Algorithmic Game Theory

Auction Games, II

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Recall:

- Finite set of players 1,…,n
- strategy sets S_i for player i: bid on some items
  not a finite set
- Resulting in strategy vector: s=$(s_1, \dots, s_n)$ for each $s_i \in S_i$
- Utility player i: $u_i(s)$ or $u_i(s_i, s_{-i})$
  - We assume quasi-linear utility, and no externalities:
    - If player wins set if items A_i and pays p_i her value is $v_i(A_i) - p_i$
- Pure Nash equilibrium if $u_i(s) \geq u_i(s_i', s_{-i})$ for all players and all alternate strategies $s'_i \in S_i$

Robust Analysis: first price auction

No regret: $u_i(b) \geq \quad u_i\left(\frac{1}{2}v_i, b_{-i}\right) \geq \frac{1}{2}v_i - p$

either i wins or price above p $\geq \frac{1}{2}v_i$

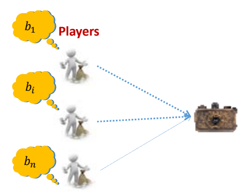
- Apply this to the top value
+ winner doesn't regret paying
$\Rightarrow$ winner has value $\geq \frac{1}{2} \max_i v_i$

No need to bid $\frac{1}{2}v_i$ …
Just don't regret this!

Bayes Nash analysis

Strategy: bid as a function of value $b_i(v)$

Nash: $E_{v_{-i}b}[u_i(b(v))|v_i] \geq E_{v_{-i}b_{-i}}[u_i(b_i', b_{-i}(v_{-i}))|v_i]$
for all b_i'



Same bound on price of anarchy,
same prof (take expectation)

Smoothness for auctions

Auction game is (λ,μ)-smooth if for some μ>1, λ>0 and some strategy s* and all s we have

$$\sum_i u_i(s_i^*, s_{-i}) \geq \lambda opt - \mu\, R(s)$$

R(s) = revenue at bid vector s (usually μ=1)

Theorem: [Syrgkanis-T'13] Price of anarchy for any (λ,μ)-auction game is at most μ/λ

Social welfare: $\sum_i u_i(s) + R(s)$

Smoothness for auctions

for some μ>1, λ>0 and some strategy s* and all s we have

$$\sum_i u_i(s_i^*, s_{-i}) \geq \lambda opt - \mu\, R(s)$$

R(s) = revenue at bid vector s (usually μ=1)

Price of Anarchy: full information.

Social welfare:

$$\sum_i u_i(s) + R(s) \geq \sum_i u_i(s_i^*, s_{-i}) + R(s) \geq \lambda Opt - \mu R(s) + R(s)$$

Smoothness and Bayesian games

We had $b_i^*(v) = v_i/2$. Depends only on the players own value!

Theorem: Auction is (λ,μ)-smooth and $b_i^*$is a function of v_i only, then price of anarchy bounded by $\mu\ /\lambda$ for arbitrary (private value) type distributions

Proof: just take expectations!

All pay auction

Claim: all pay auction is (1/2, 1)-smooth

Max value player: s_i*(v) uniform random [0,v].

All others: bid s_i*(0)

i not the top value: $u_i(s_i^*, s_{-i}) = 0$

i is the top value, and suppose max other bid is b.

If b>v_i we are set: $\sum_i u_i(s_i^*, s_{-i}) \geq -\frac{v_i}{2} \geq \frac{1}{2} Opt\ - b$

Else expected value for player i

$\mathrm{E}(\mathrm{u_i}(\mathrm{s_i^*}, \mathrm{s}_{-i})) = -\frac{v_i}{2} + v_i \frac{v_i - b}{v_i} \geq \frac{1}{2} v_i - b$

Bayesian extension theorem

Theorem [Syrgkanis-T'13] Auction game is (λ,μ)-auction smooth, and values are drawn from independent distribution, than the Price of anarchy in the Bayesian game is at most $\mu\ /\lambda$

Extension theorem: OK to only think about the full information game!

Proof idea: bid b*(v)….

Trouble: depends on other players and hence we don't know……

Bayesian extension theorem

- Notation v=$(v_1, \ldots v_n)$ value vector and use $b_i^*(v) = b_i^*(v_i, v_{-i})$

Idea: random sample opponent w_{-i}, and bid $b_i^*(v_i, w_{-i})$

Any fixed value v_i, and any player i we get

$$E_{w_{-i}b_{-i}}(u_i(b_i^*(v_i, w_{-i}), b_{-i}|v_i) \geq E_{b_{-i}}(u_i(b))$$

Rename $w_{-i} = v_{-i}$, and also take expectation over v_i

$$E_{vb}(u_i(b_i^*(v), b_{-i}) \geq E_{vb}(u_i(b))$$

Bayesian extension theorem (cont)

$$E_{vb}(u_i(b_i^*(v), b_{-i}) \leq E_{vb}(u_i(b))$$

Recall smoothness: for all fixed v and b

$$\sum_i u_i(b_i^*(v), b_i|v_i) \geq \text{\lamba } Opt(v) - \mu R(b)$$

Combine and take expectation over b and v (these are independent in the above!!!)

$$E_{vb}(\sum_i u_i(b)) \geq E_{vb}(\sum_i u_i(b_i^*(v), b_{-i})) \geq \lambda E_v(Opt(v) - \mu\ E_b(R(b))$$

Second price auction

Other pricing schemes:

Highest bid wins, and pays second highest bid, third highest, etc

Similar conclusion if we change $(\mu + 1)/\lambda$

- $\sum_i u_i(s_i^*, s_{-i}) \geq \lambda opt - \mu\ b(s)$ where b(s) is the sum of highest bids on items
- And assume no overbidding!

Unit demand bidders

- Values v_{ij} value of item j for player i. If i gets a set of items A_i her value is $\mathrm{v_i(A_i)} = \max_{j \in A_i} v_{ij}$ (free disposal)

Opt = matching!

(1/2, 1)-smooth: bid $\frac{v_{ij}}{2}$ on item j assigned in Opt

Homework

- All pay variant of multi-item auction from yesterday, also (1/2, 1)-smooth
- All pay auction with two bidder, each with value uniform [0,1] independent. What is the symmetric Nash?
- All pay auction with n bidder, each with value uniform [0,1] independent. What is the symmetric Nash?
- All pay auction with two bidder, each with value uniform [1,2] independent. What is the symmetric Nash?

All pay auction

Highest bidder wins, but all pay!

Example: n players uniform value [0,1], symmetric bidding b(v)

Need v $= argmax_z \; - b(z) + z^{n-1} v$,

We get $b'(v) = (n-1)v^{n-1}$, so $b(v) = \frac{n-1}{n} v^n$

All pay auction

Highest bidder wins, but all pay!

Example: two players uniform value [1,2], symmetric bidding b(v-1)
Value v=1+x, pretend, its 1+z

Need x $= argmax_z \; - b(z) + z(1+x)$,

We get $b'(x) = 1 + x$, so $b(x) = x + \frac{1}{2}x^2$,
so for value v bid $(v-1) + \frac{1}{2}(v-1)^2 = \frac{1}{2}v^2 - \frac{1}{2}$ in [0,1.5]

Homework

Recall setup from yesterday: multiple items concave values.

- From yesterday: Auction A is (1/2, 1)-smooth
- Auction C: all pay. All pay value $v_i(K)$ no matter how many items they get! Show that Auction A is (1/2, 1)-smooth (and hence has a price of anarchy of at most 2
- Auction B: is (1/2, 1)-smooth with bids not prices