_n)$
- What we know:
  $c_i(s) \le c_i(s_i', s_{-i})$ for all i and all $s_i' \in S_i$

Use it for all players and sum
$c(s) = \sum_i c_i(s) \le \sum_i c_i(s_i^*, s_{-i})$

Proof smooth games

Nash property gave us (s is Nash, s* optimum)

$c(s) = \sum_i c_i(s) \le \sum_i c_i(s_i^*, s_{-i})$

Game is smooth if for some μ<1 and λ>0 and all s and s*
$\sum_i c_i(s_i^*, s_{-i}) \le \lambda c(s^*) + \mu\, c(s)$ (λ,μ)-smooth

Theorem: Price of anarchy for any (λ,μ)-smooth game is at most $\lambda/(1-\mu)$

Proving smoothness for flows

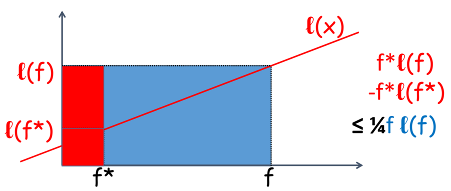
- What we need $\sum_i c_i(s_i^*, s_{-i}) \le \lambda c(s^*) + \mu\, c(s)$
- $\sum_e f_e^* c_e(f_e + 1) \le \lambda \sum_e f_e^* c_e(f_e^*) + \mu \sum_e f_e c_e(f_e)$

Non-atomic flow, when each user is small enough…

- Nash : each flow on shortest path
- Smoothness without the +1
- True edge-by-edge

Linear delay is smooth

Claim: f*•ℓ(f) ≤ f*•ℓ(f*) + ¼ f•ℓ(f)
assuming ℓ(f) linear: λ =1; μ=¼



Linear delay atomic flow

- Need to prove that for all integers x and y

$$x(y+1) \le \frac{5}{3}x^2 + \frac{1}{3}y^2$$

$$3xy + 3x \le 5x^2 + y^2$$

Examples of "smoothness bounds"

- Atomic game (players with >0 traffic) with linear delay (5/3,1/3)-smooth (Awerbuch-Azar-Epstein & Christodoulou-Koutsoupias'05)
  ⇒ 2.5 price of anarchy

Non-atomic (very small) players:

- Monotone increasing congestion costs (1,1) smooth
  ⇒ Nash cost ≤ opt of double traffic rate (Roughgarden-T'02)
- affine congestion cost are (1, ¼) smooth (Roughgarden-T'02)
  ⇒ 4/3 price of anarchy

Resulting bounds are often tight

Homework problem

- Prove that non-atomic congestion games (ignoring the +1 in the Nash condition) with increasing delay functions are (1,1) smooth
- Do these games have a good price of anarchy?
- Prove that the following: consider a non-atomic congestion game, and the same game with twice as much flow. Show that

Cost of Nash ≤ cost of opt with twice as much flow

More generally, how does cost of Nash compare to opt that carries (1+δ) times as much flow?