Exercises: Day 2

Exercise 1 (*) Let $\mathcal{P}_n$ be a Poisson point process with n times Lebesgue measure as its mean measure in $[0,1]^2$, and let x be another independent point uniformly distributed in $[0,1]^2$. Denote by $B(x,r)$ the ball of radius r centered at x (intersected with $[0,1]^2$). Let $r = \sqrt{\frac{\log n - \frac{1}{2}\log\log n}{\pi n}}$ and let $\mu = \frac{(\log\log n)^2}{\sqrt{n}\log n}$. Show that

$$\Pr\left(|\mathcal{P}_n \cap B(x,\mu) \backslash \{x\}| \geq 1,\, \mathcal{P}_n \cap (B(x,r) \setminus B(x,\mu)) = \emptyset\right) = o(1/n).$$

Exercise 2 (*) Show that $p_c(\mathbb{Z}) = 1$.

Exercise 3 (**) Site percolation in $\mathbb{Z}^2$ is defined as follows: each site is open with probability p. Two open sites at distance 1 are connected by an edge. Define $p_c(\text{site})$ be

$$p_c(\text{site}) = \inf\{p \in (0,1) : \theta(p) > 0\},$$

with $\theta(p) = \Pr\left((0,0) \text{ is in an infinite cluster}\right)$. Show that

$$p_c(\text{bond}) \leq p_c(\text{site}) \leq 1 - (1 - p_c(\text{bond}))^4.$$