The Traveling Salesman: Classical Tools and Recent Advances

Lecture 2: Graph-TSP

Anke van Zuylen

XV Summer School in Discrete Mathematics

Valparaiso, January 6-10, 2020

WILLIAM & MARY
CHARTERED 1693

Aside: Some Comments on Finding Optimal Solutions to Integer Programs

Solving Integer Programs

How did Cook, Espinoza, Goycoolea and Helsgaun find the optimal tour through UK's 24,727 pubs???

Difficult part is typically not finding the optimal tour, but verifying that no other tour is cheaper.

The key idea of the method to do so is based on a paper by Dantzig, Fulkerson and Johnson from 1954.

Optimal tour through UK's 24,727 pubs – Cook, Espinoza, Goycoolea and Helsgaun 2015

Dantzig, Fulkerson and Johnson

The Cutting Plane Method

To find optimal tour, or verify a given tour is optimal:

- Solve a linear programming relaxation of the problem of interest.
- Repeat:
  - If the optimal LP solution is integer, and corresponds to a tour, then we are done.
  - If the optimal LP solution is not a tour, find a cutting plane: a linear constraint that is satisfied by every tour, but not by the current optimal LP solution.
    Add the cutting plane to the LP and resolve.

  Until the optimal solution is integer tour, or the LP optimal value is equal to the cost of the given tour.

Dantzig, Fulkerson and Johnson found optimal tour of 49 cities in the US by hand using this method.

After the Dantzig-Fulkerson-Johnson Paper, TSP is Still Hard

Proctor and Gamble competition in 1962.

Example of Cutting Plane Method

For the TSP, we could first solve:

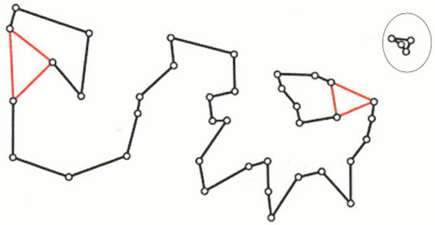
$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} c(e)x(e) \\
\text{subject to:} \quad & \\
& x(\delta(i)) = 2 \qquad \forall i \in V, \\
& 0 \leq x(e) \leq 1 \qquad \forall e \in E.
\end{aligned}
$$

If the optimal solution x^* is integer, then either

- it is a tour, and we have found the optimal tour,
- or, it is has two or more "subtours" on subsets of the vertices. Let S be one such subset, then we can add the constraint $x(\delta(S)) \geq 2$ to the LP. This constraint is satisfied by all tours, but not by the current LP solution.

Example of Cutting Plane Method

If the optimal solution x^* is not integer, we can similarly try to find a constraint that is not satisfied by x^*, but that is satisfied by every tour.



From Cook, p. 128

Integrality Gap

Recall that the integrality gap of the subtour LP is at least $\frac{4}{3}$, so once x^* satisfies $x^*(\delta(S)) \geq 2$ for all $S \subset V$, more work is still needed (for example, branch-and-bound, adding more cutting planes, or a combination of these approaches) to find the optimal solution.

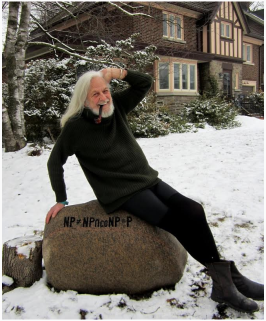
Polyhedral Combinatorics

For some combinatorial optimization problems, we known a linear program for which the extreme points of the feasible region are integer vectors and correspond to solutions to the combinatorial optimization problem.

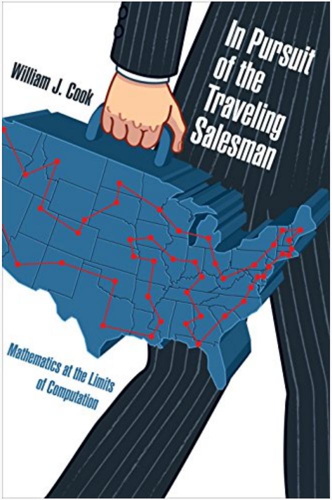
In that case, we say that we have a polyhedral description of the problem of interest.

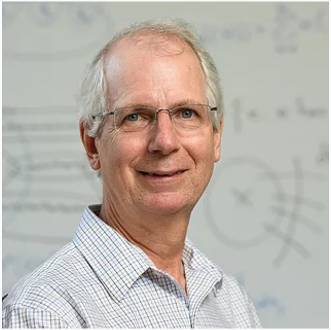
Last time, we used the fact that Edmonds showed that this is the case for:

- minimimum spanning tree,
- minimum-cost perfect matching problem.



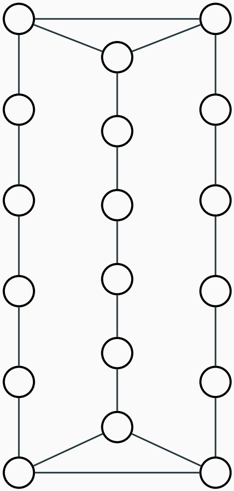
Recommended Reading





Approximating the Optimum: Graph-TSP Instances

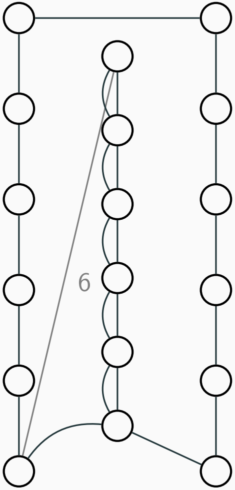
Graph-TSP Instances



A Graph-TSP instance is defined by a connected unweighted graph $G = (V, E)$.

$c(i, j)$ is the shortest path distance between i and j.

Graph-TSP Instances

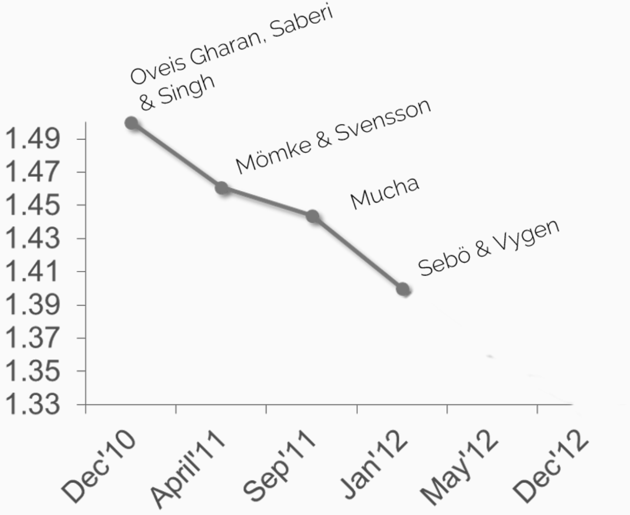


Finding an optimal TSP solution for a graph-TSP instance is equivalent to finding a multisubset F of E such that

- (V, F) is connected,
- $deg_F(v)$ is even for all $v \in V$,

and $|F|$ is minimum.

Recent Progress on Graph-TSP



Today's Result

We will prove the following theorem.

Theorem (Mömke, Svensson (2011))

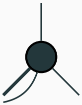
If G is cubic and 2-edge-connected, then there is a $\frac{4}{3}$-approximation algorithm for the Graph-TSP problem on G.

Cubic means all vertices have degree three. 2-edge-connected means that removing any one edge from the graph does not disconnect the graph.

Building Some Intuition

Idea

First idea: Since all vertices are odd-degree, correct all parities by adding a perfect matching to G. Then all vertices have even degree and G is connected.



Lemma

Lemma (Naddef, Pulleyblank (1981))

Given a 2-edge-connected, cubic graph G with costs $c(e)$ on the edges, there is a perfect matching of cost at most $\frac{1}{3}\sum_{e\in E} c(e)$.

Set $c(e) = 1$ for all $e \in E$. Then number of edges in E plus perfect matching M is

$$|E| + c(M) \leq |E| + \frac{1}{3}|E| \quad =$$

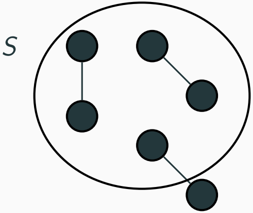
Proof of Naddef-Pulleyblank lemma

Lemma (Naddef, Pulleyblank (1981))

Given a 2-edge-connected, cubic graph G with costs $c(e)$ on the edges, there is a perfect matching of cost at most $\frac{1}{3}\sum_{e \in E} c(e)$.

The minimum-cost perfect matching can be found as the solution to the following LP:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} c(e) z(e) & \\
& z(\delta(i)) = 1 & \forall i \in V \\
& z(\delta(S)) \geq 1 & \forall S \subset V, |S| \text{ odd.}
\end{aligned}
$$

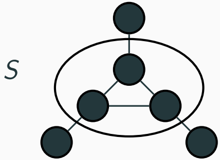


Proof of the Naddef-Pulleyblank lemma

Lemma (Naddef, Pulleyblank (1981))

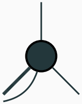
Given a 2-edge-connected, cubic graph G with costs $c(e)$ on the edges, there is a perfect matching of cost at most $\frac{1}{3}\sum_{e\in E} c(e)$.

$$\begin{aligned} \text{Min} \sum_{e\in E} c(e)z(e) & \\ z(\delta(i)) = 1 & \quad \forall i \in V \\ z(\delta(S)) \geq 1 & \quad \forall S \subset V, |S| \text{ odd} \\ z(e) \geq 0 & \quad \forall e \in E. \end{aligned}$$



New Idea

Second idea: Adding an edge has the same effect on the parity of a vertex degree as removing an edge. Since all vertices are odd-degree, we can also correct all parities by removing a perfect matching from G.



But then the resulting graph is not Eulerian because it is typically not connected!!

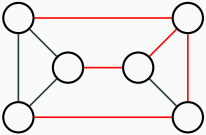
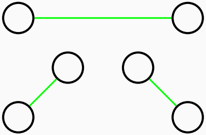
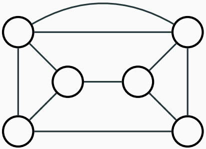
Better New Idea

Third idea: Can we figure out which edges we can safely remove and maintain connectivity?

If so, take a perfect matching, *remove* the edges which are removable and *add* a copy of the edges in the matching which are not removable.

Finding Removable Edges: First Attempt

- Find a spanning tree T of $G = (V, E)$.
- Let $R = E \setminus T$ (i.e., all edges except those in T are removable).
- Let $c(e) = 1$ for edges in T, and $c(e) = -1$ for edges in R. Compute a minimum-cost perfect matching M.
- Form F by taking E, removing the edges in $M \cap R$, and adding a copy of the edges in $M \cap T$.

Analysis

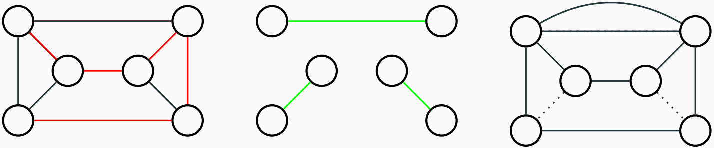
Lemma

$$|F| \leq \frac{5}{3}|V| - \frac{2}{3}$$

Proof: $|F| = |E| + c(M) \leq \ldots$

Finding Removable Edges: Second Attempt

Our spanning tree was a bit unfortunate; we could swap one edge of the tree to another edge so that the spanning tree does not intersect with the matching.



Mömke and Svensson introduce the idea of removable pairings: a systematic approach to getting many removable edges.

Removing and Adding Edges for the Traveling Salesman Problem

Removable pairing

Definition (Mömke, Svensson (2011))

Given G, $R \subseteq E$ are removable edges, and $P \subseteq R \times R$ is a removable pairing if:

(i) Any edge is in at most one pair of P;

(ii) Edges in a pair have a common endpoint of degree at least 3;

(iii) If we remove edges in R from G with at most one edge per pair in P removed, the resulting graph is still connected.

Theorem

Theorem (Mömke, Svensson (2011))

Given a 2-edge-connected cubic graph G and removable pairing (R, P), there is an Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

Similar to before, the algorithm is

- Let $c(e) = 1$ for edges in $E \setminus R$, and $c(e) = -1$ for edges in R. Compute a minimum-cost perfect matching M.
- Form F by taking E, removing the edges in $M \cap R$, and adding a copy of the edges in $M \setminus R$.

Need to show (V, F) is Eulerian, and $|F| \leq \frac{4}{3}|E| - \frac{2}{3}|E|$.

Theorem

Theorem (Mömke, Svensson (2011))

Given 2-edge-connected cubic graph G and a removable pairing (R, P), there is an Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

Proof of second claim: $|F| \leq \frac{4}{3}|E| - \frac{2}{3}|R|$.

Theorem

Theorem (Mömke, Svensson (2011))

Given a 2-edge-connected cubic graph G and removable pairing (R, P), there is an Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

Proof of first part: (V, F) is Eulerian (connected and has even degree at all nodes).

Final Lemma: Removable Pairing in a Cubic 2-Edge-Connected Graphs

Lemma (Mömke, Svensson (2011))

In any cubic, 2-edge-connected graph G, there is a removable pairing (R, P) with $|R| \geq |V|$.

Therefore, we can find an Eulerian graph with total number of edges at most

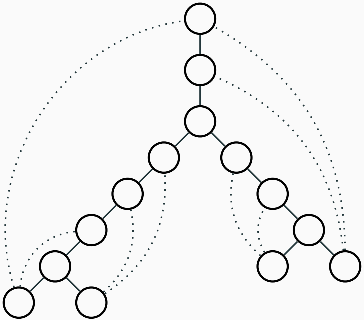
$$\frac{4}{3}|E| - \frac{2}{3}|R| \quad =$$

Proof of Final Lemma

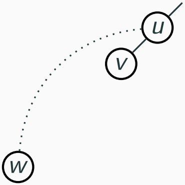
Lemma (Mömke, Svensson (2011))

In any cubic, 2-edge-connected graph G, there is a removable pairing (R, P) with $|R| \geq |V|$.

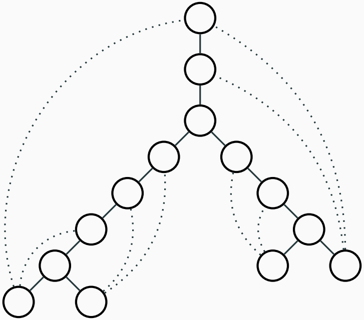
Start by considering a depth-first search tree T of the graph G.



Proof of Final Lemma



For each back edge (u, w) with u the ancestor of w, tree edge (u, v), make (u, w) and (u, v) a removable pair. For root put only one back edge in the pair.



Proof of Final Lemma: Number of Removable Edges $|R|$

Number of back edges is

$|R| =$

Proof of Final Lemma: (R, P) is a Removable Pairing

Recall the definition:

Definition (Mömke, Svensson (2011))

Given G, $R \subseteq E$ are removable edges, and $P \subseteq R \times R$ is a removable pairing if:

(i) Any edge is in at most one pair of P;

(ii) Edges in a pair have a common endpoint of degree at least 3;

(iii) If we remove edges in R from G with at most one edge per pair in P removed, the resulting graph is still connected.

Proof of Final Lemma: (R, P) is a Removable Pairing

Need to show that G stays connected if we remove at most one edge per pair. Prove by induction bottom up on subtrees; let T_u be subtree rooted at vertex u. If u is a leaf, there is nothing to prove. Otherwise, consider three cases:

Case 1: u has two children v and w, one parent in T.

Proof of Final Lemma: (R, P) is a Removable Pairing

Need to show that G stays connected if we remove at most one edge per pair. Prove by induction bottom up on subtrees; let T_u be subtree rooted at vertex u. If u is a leaf, there is nothing to prove. Otherwise, consider three cases:

Case 2: u has one child v, one parent in T.

Proof of Final Lemma: (R, P) is a Removable Pairing

Need to show that G stays connected if we remove at most one edge per pair. Prove by induction bottom up on subtrees; let T_u be subtree rooted at vertex u. If u is a leaf, there is nothing to prove. Otherwise, consider three cases:

Case 3: u is the root of T.

Extensions

- Mömke and Svensson show it is possible to extend the theorem to subcubic graphs[1].
- Recall that we showed in the first lecture the integrality gap $\sup \frac{OPT}{OPT_{LP}} \geq \frac{4}{3}$, and that the example demonstrating this gap is a graph-TSP instance where G is 2-edge-connected and subcubic. Mömke and Svensson's result shows that $OPT \leq \frac{4}{3}|V| \leq \frac{4}{3}OPT_{LP}$ for these instances, so this is tight.
- For graph-TSP on arbitrary graphs, Mömke and Svensson show a 1.461-approximation algorithm (improved by Mucha (2012) to $\frac{13}{9}$).
- Sebő and Vygen (2012) add several new ideas and get a 1.4-approximation algorithm.

[1] Every vertex degree is at most 3