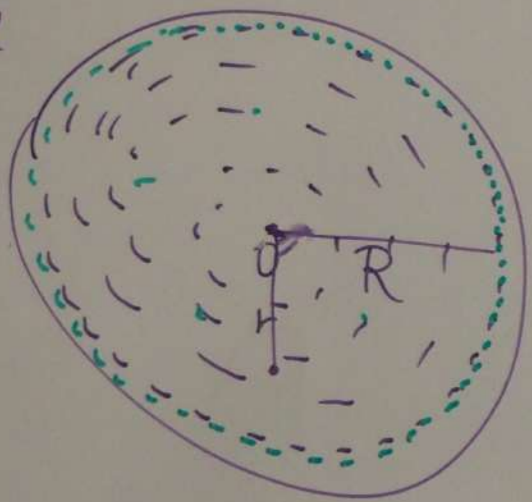
Random hyperbolic graphs

α small

α large



Poisson(n) or Uniform(n) vertices

$v = (r_v, \theta_v)$: $\theta_v \in_{uar} [0, 2\pi)$

$r_v \propto f(r_v) = \begin{cases} \frac{\alpha \sinh(\alpha r_v)}{\cosh(\alpha R)-1}, & 0 \le r_v \le R \\ 0 & \text{otw} \end{cases}$

$R = 2\log(n/\nu)$, $\nu > 0$

$\alpha \ge 1/2$ (in order to make avg degree constant)

$\mu(B_0(r)) = e^{-\alpha(R-r)} \Rightarrow$ exp. # points in $B_0(r) = ne^{-\alpha(R-r)}$

$n\mu(B_r(R) \cap B_0(R)) = ne^{-r/2}$ (exp. degree of vertex of radius r) for $\alpha > 1/2$: linear number of isolated vertices (whp)

It turns out: $\alpha = \frac{1}{2}$ threshold for connectivity (for $\alpha > 1/2$, whp G not connected, for $\alpha < 1/2$, whp G is connected)

$\alpha = 1$ threshold for giant component (for $\alpha < 1$, whp G has giant comp, $\alpha > 1$, whp G has no giant comp)

$\frac{1}{2} < \alpha < 1$: diameter of giant component: $\Theta(\log n)$

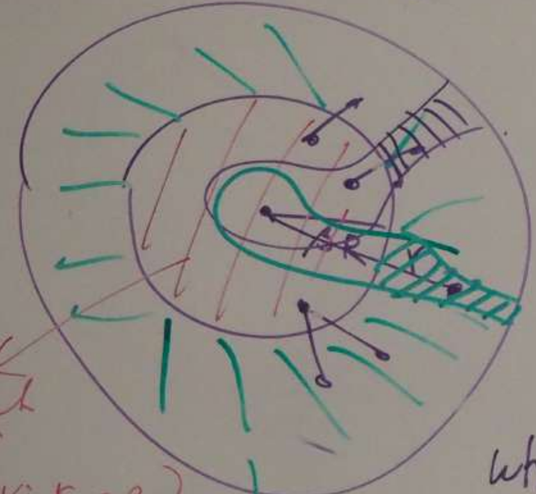
typical distance: $\Theta(\log \log n)$

Degree distribution: Thm: whp, the number of vertices of degree k (for k fixed) is $\Theta\left(n k^{-(2\alpha+1)}\right)$.

(power law distribution with exponent between 2 and 3)

Sketch of proof:

Fix $0<\beta<1$.



inner set
$I(\beta)=\{v : r_v \le \beta R\}$

Outer set
$O(\beta)=\{v : r_v > \beta R\}$

By Exercise 1a, whp, $|I(\beta)| \le \max\left(\sqrt{n},\, n^{1-2\alpha(1-\beta)}\right)$

By Exercise 1a, whp, # edges from $I(\beta)$ to $O(\beta)$ $[:= e(I(\beta), O(\beta))]$ satisfies

$$e(I(\beta), O(\beta)) = O\left(n^{1-(2\alpha-1)(1-\beta)}\right)$$

For an outer vertex at radial coordinate $r > \beta R$, what is its expected degree inside the outer set?

$$\mathbb{P}\left(\text{degree of such vertex is } k \text{ in the uniform model}\right) = \binom{n-1}{k} q_r^{k} (1-q_r)^{n-1-k}$$

$q_r = \mathbb{P}\left(\text{vertex falls in } B_r(R) \cap B_0(R) \setminus B_0(\beta R)\right)$

So, defining $D_k(\beta)$ = # vertices in $O(\beta)$ with exactly k neighbors in $O(\beta)$, we have

$$\mathbb{E}[D_k(\beta)] = n \int_{\beta R}^{R_n} \binom{n-1}{k} q_r^k (1-q_r)^{n-1-k} f(r)\,dr,$$ where $f(r)$ density function

By Exercise 1b), $\mathbb{E}[D_k(\beta)] = \Theta(n k^{-(2\alpha+1)})$.

$$D_k(\beta) - e(I(\beta), O(\beta)) \le D_k \le D_k(\beta) + \underbrace{e(I(\beta), O(\beta))}_{} + \underbrace{|I(\beta)|}_{}$$

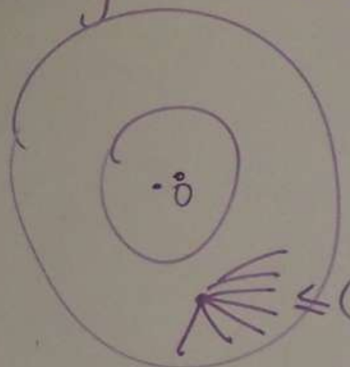
small compared to $D_k(\beta)$

$$\Rightarrow \mathbb{E}[D_k] = \mathbb{E}[D_k(\beta)](1+o(1))$$

Concentration results: By Chernoff bounds, with probability at least $1 - e^{-\Omega(n^{1-\beta})}$, the degree of a vertex in the outer set is $\le C n^{1-\beta}$.

Assuming this claim, every vertex in the outer set has degree $\le C n^{1-\beta}$,

every such vertex can change D_k by at most $C n^{1-\beta}$



Azuma | McDiarmid's bound:

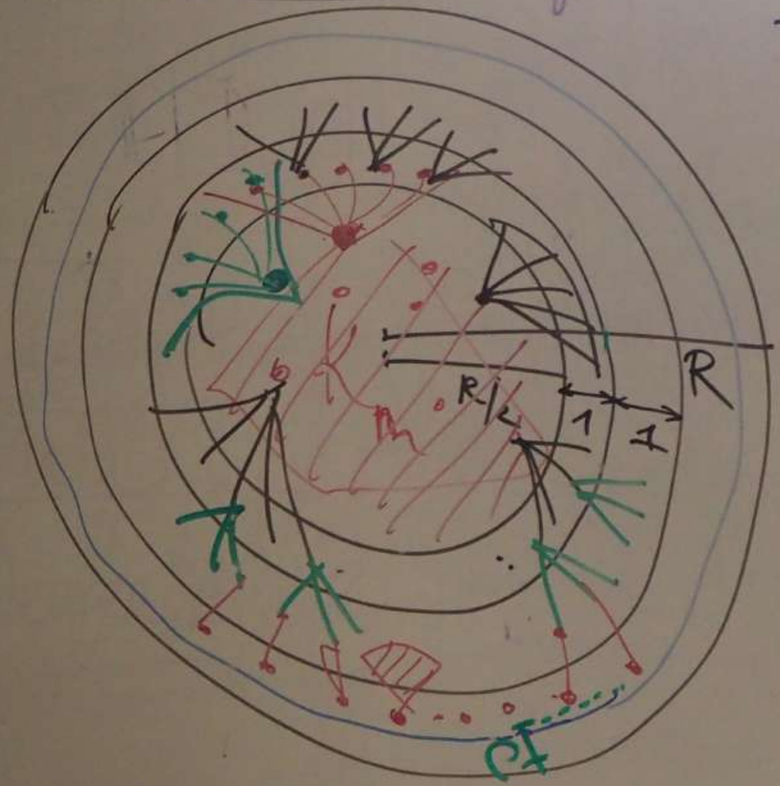
Assume $\left| f(\underbrace{X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n}_{X}) - f(X_1, \dots, X_{i-1}, X_i', X_{i+1}, \dots, X_n) \right| \le c_i$ $\forall i$.

Then $\forall t > 0$, $\mathbb{P}\left[|f(X) - \mathbb{E} f(X)| \ge t \right] \le e^{-t^2 / \sum_{i=1}^{n} c_i^2}$.

Plugging in $c_i = n^{1-\beta}$ $\forall i$, $t = n^{3/2 - 2\beta + \varepsilon}$; $t^2 \to n^{3-2\beta+2\varepsilon}$, $\sum_{i=1}^{n} c_i^2 \to n^{1-2(1-\beta)}$

$$\mathbb{P}\left[|D_k - \mathbb{E}(D_k)| \ge n^{3/2 - 2\beta + \varepsilon} \right] \le n^{-2\varepsilon} \to 0$$

Emergence of giant component, $\frac{1}{2} < \alpha < 1$



Thm: Whp, the largest component of RHG for $\alpha \in (1/2, 1)$ is $\Theta(n)$.

Sketch of proof: $D_0 = \{v : r_v \le R/2\}$

For $i \ge 1$, let $R_i := R/2 + i$.

Let $D_i = \{v : R_{i-1} \le r_v \le R_i\}$

We know that $\mathbb{E}[|D_i|] = \Theta\left(n e^{-\alpha(R-R_i)}\right)$.

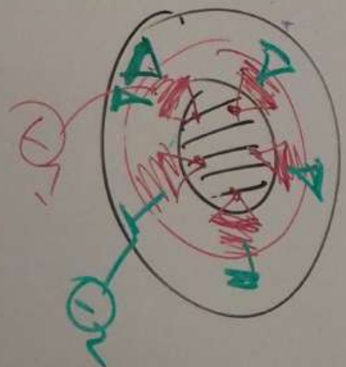
By Chernoff bounds, $\Big[$ $X \sim Po(k)$, $\mathbb{P}(X \geq (1+\delta)\mathbb{E}(X)) \leq \left(\frac{e^{-\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mathbb{E}(X)}$

$\mathbb{P}(X \leq (1-\delta)\mathbb{E}(X)) \leq e^{-\frac{\delta^2 \mathbb{E}(X)}{3}}$ $\Big]$

whp $|D_i| = (1+o(1))\,\mathbb{E}[|D_i|]$.

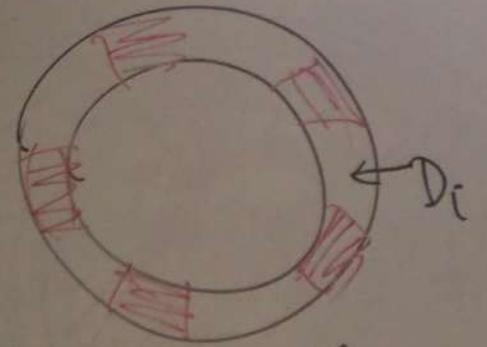
since $\mathbb{E}[|D_i|] \geq \mathbb{E}[|D_0|] = \Theta(n^{1-\alpha})$

Define <u>active angles</u>: $\Theta_0 = 2\pi$; Θ_i = union of angles with at least one neighbor in Θ_{i-1}



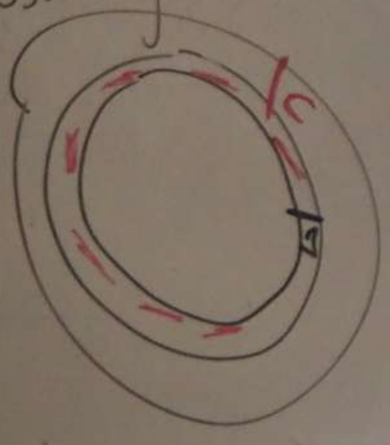
<u>Lemma (assumed)</u>: Conditional under $\Theta_{i-1} \geq \pi$, $\Theta_i \geq \left(1-e^{-\Theta(R_i)}\right)\cdot\Theta_{i-1}$ with probability $1-o(1/R)$.

Define $D_i' \subseteq D_i$ as the vertices in "active angles" of D_i.



$\mathbb{E}[|D_i'|] \geq \mathbb{E}[|D_i|] \cdot \frac{\Theta_i}{2\pi}$, again concentration by Chernoff bounds

Assuming the lemma holds, $\theta_i \geq \pi$ as long as $i \leq k/2 - C$ for some $C > 0$



$\Rightarrow$ in the last layer we have $\Theta(n)$ vertices in total (concentrated)

Half of them (in expectation) fall into active angles

w.h.p. $\left(\frac{1}{2} - \varepsilon\right) \cdot$ total # of vertices falls in active angles.