Edge colouring multigraphs

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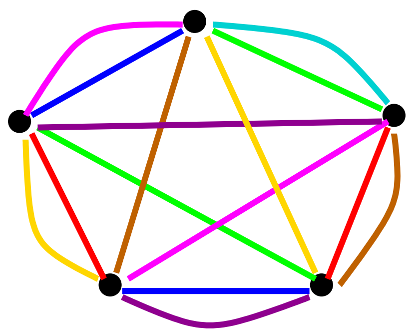
Multigraphs

For a multigraph G, we denote by

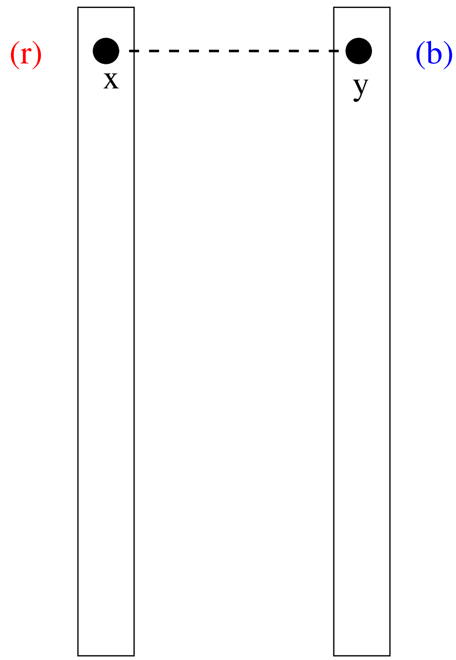
- $\Delta(G)$ the maximum degree of G,
- $\mu(G)$ the maximum edge multiplicity of G, and
- $\chi'(G)$ the chromatic index of G.



Every G satisfies $\chi'(G) \geq \Delta(G)$.

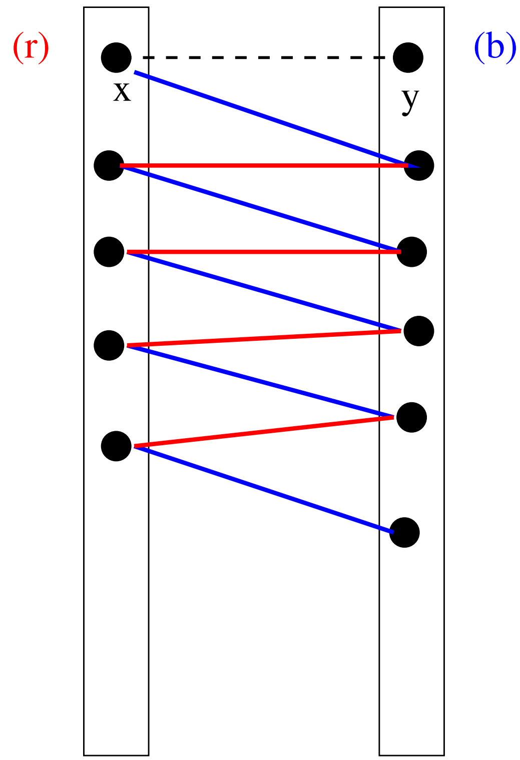
$\chi'(G) = \Delta(G)$ for bipartite multigraphs

Suppose we have coloured some of the edges of a bipartite multigraph G with Δ colours, and $e = xy$ is not yet coloured.



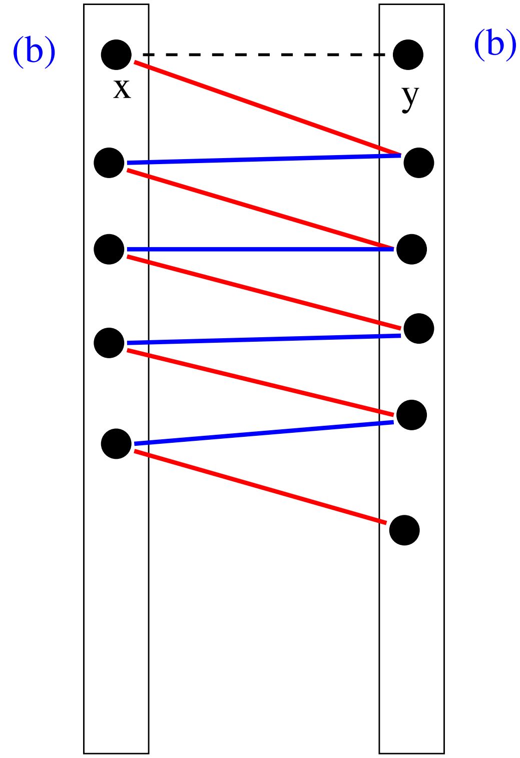
If x and y are missing the same colour then we can immediately colour e as well.

Otherwise there is a blue-red alternating path P starting at x.



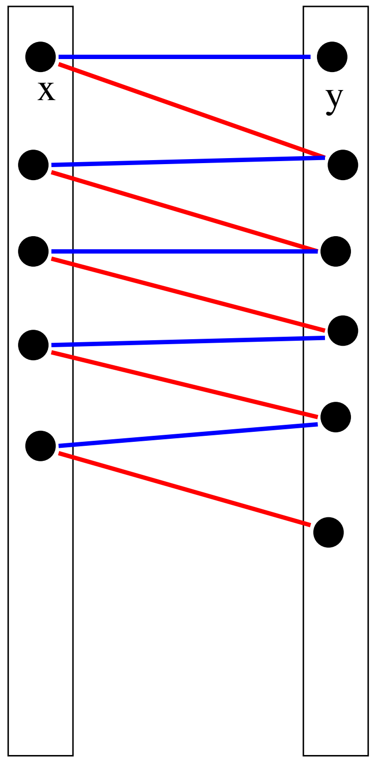
Note that P cannot end at y.

Switch the colours on P to get a new partial edge colouring of G that colours the same set of edges.



Now x and y are missing the same colour.

Improve the colouring by giving e colour blue.



What makes $\chi'(G) > \Delta(G)$?

We cannot expect an efficient characterisation, in particular not in the case $\mu = 1$.

Holyer's Theorem (1981). It is NP-complete to determine if a given graph has chromatic index Δ or $\Delta + 1$.

Classical upper bounds

Shannon's Theorem (1949). For every multigraph G we have

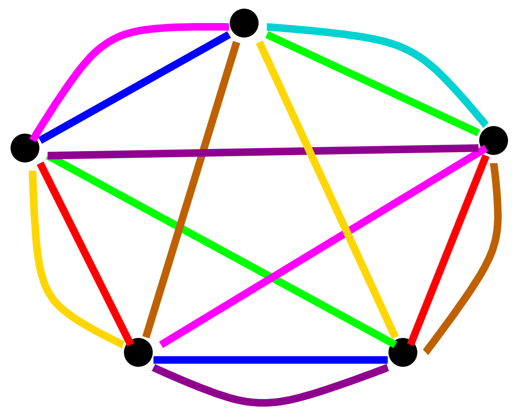
$$\chi'(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor.$$

Vizing's Theorem (1964). For every multigraph G we have

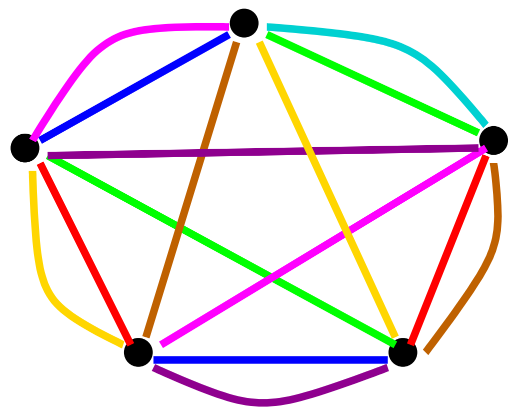
$$\chi'(G) \leq \Delta + \mu.$$

Equality holds in Shannon's Theorem if and only if G contains a triangle with $\lfloor \frac{3\Delta}{2} \rfloor$ edges. (Proved by Vizing in his 1968 doctoral dissertation.)

What makes $\chi'(G)$ large?



What makes $\chi'(G)$ large?



There is no edge colouring of this graph with seven colours, because each colour class has size at most 2 and there are 15 edges.

Another lower bound for $\chi'(G)$

If G contains an odd subset S of vertices such that

$$|E[S]| > \frac{(|S| - 1)}{2} t$$

then the set $E[S]$ of edges induced by S cannot be coloured with t colours. Therefore

$$\chi'(G) > t.$$

A conjecture of Goldberg and Seymour

Conjecture (1973, 1979). Let G be a multigraph with

$$\chi'(G) \geq \Delta + 2.$$

Then there exists an odd subset $S \subseteq V(G)$ with $|S| \geq 3$, such that

$$|E[S]| > \frac{(|S|-1)}{2}(\chi'(G)-1).$$

In other words, if $\chi'(G)$ is large, then G contains a dense odd subset S of vertices.

It is easy to check that if such an S exists then $|S| \leq \Delta$.

The conjecture restated

For a vertex subset S, define $\rho(S)$ to be the quantity

$$\rho(S) = \frac{|E[S]|}{\lfloor |S|/2 \rfloor}.$$

The parameter $\rho(G)$ is defined by

$$\rho(G) = \max\{\rho(S) : S \subseteq V(G), |S| \geq 2\}.$$

(It is always attained on an odd subset.)

Then $\chi'(G) \geq \lceil \rho(G) \rceil$ for every G.

Conjecture (1973, 1979). For every multigraph G

$$\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \rho(G) \rceil\}.$$

The best partial (published) result currently known is due to Scheide (2009) (also proved independently by Chen, Yu and Zang 2011).

Theorem (Scheide). For every multigraph G

$$\chi'(G) \le \max\{\Delta(G) + \sqrt{\frac{\Delta(G)-1}{2}}, \lceil \rho(G) \rceil\}.$$

In a recent development:

Theorem (Chen, Gao, Postle, Kim, Shan). For every multigraph G

$$\chi'(G) \le \max\{\Delta(G) + \sqrt[3]{\frac{\Delta(G)}{2}}, \lceil \rho(G) \rceil\}.$$

Fractional chromatic index

A fractional edge colouring of a multigraph G is a function w that assigns a nonnegative weight to each matching M in G, such that for each edge e we get

$$\sum_{M \ni e} w_M = 1.$$

So every edge colouring is a fractional edge colouring with all weights either zero or one.

The value of w is the sum $\sum_M w_m$ over all matchings M in G. If w represents an edge colouring then its value is the number of colours used.

The fractional chromatic index $\chi'^*(G)$ is the mimimum value of a fractional edge colouring of G.

Facts about fractional chromatic index

Fact. For every G

$$\Delta(G) \leq \chi'^{*}(G) \leq \chi'(G).$$

(This follows directly from the definition.)

Fact. The fractional chromatic index can be computed in polynomial time (even though the number of matchings is exponential).

Fact. For every G

$$\chi'^{*}(G) = \max\{\Delta(G), \rho(G)\}.$$

(This follows from the matching polytope theorem of Edmonds.)

Algorithmic consequences of Goldberg-Seymour

The **Facts** tell us

- if $\chi'^*(G) = \Delta(G)$ then $\lceil \rho(G) \rceil \leq \Delta(G)$, and
- if $\chi'^*(G) > \Delta(G)$ then $\lceil \rho(G) \rceil = \lceil \chi'^*(G) \rceil$ can be computed in polynomial time.

Thus IF the Goldberg-Seymour conjecture is true, in other words if $\chi'(G) \leq \max\{\Delta(G)+1, \lceil \rho(G) \rceil\}$ holds, we find

IF $\chi'(G) \geq \Delta + 2$ then $\chi'(G) = \lceil \chi'^*(G) \rceil$ can be computed in polynomial time.

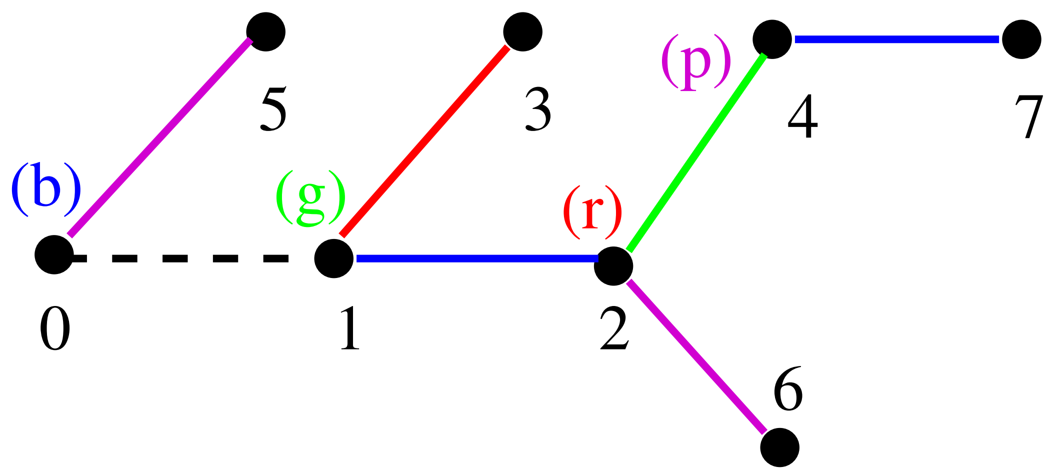
In other words, the ONLY computationally difficult problem for $\chi'(G)$ is distinguishing between $\Delta(G)$ and $\Delta(G) + 1$.

Tashkinov trees

Let G be a multigraph, and let φ be a partial $(\chi' - 1)$-edge-colouring of G. A tree T in G is a φ-Tashkinov tree if

its first edge is uncoloured, and

each subsequent edge is coloured with a colour that is missing at a previous vertex.



Origin of Tashkinov trees

The Tashkinov tree method generalises an argument of Kierstead (1984), which in turn generalises the method of alternating paths. It was introduced to prove the following approximate version of the Goldberg-Seymour conjecture.

Theorem (Tashkinov 2000). For every multigraph G

$$\chi'(G) \leq \max\{\Delta(G) + \frac{\Delta(G)}{10}, \lceil \rho(G) \rceil\}.$$

The same method was used also by various other authors (e.g. Favrholdt, Stiebitz, Toft, Scheide, Chen, Yu, Zang, Gao, Postle, Kim, Shan) to prove other results related to the Goldberg-Seymour conjecture, including a sequence of improvements leading to the current best bound.

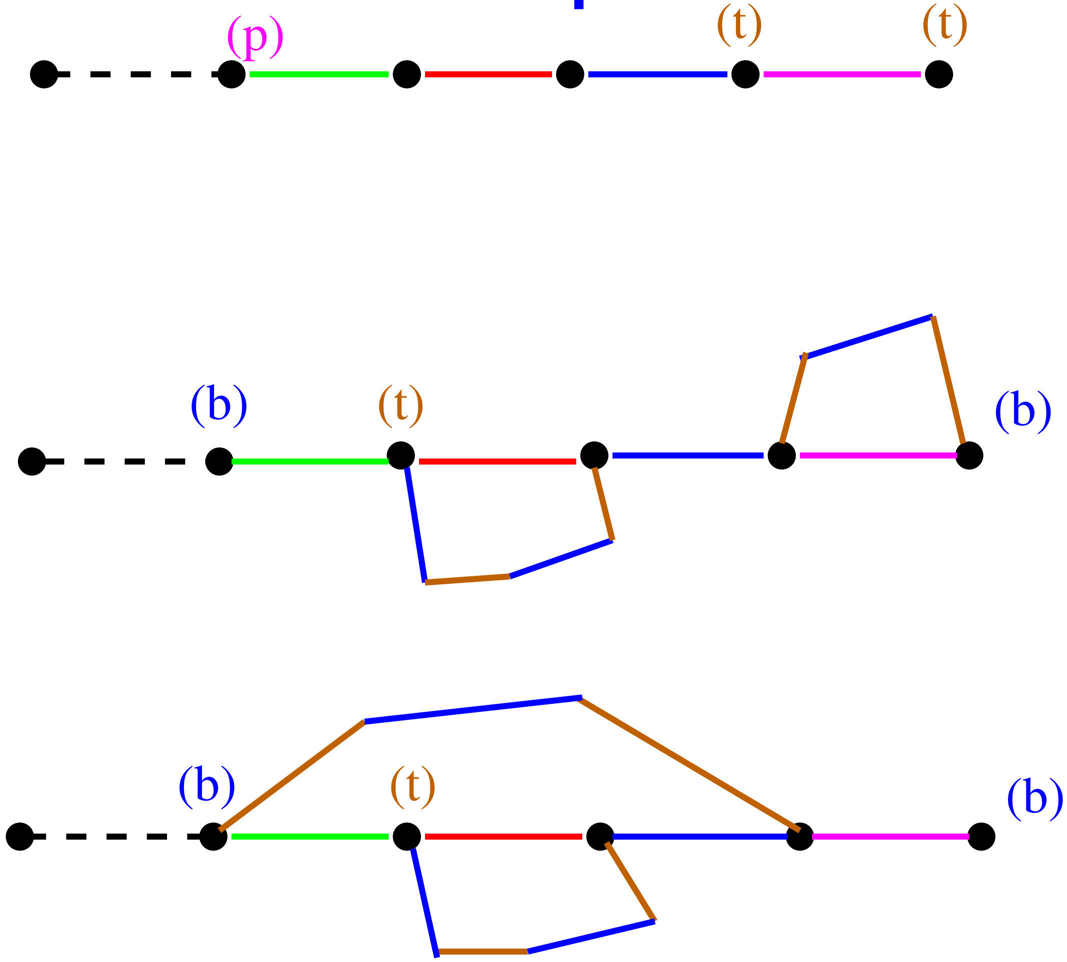
Key property of Tashkinov trees

Let G be a multigraph with $\chi'(G) \geq \Delta + 2$. Let T be a φ-Tashkinov tree, where φ is a partial $(\chi' - 1)$-edge-colouring of G, that colours the maximum possible number of edges.

Theorem (Tashkinov 2000). No colour is missing at two different vertices of T.

We say that T is φ-elementary.

Idea of proof

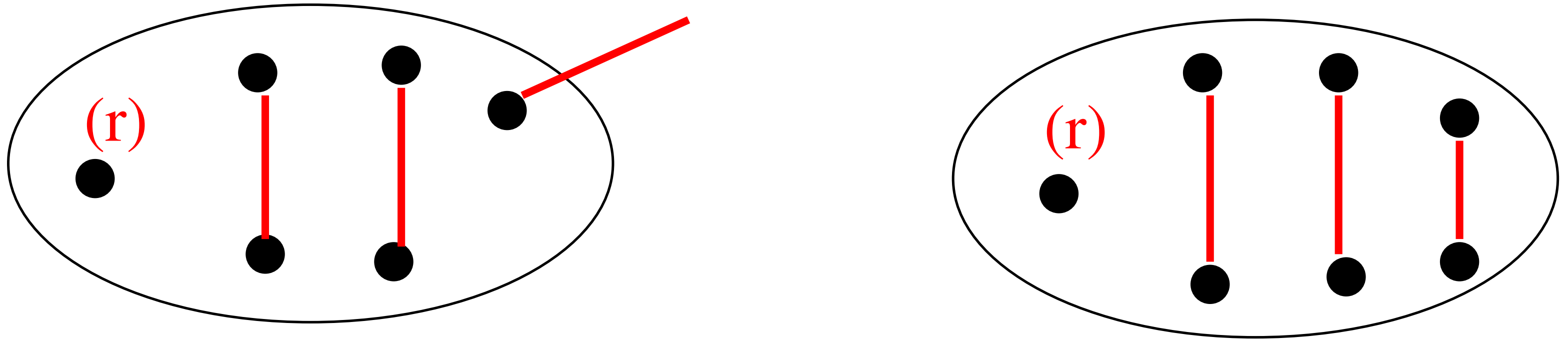


Consequences

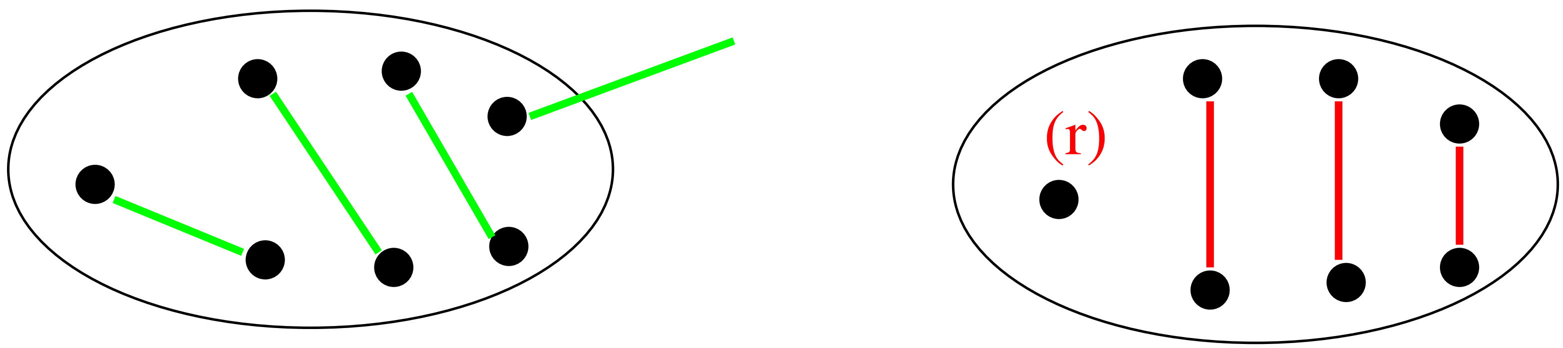
If T is maximal then

1. $S = |V(T)|$ is odd

2. every colour missing at a vertex of T occurs on exactly $(|S| - 1)/2$ edges of S.



If EVERY colour occurs on exactly $(|S|-1)/2$ edges of S then S is the set we are looking for: it induces more than $(|S|-1)(\chi'-1)/2$ edges.



A colour that appears on more than one edge leaving S is called defective.

Using Tashkinov trees

Suppose that G is a multigraph with $\chi'(G) > \Delta + x$, and we want to show that there exists an odd set S with more than $(|S| - 1)(\chi' - 1)/2$ edges.

- Fix a best partial colouring φ with $\chi' - 1$ colours.

- Construct a best φ-Tashkinov tree T starting from an uncoloured edge.

- If there are no defective colours for $S = V(T)$ then we are done.

Size matters

LARGE Tashkinov trees are GOOD:

- a maximal φ-Tashkinov tree is elementary (no colour is missing at more than one vertex)
- each vertex is missing at least x colours (or $x+1$ if it is incident to the uncoloured edge)
- the total number of colours missing on the vertices of T is $x|V(T)|+2$.

Thus if $x|V(T)|+2 > \Delta + x$ we are done.

Example: Shannon's Theorem

Suppose on the contrary that $\chi'(G) \geq \frac{3\Delta}{2}+1$. Fix a best partial colouring φ with $\frac{3\Delta}{2}$ colours, and let T be a best φ-Tashkinov tree.

Then the total number of colours missing on the vertices of T is at least

$$\frac{\Delta}{2}|V(T)| + 2.$$

Since this is at most the total number of colours, we find

$$|V(T)| < 3.$$

But $|V(T)|$ is odd and at least two.

CONTRADICTION.

Example: Equality in Shannon's Theorem

Suppose that $\chi'(G) = \frac{3\Delta}{2}$. Fix a best partial colouring φ with $\frac{3\Delta}{2} - 1$ colours, and let T be a best φ-Tashkinov tree.

Then the total number of colours missing on the vertices of T is at least

$$(\frac{\Delta}{2} - 1)|V(T)| + 2.$$

Since this is at most the total number $\frac{3\Delta}{2} - 1$ of colours, we find

$$|V(T)| = 3.$$

So all $\frac{3\Delta}{2} - 1$ colours appear on the subgraph induced by $V(T)$, plus the uncoloured edge.

Therefore this subgraph is a triangle with multiplicity $\frac{\Delta}{2}$.

Using Tashkinov trees

- All arguments are based on alternating paths.

- All arguments give polynomial-time algorithms for finding an edge colouring with the guaranteed number of colours OR a dense odd set preventing such a colouring.

Seymour's exact conjecture

Conjecture (1979). For every PLANAR multigraph G we have

$$\chi'(G) = \max\{\Delta(G), \lceil \rho(G) \rceil\}.$$

Let G be a cubic (i.e. 3-regular) multigraph with at least three vertices. If G is bridgeless then every odd vertex subset S has $3|S| - 2|E[S]| \geq 3$ edges to $V(G) \setminus S$.

So for odd S we find $\frac{|E[S]|}{\lfloor |S|/2 \rfloor} = \frac{2|E[S]|}{|S|-1} \leq 3$, otherwise $2|E[S]| > 3|S|-3$ which contradicts the above statement.

Therefore recalling $\rho(G)$ is attained on an odd set

$$\rho(G) = \max_{S \subset V(G), |S| \geq 3 \text{ odd}} \left\{ \frac{|E[S]|}{\lfloor |S|/2 \rfloor} \right\} \leq 3.$$

Thus if Seymour's exact conjecture holds then every planar cubic bridgeless multigraph is 3-edge-colourable.

This is equivalent to the Four-Colour Theorem.

Theorem (Marcotte 2001). If a multigraph G has no K_5^- or $K_{3,3}$ as a minor then

$$\chi'(G) = \max\{\Delta(G), \lceil \rho(G) \rceil\}.$$

Goldberg's exact conjecture

Goldberg also proposed the following sharp version of his conjecture, for multigraphs with $\rho(G) \le \Delta(G) - 1$.

Conjecture (1973). For every multigraph G, if $\rho(G) \le \Delta(G) - 1$ then $\chi'(G) = \Delta(G)$.

Theorem (PH, Kierstead). Let G be a multigraph with maximum degree Δ, and let ε be given where $0 < \varepsilon < 1$. Let $k = \lfloor \log_{1+\varepsilon} \Delta) \rfloor$. If

$$\rho(S) \le (1 - \varepsilon)(\Delta + k)$$

for every $S \subseteq V(G)$ with $|S| < \Delta/k + 1$ then

$$\chi'(G) \le \Delta + k.$$

Classical upper bounds

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Equality holds in Shannon's Theorem if and only if G contains a triangle with $\lfloor \frac{3\Delta}{2} \rfloor$ edges. (Proved by Vizing in his 1968 doctoral dissertation.)

Equality in Vizing's Theorem

Conjecture (1973, 1979). Let G be a multigraph with

$$\mu \geq 2.$$

Then $\chi'(G) = \Delta + \mu$ if and only if there exists an odd subset $S \subseteq V(G)$ with $|S| \geq 3$, such that

$$|E[S]| > \frac{(|S|-1)}{2}(\Delta + \mu - 1).$$

This would mean that $\mu = 1$ is the ONLY value of μ for which there is no characterisation.

A partial characterisation

Theorem (PH, J. McDonald 2012). Let G be a multigraph with

$$\mu \geq \log_{5/4}(\Delta) + 1.$$

Then $\chi'(G) = \Delta + \mu$ if and only if there exists an odd subset $S \subseteq V(G)$ with $|S| \geq 3$, such that

$$|E[S]| > \frac{(|S| - 1)}{2}(\Delta + \mu - 1).$$

Aims for this course

- Details of Tashkinov's Theorem
- Applications of Tashkinov's Theorem, including: simple applications, Scheide's Theorem, and the case of equality in Vizing's Theorem
- Kierstead's Theorem (when a Tashkinov tree is a path)
- Proof of Tashkinov's Theorem
- The Goldberg-Seymour Conjecture for random multigraphs