CUT-GENERATING FUNCTIONS

for INTEGER VARIABLES

Gérard Cornuéjols

Tepper School of Business
Carnegie Mellon University, Pittsburgh

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Pure Integer Linear Programming

Consider a simplex tableau of the linear programming relaxation. Select n rows of the tableau. Let $\{x_i\}_{i=1}^n$ denote the basic variables and $\{y_j\}_{j=1}^m$ the nonbasic variables.

$$\begin{aligned} x = & f + \sum_{j=1}^{m} r^j y_j, \\ & x \in \mathbb{Z}_+^n, \\ & y_j \in \mathbb{Z}_+, \forall j \in \{1, \dots, m\}, \end{aligned}$$

where $f \in \mathbb{R}_+^n$ and $r^j \in \mathbb{R}^n$ for $j \in \{1, \dots, m\}$. We assume $f \notin \mathbb{Z}^n$; therefore, the basic solution $x = f$, $y = 0$ is not feasible.

Goal: Generate cutting-planes that cut off this infeasible solution.

A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a cut-generating function if the inequality $\sum_{j=1}^m \pi(r^j) y_j \geq 1$ holds for all feasible solutions (x, y) for any number m of nonbasic variables and any choice of nonbasic columns r^j.

Minimal Cut-Generating Functions

We say that a cut-generating function π' dominates another cut-generating function π if $\pi \geq \pi'$, that is, $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$.

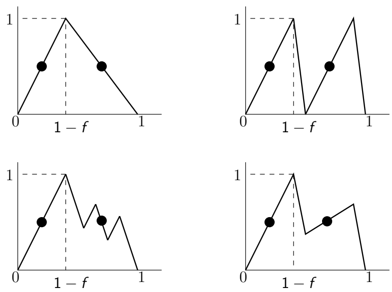
A cut-generating function π is minimal if there is no cut-generating function π' distinct from π that dominates π.

The following theorem shows that minimal cut-generating functions indeed always exist.

THEOREM
Every cut-generating function is dominated by a minimal cut-generating function.

The Gomory-Johnson Model

Examples of minimal cut-generating functions when $n = 1$



$$x = f + \sum_{j=1}^{m} r^j y_j,$$
$$x \in \mathbb{Z}^n,$$
$$y_j \in \mathbb{Z}_+, \forall j \in \{1, \ldots, m\}.$$

A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is periodic if $\pi(r) = \pi(r + w)$ for all $r \in [0, 1]^n$ and $w \in \mathbb{Z}^n$.

π is said to satisfy the symmetry condition if $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^n$.

π is subadditive if $\pi(a + b) \leq \pi(a) + \pi(b)$.

The Gomory-Johnson Model

$$x = f + \sum_{j=1}^{m} r^j y_j,$$
$$x \in \mathbb{Z}^n,$$
$$y_j \in \mathbb{Z}_+, \forall j \in \{1, \ldots, m\},$$

THEOREM (Gomory and Johnson 1972)

Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a non-negative function.
Then π is a minimal cut-generating function if and only if
$\pi(0) = 0$,
π is periodic,
subadditive,
and satisfies the symmetry condition.

Can we get a similar theorem if we do not relax $x \in \mathbb{Z}^n_+$ to $x \in \mathbb{Z}^n$?

The Gomory-Johnson Model in the Mixed Case

$$
\begin{aligned}
x &= f + \sum_{j=1}^{\ell} \rho^j z_j + \sum_{j=1}^{m} r^j y_j \\
x &\in \mathbb{Z}^n \\
& \qquad z \geq 0, \qquad y \in \mathbb{Z}^m_+.
\end{aligned}
$$

We are interested in pairs of functions $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\pi : \mathbb{R}^n \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{\ell} \psi(\rho^j) z_j + \sum_{j=1}^{m} \pi(r^j) y_j \geq 1$$

is valid for every choice of integers ℓ, m and vectors $\rho^1, \ldots, \rho^\ell \in \mathbb{R}^n$ and $r^1, \ldots, r^m \in \mathbb{R}^n$.

THEOREM Johnson 1974

Let (ψ, π) is a cut-generating function pair.
The pair (ψ, π) is minimal if and only if
π is a minimal cut-generating function and ψ is defined as

$$\psi(r) := \lim_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}, \text{ for all } r \in \mathbb{R}^n.$$

Back to Pure Integer Linear Programming

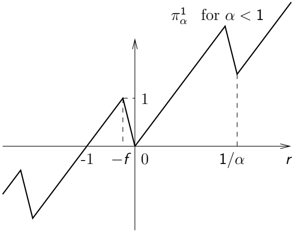
$$
\begin{aligned}
x =& f + \sum_{j=1}^{m} r^j y_j, \\
& x \in \mathbb{Z}_+^n, \\
& y_j \in \mathbb{Z}_+, \forall j \in \{1, \ldots, m\},
\end{aligned}
$$

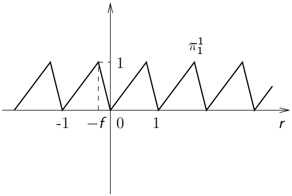
where $f \in \mathbb{R}_+^n$ and $r^j \in \mathbb{R}^n$ for $j \in \{1, \ldots, m\}$. We assume $f \notin \mathbb{Z}^n$; therefore, the basic solution $x = f$, $y = 0$ is not feasible.

Goal: Generate cutting-planes that cut off this infeasible solution through cut-generating functions.

Generalizing the Gomory-Johnson Theorem

General cut-generating functions differ from those of the Gomory-Johnson model is that they need not be nonnegative, as the example on the left illustrates (Cornuéjols, Kis, and Molinaro 2013). The example on the right is the classical Gomory function.





Here $n = 1$ and $0 < f < 1$.

The cut-generating function π_α^1 above are minimal for any $0 \leq \alpha \leq 1$.

Generalizing the Gomory-Johnson Theorem

We consider a further generalization of the Gomory-Johnson model:

$$
\begin{aligned}
x =& f + \sum_{j=1}^{m} r^j y_j, \\
& x \in S, \\
& y_j \in \mathbb{Z}_+, \forall j \in \{1, \dots, m\},
\end{aligned}
$$

where S can be any nonempty subset of the Euclidean space.

The Gomory-Johnson model is the special case where $S = \mathbb{Z}^n$.

Models studied by Jeroslow 1979, Blair 1978, and Bachem, Johnson, and Schrader 1982 correspond to the case $S = \{0\}$.

The case where $S = \mathbb{Z}^n_+$ is of particular interest in integer linear programming. It is a main focus of this lecture.

The Infinite Model

Gomory and Johnson introduced the infinite group relaxation

$$\begin{aligned} x = & f + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in \mathbb{Z}^n, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & y \text{ has finite support}, \end{aligned}$$

where an infinite-dimensional vector is said to have finite support if it has a finite number of nonzero entries.

Here, for $S \neq \emptyset$, we consider:

$$\begin{aligned} x = & f + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in S, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & y \text{ has finite support}. \end{aligned}$$

Properties of Minimal Cut-Generating Functions

LEMMA

If π is a minimal cut-generating function, then $\pi(0) = 0$.

PROOF

Suppose $\pi(0) < 0$, and let $(\bar{x}, \bar{y})$ be a feasible solution. Then there exists some $\bar{k} \in \mathbb{Z}_{++}$ such that $\pi(0)\bar{k} < 1 - \sum_{r \in \mathbb{R}^n \setminus \{0\}} \pi(r)\bar{y}_r$ since the right-hand side of the inequality is a constant. Define $\tilde{y}$ as $\tilde{y}_0 = \bar{k}$ and $\tilde{y}_r = \bar{y}_r$ for all $r \neq 0$. Note that $(\bar{x}, \tilde{y})$ is a feasible solution. This contradicts the assumption that π is a cut-generating function since $\sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r < 1$. Thus, $\pi(0) \geq 0$.

Let $(\bar{x}, \bar{y})$ be a feasible solution, and consider $\tilde{y}$ defined as $\tilde{y}_0 = 0$ and $\tilde{y}_r = \bar{y}_r$ for all $r \neq 0$. Then $(\bar{x}, \tilde{y})$ is a feasible solution. Now define the function π' as $\pi'(0) = 0$ and $\pi'(r) = \pi(r)$ for all $r \neq 0$. Observe that $\sum_{r \in \mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$ where the inequality follows because π is a cut-generating function. This implies that π' is also a cut-generating function. Since π is minimal and $\pi' \leq \pi$, we must have $\pi = \pi'$ and $\pi(0) = 0$. $\square$

Properties of Minimal Cut-Generating Functions

LEMMA

If π is a minimal cut-generating function, then π is subadditive.

PROOF

Let $r^1, r^2 \in \mathbb{R}^n$. We need to show $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$. This inequality holds when $r^1 = 0$ or $r^2 = 0$ by the previous lemma. Assume now that $r^1 \neq 0$ and $r^2 \neq 0$. Define the function π' as $\pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$ and $\pi'(r) = \pi(r)$ for $r \neq r^1 + r^2$. We show that π' is a cut-generating function. Since π is minimal, it then follows that $\pi(r^1 + r^2) \leq \pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$.

Consider any feasible solution $(\bar{x}, \bar{y})$. Define $\tilde{y}$ as $\tilde{y}_{r^1} = \bar{y}_{r^1} + \bar{y}_{r^1+r^2}$, $\tilde{y}_{r^2} = \bar{y}_{r^2} + \bar{y}_{r^1+r^2}$, $\tilde{y}_{r^1+r^2} = 0$, and $\tilde{y}_r = \bar{y}_r$ otherwise. Note that $\tilde{y}$ is well-defined since $r^1 \neq 0$ and $r^2 \neq 0$. It is easy to verify that $\tilde{y}_r \in \mathbb{Z}_+$ for all $r \in \mathbb{R}^n$, and $\sum_{r \in \mathbb{R}^n} r\tilde{y}_r = \sum_{r \in \mathbb{R}^n} r\bar{y}_r$. This shows that $(\bar{x}, \tilde{y})$ is a feasible solution. Furthermore, $\sum_{r \in \mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$ since π is a cut-generating function. This proves that π' is a cut-generating function. $\square$

Inadequacy of the Notion of Minimality

The notion of minimality that we defined above can be unsatisfactory for certain choices of S.

PROPOSITION

If a cut-generating function is linear, then it is minimal.

PROOF

Let π be a linear cut-generating function. By above theorem, there exists a minimal cut-generating function π' such that $\pi' \leq \pi$. The two previous lemmas show that π' is subadditive and $\pi'(0) = 0$. For any $r \in \mathbb{R}^n$, the inequality $\pi' \leq \pi$ implies $\pi(r) + \pi(-r) \geq \pi'(r) + \pi'(-r) \geq \pi'(0) = 0 = \pi(r) + \pi(-r)$ where the last equality follows from the linearity of π. Hence, $\pi' = \pi$.

Inadequacy of the Notion of Minimality

For a minimal cut-generating function π, it is possible that the inequality $\sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ is implied by an inequality $\sum_{r \in \mathbb{R}^n} \pi'(r) y_r \geq 1$ arising from some other cut-generating function π'. Indeed, for $n = 1$, $f > 0$, and $S = \{0\}$, consider the linear cut-generating functions $\pi^2_\alpha(r) = \alpha r$ of with $\alpha \leq -\frac{1}{f}$. These are minimal by the previous proposition. However, the inequalities $|\alpha| f \sum_{r \in \mathbb{R}} \frac{-r}{f} y_r \geq 1$ generated from π^2_α for $\alpha < -\frac{1}{f}$ are implied by the inequality $\sum_{r \in \mathbb{R}} \frac{-r}{f} y_r \geq 1$ generated for $\alpha = -\frac{1}{f}$.

Therefore, it makes sense to define a stronger notion of minimality as follows.

A cut-generating function π' implies another cut-generating function π via scaling if there exists $\beta \geq 1$ such that $\pi \geq \beta \pi'$.

A cut-generating function π is restricted minimal if there is no cut-generating function π' distinct from π that implies π via scaling.

Restricted Minimality

A cut-generating function π' implies another cut-generating function π via scaling if there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$.

A cut-generating function π is restricted minimal if there is no cut-generating function π' distinct from π that implies π via scaling.

This notion was the one used by Jeroslow, Blair, and Bachem, Johnson, and Schrader; they just called it minimality. We call it restricted minimality to distinguish it from the notion of minimality introduced earlier. The next proposition shows that restricted minimal cut-generating functions are the minimal cut-generating functions which enjoy an additional "tightness" property.

PROPOSITION

A cut-generating function π is restricted minimal if and only if it is minimal and $\inf_x \{\pi(x - f) : x \in S\} = 1$.

The Main Theorem

Let us say that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing with respect to a set $S \subset \mathbb{R}^n$ if $\pi(r) \leq \pi(r+w)$ for all $r \in \mathbb{R}^n$ and $w \in S$.

THEOREM Yildiz and Cornuéjols 2016

Let $K \subset \mathbb{R}^n$ be a closed convex cone and $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.
Let $\pi : \mathbb{R}^n \to \mathbb{R}$.
The function π is a restricted minimal cut-generating function if and only if
$\pi(0) = 0$,
π is subadditive,
nondecreasing with respect to S,
and satisfies the symmetry condition.

The Mixed Case

We now consider a system of the form

$$\begin{aligned} x &= f + \sum_{j=1}^{\ell} \rho^j z_j + \sum_{j=1}^{m} r^j y_j \\ x &\in \mathbb{Z}_+^n \end{aligned}$$

$$z \geq 0, \qquad y \in \mathbb{Z}_+^m.$$

We are interested in pairs of functions $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\pi : \mathbb{R}^n \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{\ell} \psi(\rho^j) z_j + \sum_{j=1}^{m} \pi(r^j) y_j \geq 1$$

is valid for every choice of integers ℓ, m and vectors $\rho^1, \ldots, \rho^\ell \in \mathbb{R}^n$ and $r^1, \ldots, r^m \in \mathbb{R}^n$.

THEOREM Yildiz and Cornuéjols 2016

Let (ψ, π) is a cut-generating function pair.
The pair (ψ, π) is minimal if and only if
π is a minimal cut-generating function and ψ is defined as

$$\psi(r) := \lim_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}, \text{ for all } r \in \mathbb{R}^n.$$