Exercises: Day 4

Exercise 1 Let $\mathcal{U}_n$ be the uniform model of the random hyperbolic graph with parameters $\frac{1}{2} < \alpha < 1$ and $\nu = 1$. We use the native representation: for a vertex v, let r_v be its radius (measured from the origin) and θ_v its angle at the origin. Let $\theta_R(r_u, r_v)$ be the maximum angle two vertices with radial coordinates r_u and r_v respectively can make while still being connected.

- (*) Fix $0 < \beta < 1$, and define the inner set of vertices $I(\beta) = \{v : r_v \leq \beta R\}$ and the outer set of vertices $O(\beta) = \{v : r_v > \beta R\}$. Show that with high probability, $|I(\beta)| \leq \max\{n^{1/2}, n^{1-2\alpha(1-\beta)}\}$ and show that the number of edges from $I(\beta)$ to $O(\beta)$ is with high probability at most $O(n^{1-(2\alpha-1)(1-\beta)})$.
- (**) For a vertex u with $r_u > \beta R$ (that is, $u \in O(\beta)$), let q_R be the probability that a randomly chosen vertex has radius at least βR and is at (hyperbolic) distance at most R from u. Define $D_k(\beta)$ to be the random variable counting the number of vertices having exactly k neighbors of radius at least βR. Show that
$$\mathbb{E}[D_k(\beta)] = n \int_{\beta R}^{R} \binom{n-1}{k} q_R^k (1 - q_R)^{n-1-k} f(r) dr,$$
where $f(r)$ is the distribution of the vertices in the random hyperbolic graph. Show then that $\mathbb{E}[D_k(\beta)] = \Theta(nk^{-2\alpha-1})$.

Exercise 2 (***) Assume $0 \leq r_u, r_v \leq R$ and $r_u + r_v > R$. Show that $\theta_R(r_u, r_v) = 2e^{\frac{1}{2}(R-r_u-r_v)}(1 + o(1))$. Hint: use $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$, the definition of cosh and Taylor series expansions.