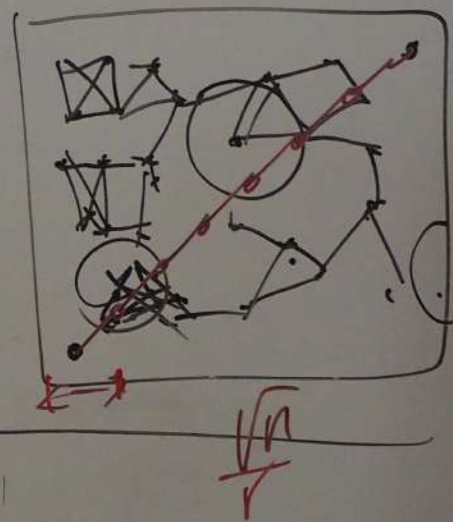
Random hyperbolic graphs

Motivation: model, mathematically analyzable, resembling real-world networks (facebook friends, server networks, ..)

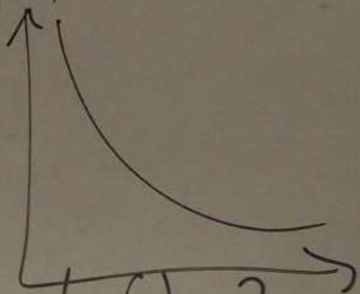
Desired properties:

- small diameter $O(\log n)$
- ultra-small networks: typical distance $\Theta(\log \log n)$
- sparse graphs (#edges is $\Theta(n)$) $\Leftrightarrow$ average degree constant

RGG in Euclidean:



- Power-law degree distribution $\mathbb{P}(\deg(k)) \propto k^{-\alpha}$,
  for some exponent $\alpha > 2$



- Positive clustering coefficient

  local clustering coeff. at v: $c_v = \begin{cases} 0 & \deg(v) < 2 \\ \dfrac{|\{\{u_1,u_2\} \in E \mid u_1, u_2 \in N(v)\}|}{\binom{\deg(v)}{2}} & \text{otw} \end{cases}$

  global clustering coeff: $\bar{c} = \frac{1}{n} \sum_v c_v$

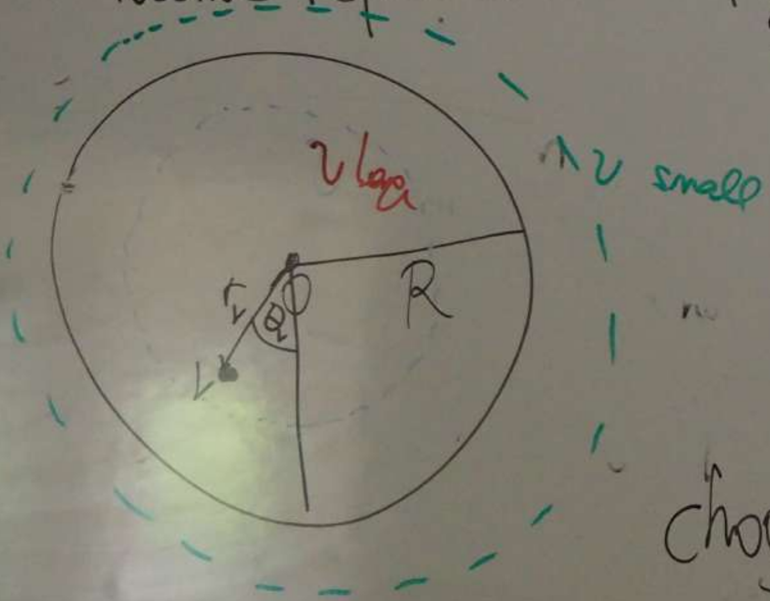
  Want $\bar{c} > 0$.

- want some geometric explanation

Random hyperbolic graphs

Hyperbolic plane: Model here with Krioukov et al.

- Native representation of hyperbolic graphs:



Poisson model with intensity n in a disk of radius $R = 2\log(n/\nu)$,

"think $\nu = 1$"

$\nu > 0$ intensity parameter

Parameter $\alpha > \frac{1}{2}$:

choose points uniformly distributed

in hyperbolic plane of curvature $-\alpha^2$

<u>Uniform model:</u> n vertices, for a vertex $v = (r_v, \theta_v)$

$\theta_v \in_{u.a.r.} [0, 2\pi)$

r_v chosen according to density function

$$f(r_v) = \begin{cases} \frac{\alpha \sinh(\alpha r_v)}{\cosh(\alpha R) - 1}, & 0 \le r_v \le R \\ 0 & r_v > R \end{cases}$$

Perimeter of a circle of radius r centered at 0 in hyperbolic space with curvature $-\alpha^2$ has length

$\alpha \cdot \sinh(\alpha r)$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Area of ball centered at 0 of radius r:

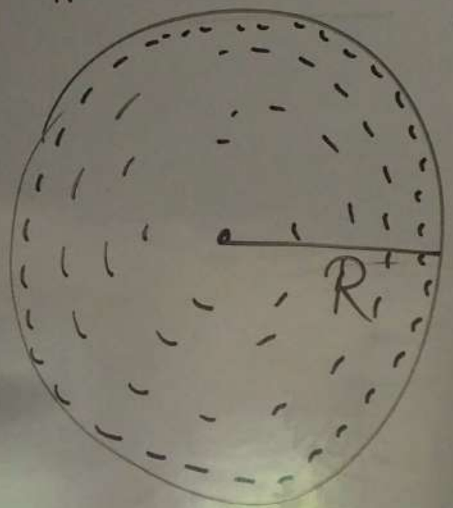
$\cosh(\alpha r) - 1$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

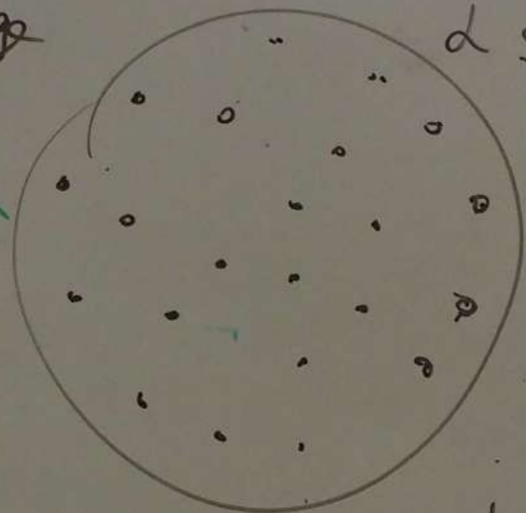
Random hyperbolic graphs

Effect of α?

α large



α small



Vertices = points of uniform/Poisson model

Edges:

$u = (r_u, \theta_u)$

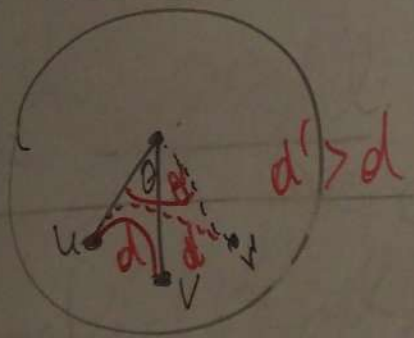
$v = (r_v, \theta_v)$

$\{u,v\} \in E \Leftrightarrow d := d_H(u,v) \leq R$

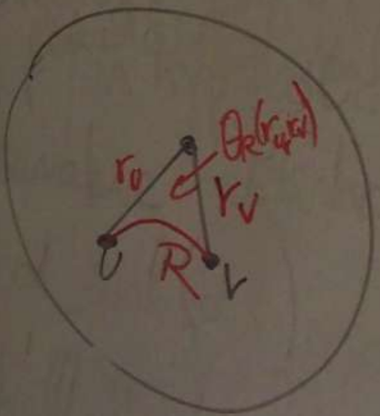
$d_H(u,v) = d \Leftrightarrow$

$\cosh d = \cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cdot \cos(\theta_u - \theta_v)$

Observation: For fixed r_u, r_v, d is monotone increasing in θ

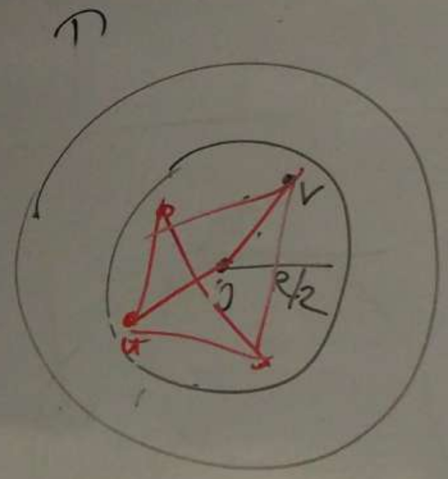


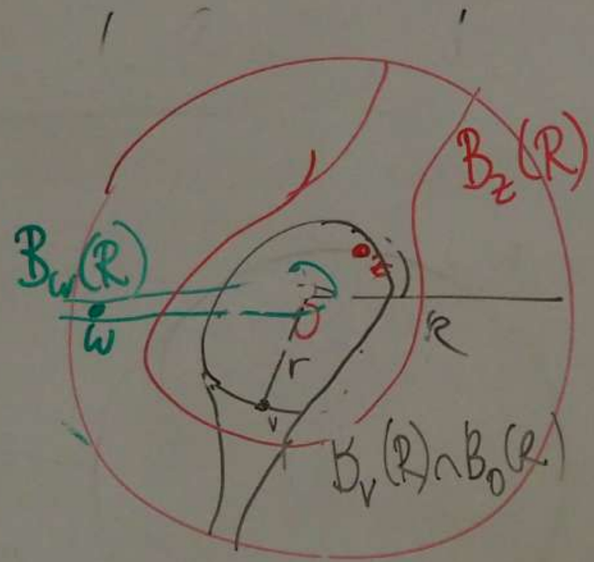
Hence, in order to know whether an edge is present, we set $d = R$, and define $\theta_R(r_u, r_v)$ = maximum angle such that two vertices with radii r_u and r_v are connected by an edge



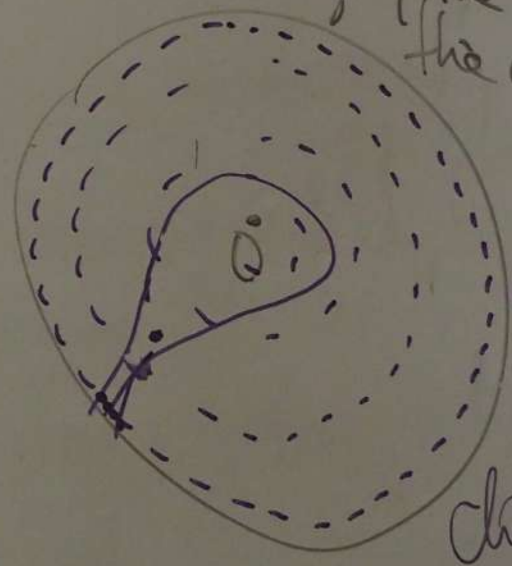
Exercise: If $0 \le r_u, r_v \le R$, $r_u + r_v \ge R$,

$$\theta_R(r_u, r_v) = (2 + o(1))\, e^{R - r_u - r_v}$$





Obs.: degree of a vertex depends on radius (the closer to the origin, the larger the degree)



most neighbors of any vertex are close boundary

How many points inside $B_0(r)$?

$$\mu(B_0(r)) = \int_0^r \frac{\alpha \sinh(\alpha x)}{\cosh(\alpha R)-1}\, dx \approx \frac{\cosh(\alpha r)}{\cosh(\alpha R)-1} = e^{-\alpha(R-r)}(1+o(1))$$

Expected # of vertices in there: $n e^{-\alpha(R-r)}$

Degree of a vertex:

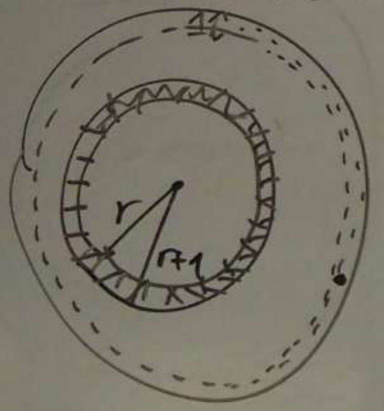
at radius r: $$\mu(B_r(R) \cap B_0(R)) = 2 \int_0^R \int_0^{\theta_R(y)} f(y)\, d\theta\, dy \approx e^{-r/2}$$

density function of radius y

Expected degree of a vertex at radius r: $\Theta(n e^{-r/2})$

Exp. degree of a vertex with r $R-C$ for some constant $C>0$ $= \Theta(1)$

Are there isolated vertices? $\mu(B_0(r+1) \setminus B_0(r)) = \Theta(\mu(B_0(r+1)))$



$\Rightarrow$ constant fraction of vertices will be in $B_0(R) \setminus B_0(R-1)$

For each such vertex, Exp(degree) = c, for some c

In the Poisson model, $\mathbb{P}(\text{degree} = 0) = e^{-c}$

$\Rightarrow$ linear number of isolated vertices

Maximum degree?

closest vertex to the origin: at a radius i_0 for which

$n\mu(B_0(i_0)) \simeq 1 \Longrightarrow i_0 = \left(1-\frac{1}{2\alpha}\right)R$

$\rightarrow$ expected maximum degree: $n^{\frac{1}{2\alpha}(1+o(1))}$, concentration by Chernoff

$\mathbb{P}(X \in (1\pm\delta)\mathbb{E}(X)) \geq 1 - 2e^{-\frac{\delta^2}{3}\mathbb{E}(X)}$