Problem Set 3 - Quasirandomness

Suppose that $0 < p < 1$ is fixed and $(G_n)_{n \in \mathbb{N}}$ with $|V(G_n)| = n$ is a sequence of graphs. Prove that the following properties are equivalent:

P_1: G_n is $(p, o(n))$-jumbled, that is, for all subsets $X, Y \subseteq V(G_n)$, $|e(X,Y) - p|X||Y|| = o(n^2)$.

P_2: $e(G_n) \geq p\binom{n}{2} + o(n^2)$, $\lambda_1(G_n) = pn + o(n)$ and $|\lambda_2(G_n)| = o(n)$, where $\lambda_i(G_n)$ is the ith largest eigenvalue, in absolute value, of the adjacency matrix of G_n.

P_3: For all graphs H, the number of labeled induced copies of H in G_n is $(1-p)^{\binom{t}{2}-\ell} p^\ell n^t + o(n^t)$, where $t = v(H)$ and $\ell = e(H)$.

P_4: $e(G_n) \geq p\binom{n}{2} + o(n^2)$ and the number of labeled cycles of length 4 in G_n is at most $p^4 n^4 + o(n^4)$.

P_5: $\sum_{u,v} |\mathrm{codeg}(u,v) - p^2 n| = o(n^3)$, where, given vertices $u, v \in V(G_n)$, $\mathrm{codeg}(u,v) = |\{x \in V(G_n) : ux, vx \in E(G_n)\}|$.

In P_4, can we replace cycles of length 4 with cycles of length 3?