Lecture 3

Corner Polyhedron, Intersection Cuts, Maximal Lattice-Free Convex Sets

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Mixed Integer Linear Programming

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax = b & \\ & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\ & x_j \geq 0 & \text{for } j = 1, \ldots, n. \end{array}$$

- First solve the LP relaxation. Basic optimal tableau:

$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for } i \in B.$$

- If $\bar{b}_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add cutting planes:

For example Gomory 1963 Mixed Integer Cuts, Marchand and Wolsey 2001 MIR inequalities, Balas, Ceria and Cornuéjols 1993 lift-and-project cuts, are used in commercial codes.

Corner Polyhedron [Gomory 1969]

Initial formulation:

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B. \\ x_j & \in & \mathbb{Z} & \text{for } j = 1, \dots, p \\ x_j & \geq & 0 & \text{for } j = 1, \dots, n. \end{array}$$

Corner formulation:

Drop the nonnegativity restriction on all the basic variables x_i, $i \in B$.

Note that in this relaxation we can drop the constraints $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for all $i \in B \cap \{p+1, \dots, n\}$ since these variables x_i are continuous and only appear in one equation and no other constraint. Therefore from now on we assume that all basic variables in are integer, i.e. $B \subseteq \{1, \dots, p\}$.

Corner Polyhedron

We assume $B \subseteq \{1, \ldots, p\}$. The *corner formulation* is

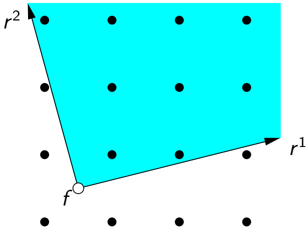
$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1, \ldots, p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

The convex hull of its feasible solutions is called the *corner polyhedron* relative to the basis B and it is denoted by $\text{corner}(B)$.

Note Any valid inequality for the corner polyhedron is valid for the initial formulation.

Let $P(B)$ be the linear relaxation of the corner polyhedron. $P(B)$ is a polyhedron whose vertices and extreme rays are simple to describe and this will be useful in generating valid inequalities for $\text{corner}(B)$.

Corner Polyhedron Example



Feasible set $\left\{ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathbb{Z}^2 : \right.$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

$$\text{where} \quad \left. x_1 \geq 0, x_2 \geq 0 \right\}$$

Restricted to the (x_3, x_4)-space, $P(B)$ is the blue region.
The feasible solutions are the integer points in the blue region, and $\text{corner}(B)$ is the convex hull of these points.

The Polyhedron $P(B)$

$P(B)$ has a unique vertex $\bar{x}$ where $\bar{x}_i = \bar{b}_i, i \in B,\ x_j = 0, j \in N$.
The recession cone of $P(B)$ is

$$\begin{array}{rcll} x_i & = & -\sum_{j\in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Since the projection of this cone onto $\mathbb{R}^N$ is defined by the inequalities $x_j \geq 0$, $j \in N$, its extreme rays are the vectors satisfying at equality all but one nonnegativity constraints. Thus there are $|N|$ extreme rays, $\bar{r}^j$ for $j \in N$, defined by

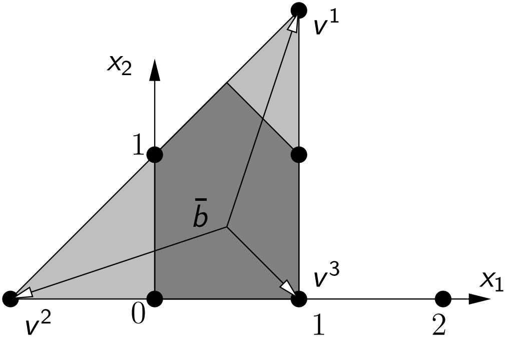
$$\bar{r}^j_h = \begin{cases} -\bar{a}_{hj} & \text{if } h \in B, \\ 1 & \text{if } h = j, \\ 0 & \text{if } h \in N \setminus \{j\}. \end{cases}$$

Remark The vectors $\bar{r}^j$, $j \in N$ are linearly independent. Hence $P(B)$ is an $|N|$-dimensional polyhedron whose affine hull is defined by the equations $x_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x_j$ for $i \in B$.

Corner Polyhedron Example

Consider the pure integer program

$$
\begin{array}{rcl}
\max \frac{1}{2}x_2 + x_3 & & \\
x_1 + x_2 + x_3 & \leq & 2 \\
x_1 \qquad - \frac{1}{2}x_3 & \geq & 0 \\
x_2 - \frac{1}{2}x_3 & \geq & 0 \\
x_1 \qquad + \frac{1}{2}x_3 & \leq & 1 \\
-x_1 + x_2 + x_3 & \leq & 1 \\
x_1, x_2, x_3 & \in & \mathbb{Z} \\
x_1, x_2, x_3 & \geq & 0.
\end{array}
$$

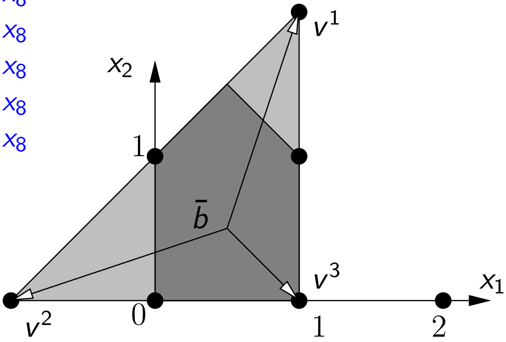


This problem has 4 feasible solutions $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$, all satisfying $x_3 = 0$. The intersection of the 5 inequalities in the formulation with the plane $x_3 = 0$ is the darker region in the figure.

Corner Polyhedron Example

We first write the problem in standard form by introducing continuous slack or surplus variables $x_4, \ldots, x_8$. Solving the LP relaxation, we get

$$
\begin{array}{lclllll}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_4 & = & 0 & -\frac{1}{2}x_6 & +\frac{3}{2}x_7 & +\frac{1}{2}x_8 \\
x_5 & = & 0 & +\frac{1}{2}x_6 & -\frac{1}{2}x_7 & +\frac{1}{2}x_8
\end{array}
$$

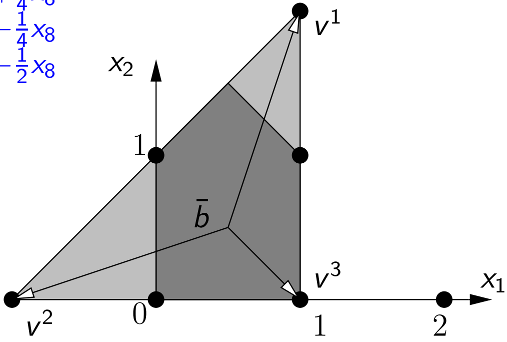


The optimal basic solution is $x_1 = x_2 = \frac{1}{2}$, $x_3 = 1$, $x_4 = \ldots = x_8 = 0$.

Corner Polyhedron Example

Relaxing the nonnegativity of the basic variables and dropping the two constraints relative to the continuous basic variables x_4 and x_5, we obtain the corner formulation:

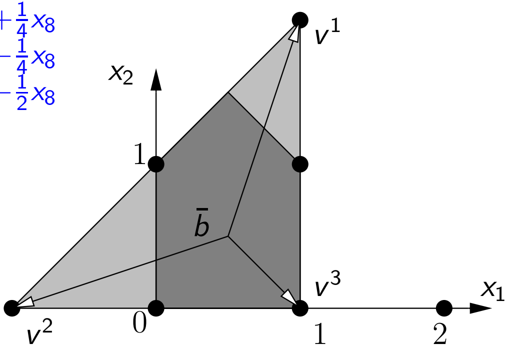
$$
\begin{array}{rcrrrr}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_1, & x_2, & x_3 & & \in & \mathbb{Z} \\
x_6, & x_7, & x_8 & & \geq & 0.
\end{array}
$$



Corner Polyhedron Example

Let $P(B)$ be the linear relaxation of corner formulation. The projection of $P(B)$ in the space of original variables x_1, x_2, x_3 is a polyhedron with unique vertex $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. The extreme rays of its recession cone are $v^1 = (\frac{1}{2}, \frac{3}{2}, -1)$, $v^2 = (-\frac{3}{2}, -\frac{1}{2}, -1)$ and $v^3 = (\frac{1}{2}, -\frac{1}{2}, -1)$.

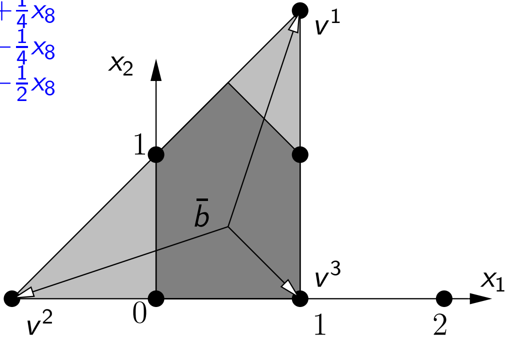
$$
\begin{array}{rcrrrr}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_6, & x_7, & x_8 & \geq & 0. &
\end{array}
$$



Corner Polyhedron Example

In the figure, the shaded region (both light and dark) is the intersection of $P(B)$ with the plane $x_3 = 0$.

$$\begin{array}{ccccccc} x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\ x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\ x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\ x_6, & x_7, & x_8 & & \geq & 0. \end{array}$$



Let P be defined by the inequalities of the initial formulation that are satisfied at equality by the point $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. The intersection of P with the plane $x_3 = 0$ is the dark shaded region.

Intersection Cuts [Balas 1971]

Consider a closed convex set $C \subseteq \mathbb{R}^n$ such that the interior of C contains $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

For each of the $|N|$ extreme rays of $P(B)$, define

$$\alpha_j = \max\{\alpha \geq 0 : \ \bar{x} + \alpha \bar{r}^j \in C\}.$$

Since $\bar{x}$ is in the interior of C, $\alpha_j > 0$.

When the half-line $\{\bar{x} + \alpha \bar{r}^j \, : \, \alpha \geq 0\}$ intersects the boundary of C, then α_j is finite, and the point $\bar{x} + \alpha_j \bar{r}^j$ belongs to the boundary of C.

When $\bar{r}_j$ belongs the recession cone of C, we have $\alpha_j = +\infty$. Define $\frac{1}{+\infty} = 0$.

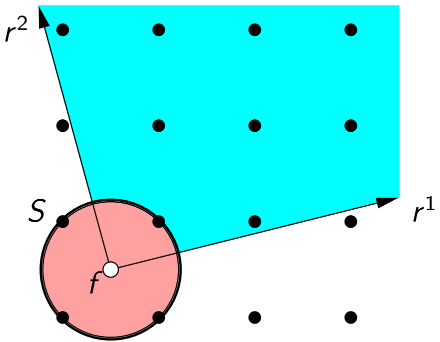
The inequality $$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1$$

is the *intersection cut* defined by C.

Intersection Cuts

Assume $f \notin \mathbb{Z}^2$.

Want to cut off the basic solution $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f,\ x_1 = 0, x_2 = 0$.



Feasible set $\{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathbb{Z}^2 :$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

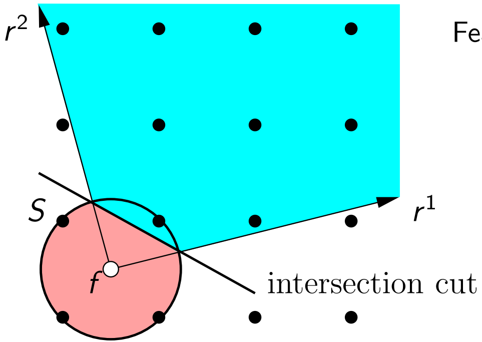
where $x_1 \geq 0, x_2 \geq 0\}$

Any convex set S with $f \in int(S)$ and no integer point in $int(S)$.

Intersection Cuts

Assume $f \notin \mathbb{Z}^2$.

Want to cut off the basic solution $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f,\ x_1 = 0, x_2 = 0$.



Feasible set $\{ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathbb{Z}^2 :$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

where $x_1 \geq 0, x_2 \geq 0\}$

Any convex set S with $f \in int(S)$ and no integer point in $int(S)$.

Intersection cut is obtained by intersecting the rays with the boundary of S: $\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{1}{4}$. Thus $4x_1 + 4x_2 \geq 1$.

Intersection Cuts

The *corner formulation* introduced by Gomory is

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1, \ldots, p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Basic solution $\bar{x}$ where $\bar{x}_i = \bar{b}_i, i \in B,\ x_j = 0, j \in N$.

Assume $B \subseteq \{1, \ldots, p\}$ and $\bar{b} \notin \mathbb{Z}^{|B|}$.

THEOREM Let $C \subset \mathbb{R}^n$ be a closed convex set whose interior contains the point $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. The intersection cut defined by C is a valid inequality for $\mathrm{corner}(B)$.

PROOF See course notes

Intersection Cuts $\equiv$ Corner Polyhedron

We just stated that intersection cuts are valid for $\mathrm{corner}(B)$.

The following theorem provides a converse statement.

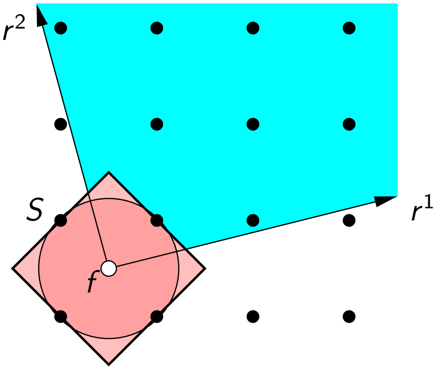
Inequalities $\sum_{j\in N} \gamma_j x_j \geq \gamma_0$ with $\gamma_j \geq 0, j \in N$ and $\gamma_0 \leq 0$ are implied by the nonnegativity constraints $x_j \geq 0,\ j \in N$ and will be called *trivial*.

Every nontrivial valid inequality for $\mathrm{corner}(B)$ can be written in the form $\sum_{j\in N} \gamma_j x_j \geq 1$ with $\gamma_j \geq 0, j \in N.$.

THEOREM Every nontrivial facet defining inequality for $\mathrm{corner}(B)$ is an intersection cut.

PROOF See course notes

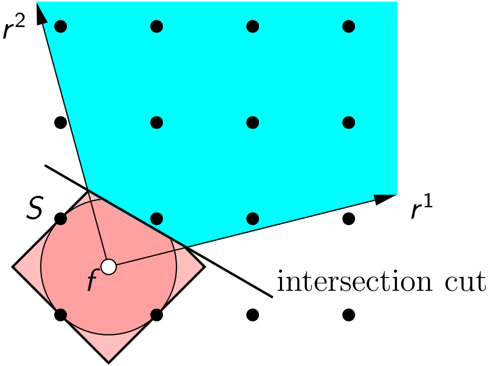
A Better Intersection Cut for our Example



Bigger convex set:

Square S such that $f \in int(S)$ with no integral point in $int(S)$.

A Better Intersection Cut for our Example



Bigger convex set:

Square S such that $f \in int(S)$ with no integral point in $int(S)$.

Better cut: $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{3}$. Thus $3x_1 + 3x_2 \geq 1$.

Intersection Cuts

- If $C_1 \subset C_2$ are two closed convex sets whose interiors contain $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then the intersection cut relative to C_2 dominates the intersection cut relative to C_1 for all $x \in \mathbb{R}^n$ such that $x_j \geq 0$, $j \in N$.

- Any closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is contained in an inclusion *maximal* such set.

- How can we construct a (maximal) convex set C whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$?

In the space $\mathbb{R}^p$, construct a $\mathbb{Z}^p$-free (maximal) convex set K whose interior contains the projection of $\bar{x}$.

The cylinder $C = K \times \mathbb{R}^{n-p}$ is a (maximal) convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

EXAMPLE: Intersection Cuts from Split Disjunctions

Consider a split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$, where $\pi \in \mathbb{Z}^p \times \{0\}^{n-p}$ and $\pi_0 \in \mathbb{Z}$, and $\pi_0 < \pi\bar{x} < \pi_0 + 1$.

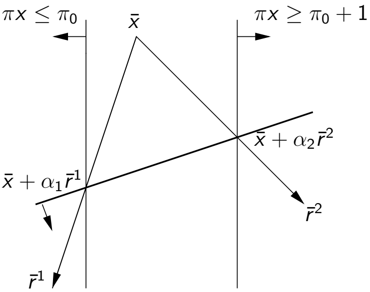
$K := \{x \in \mathbb{R}^p \ : \ \pi_0 \leq \sum_{j=1}^{p} \pi_j x_j \leq \pi_0 + 1\}$ is $\mathbb{Z}^p$-free and convex.

The set $C := K \times \mathbb{R}^{n-p} \ = \{x \in \mathbb{R}^n \ : \ \pi_0 \leq \pi x \leq \pi_0 + 1\}$ is $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free.

Define $\epsilon := \pi\bar{x} - \pi_0$. We have $0 < \epsilon < 1$. For $j \in N$, define:

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j < 0, \\ \frac{1-\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\bar{r}^j$ are the rays of $P(B)$.



Intersection cut $\quad \sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1.$

Gomory Mixed Integer Cuts from the Tableau

Let $x_i, i \in B$ be a basic integer variable, and suppose $\bar{x}_i = \bar{b}_i$ is fractional. We define $\pi_0 := \lfloor \bar{x}_i \rfloor$, and for $j = 1, \dots, p$,

$$\pi_j := \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in N \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in N \text{ and } f_j > f_0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

For $j = p+1, \dots, n$, we define $\pi_j := 0$.

Next we derive the intersection cut from the split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ as shown in the previous slide. We need to compute $\alpha_j,\ j \in N$ using our formula:

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j < 0, \\ \frac{1-\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\bar{r}^j$ are the rays of $P(B)$.

Gomory Mixed Integer Cuts from the Tableau

Let $f_0 = \bar{b}_i - \lfloor \bar{b}_i \rfloor$ and $f_j = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$. We have

$$\epsilon = \pi \bar{x} - \pi_0 = \sum_{h \in B} \pi_h \bar{x}_h - \pi_0 = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f_0.$$

Let $j \in N$. We have $\pi \bar{r}^j = \pi_j \bar{r}^j_j + \pi_i \bar{r}^j_i$ since $\bar{r}^j_h = 0$ for all $h \in N \setminus \{j\}$ and $\pi_h = 0$ for all $h \in B \setminus \{i\}$. Therefore

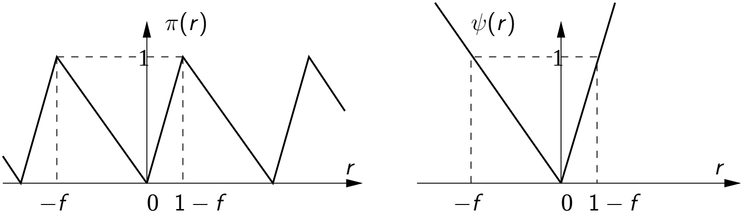
$$\pi \bar{r}^j = \begin{cases} \lfloor \bar{a}_{ij} \rfloor - \bar{a}_{ij} & = -f_j & \text{if } 1 \leq j \leq p \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil - \bar{a}_{ij} & = 1 - f_j & \text{if } 1 \leq j \leq p \text{ and } f_j > f_0, \\ -\bar{a}_{ij} & & \text{if } j \geq p+1. \end{cases}$$

Now α_j follows. Therefore the intersection cut associated with the split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ is

$$\sum_{\substack{j \in N,\, j \leq p \\ f_j \leq f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \in N,\, j \leq p \\ f_j > f_0}} \frac{1 - f_j}{1 - f_0} x_j + \sum_{\substack{p+1 \leq j \leq n \\ \bar{a}_{ij} > 0}} \frac{\bar{a}_{ij}}{f_0} x_j - \sum_{\substack{p+1 \leq j \leq n \\ \bar{a}_{ij} < 0}} \frac{\bar{a}_{ij}}{1 - f_0} x_j \geq 1.$$

This is the GMI cut, since $f_i = 0$ for $i \in B$.

Gomory Functions



The Gomory formula looks complicated, and it may help to think of it as an inequality of the form

$$\sum_{j=1}^{p} \pi(\bar{a}_{ij})x_j + \sum_{j=p+1}^{n} \psi(\bar{a}_{ij})x_j \geq 1$$

where the functions π and ψ, are

$$\pi(a) := \min\{\frac{f}{f_0}, \frac{1-f}{1-f_0}\} \quad \text{and} \quad \psi(a) := \max\{\frac{a}{f_0}, \frac{-a}{1-f_0}\}.$$

with $f = a - \lfloor a \rfloor$.

Maximal lattice-free convex sets

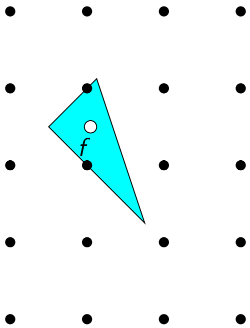
As observed earlier, the best possible intersection cuts are the ones defined by full-dimensional *maximal* $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$*-free convex sets* in $\mathbb{R}^n$.

LEMMA Let C be a full-dimensional maximal $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$-free convex set and let K be its projection onto $\mathbb{R}^p$. Then K is a full-dimensional maximal $\mathbb{Z}^p$-free convex set and $C = K \times \mathbb{R}^{n-p}$.

The next task is to understand the structure of full-dimensional maximal $\mathbb{Z}^p$-free convex sets.

Maximal Lattice-Free Convex Set

- Lattice-free convex set contains no integral point in its interior



Maximal Lattice-Free Convex Set

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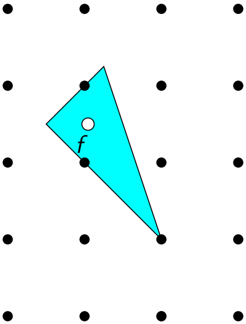
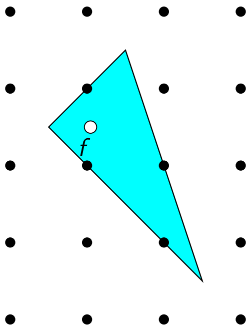


Maximal: each edge contains an integral point in its relative interior.

Maximal Lattice-Free Convex Set

- Lattice-free convex set contains no integral point in its interior



Maximal: each edge contains an integral point in its relative interior.
In the plane: it is a strip, a triangle or a quadrilateral.

Theorem [Lovász 1989]

A maximal lattice-free convex set in the plane (x_1, x_2) is one of the following:

i) Irrational line $ax_1 + bx_2 = c$ with a/b irrational;

ii) A strip $c \leq ax_1 + bx_2 \leq c + 1$
with a, b coprime integers, c integer;

iii) A triangle with an integral point in the relative interior of each edge;

iv) A quadrilateral containing exactly four integral points, one in the relative interior of each edge; The four integral points are vertices of a parallelogram of area 1.

Lovász' Theorem

THEOREM

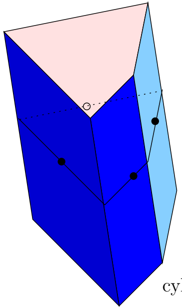
A set $K \subset \mathbb{R}^p$ is a full-dimensional maximal $\mathbb{Z}^p$-free convex set if and only if

K is a polyhedron of the form $K = P + L$

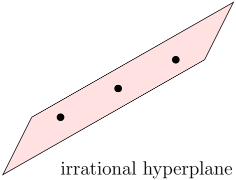
where P is a polytope, L is a rational linear space,

$\dim(P) + \dim(L) = p$,

K does not contain any point of $\mathbb{Z}^p$ in its interior and there is a point of $\mathbb{Z}^p$ in the relative interior of each facet of K.



cylinder



irrational hyperplane

Proof of Lovaśz' Theorem

Let $K \subset \mathbb{R}^p$ be a maximal $\mathbb{Z}^p$-free convex set.

We prove the theorem under the assumption that K is a bounded set. We need to show that K is a polytope and that each of its facets has an integer point in its relative interior.

Since we assume K bounded, there exist l, u in $\mathbb{Z}^p$ such that K is contained in the box $B = \{x \in \mathbb{R}^p \ : \ l_i \leq x_i \leq u_i, \ i = 1 \dots p\}$.

Since K is a convex set, for each $y \in B \cap \mathbb{Z}^p$, there exists an half-space $\{x \in \mathbb{R}^p : \ a_y x \leq b_y\}$ containing K such that $a_y y = b_y$ (separation theorem for convex sets).

Since B is a bounded set, $B \cap \mathbb{Z}^p$ is a finite set. Therefore $P = \{x \in \mathbb{R}^p : \ l_i \leq x_i \leq u_i, \ i = 1 \dots p, \ a_y x \leq b_y, \ y \in B \cap \mathbb{Z}^p\}$ is a polytope.

By construction P is $\mathbb{Z}^p$-free and $K \subseteq P$.

Therefore $K = P$ by maximality of K.

Proof of Lovaśz' Theorem

We now show that each facet of K contains an integer point in its relative interior.
Suppose, by contradiction, that facet F_t of K does not contain a point of $\mathbb{Z}^p$ in its relative interior.
Let $a_t x \leq b_t$ be the inequality defining F_t.
Given $\varepsilon > 0$, let K' be the polyhedron defined by the same inequalities that define K except the inequality $\alpha_t x \leq \beta_t$ that has been substituted with the inequality $\alpha_t x \leq \beta_t + \varepsilon$.
Since the recession cones of K and K' coincide, K' is a polytope.
Since K is a maximal $\mathbb{Z}^p$-free convex set and $K \subset K'$, K' contains points of $\mathbb{Z}^p$ in its interior.
Since K' is a polytope, the number of points in $K' \cap \mathbb{Z}^p$ is finite.
Hence there exists one such point minimizing $\alpha_t x$, say z.
Let K'' be the polytope defined by the same inequalities that define K except the inequality $\alpha_t x \leq \beta_t$ that has been substituted with the inequality $\alpha_t x \leq \alpha_t z$.
By construction, K'' does not contain any point of $\mathbb{Z}^p$ in its interior and properly contains K, contradicting the maximality of K. □

Bound on the Number of Facets of Maximal $\mathbb{Z}^p$-Free Polyhedra

Doignon 1973, Bell 1977 and Scarf 1977 show the following.

THEOREM Any full-dimensional maximal lattice-free convex set $K \subseteq \mathbb{R}^p$ has at most 2^p facets.

PROOF By Lovász' theorem, each facet F contains an integral point x^F in its relative interior. If there are more than 2^p facets, then two integral points x^F and $x^{F'}$ must be congruent modulo 2. Now their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral and it is in the interior of K, contradicting the fact that K is lattice-free. □

Exercises

In "Courses Material" on the webpage

http://eventos.cmm.uchile.cl/discretas2016/

do the following exercises in Course Notes "Cutting planes in integer programming"

Exercise 3.1

Exercise 3.6

Exercise 3.8

Optional: Exercise 3.9