Reconfiguration of vertex colouring and forbidden induced subgraphs

M. Belavadi[1], K. Cameron[2], O. Merkel[3]

[1] *Wilfrid Laurier University, Waterloo, Canada mbelavadi@wlu.ca*
[2] *Wilfrid Laurier University, Waterloo, Canada, kcameron@wlu.ca*
[3] *Wilfrid Laurier University, Waterloo, Canada, owenmerkel@gmail.com*

Let G be a finite simple graph with vertex-set $V(G)$ and edge-set $E(G)$. For a positive integer k, a *k-colouring* of G is a mapping $\alpha: V(G) \to \{1,2,\ldots,k\}$ such that $\alpha(u) \neq \alpha(v)$ whenever $uv \in E(G)$. We say that G is *k-colourable* if it admits a k-colouring and the *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G is k-colourable.

The reconfiguration graph of the k-colourings, denoted $\mathscr{R}_k(G)$, is the graph whose vertices are the k-colourings of G and two colourings are adjacent in $\mathscr{R}_k(G)$ if they differ in colour on exactly one vertex. We say that G is *k-mixing* if $\mathscr{R}_k(G)$ is connected and the *k recolouring diameter* of G is the diameter of $\mathscr{R}_k(G)$. Given two k-colourings α and β of G, deciding whether there exists a path between the two colourings in $\mathscr{R}_k(G)$ was proved to be PSPACE-complete for all $k > 3$ [1]. The problem remains PSPACE-complete for graphs with bounded bandwidth and hence bounded treewidth [2]. In this paper, we investigate the connectivity and diameter of $\mathscr{R}_{k+1}(G)$ for a k-colourable graph G restricted by forbidden induced subgraphs. We explore the structural properties of graph classes defined by forbidden induced subgraph to study the diameter and connectivity of the reconfiguration graph. A graph G is H-free if no induced subgraph of G is isomorphic to H. Let P_n, C_n, and K_n denote the path, cycle, and complete graph on n vertices, respectively. For two vertex-disjoint graphs G and H, the *disjoint union* of G and H, denoted by $G+H$, is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

We show that $\mathscr{R}_{k+1}(G)$ is connected for every k-colourable H-free graph G if and only if H is an induced subgraph of P_4 or $P_3 + P_1$. We also start an investigation into this problem for classes of graphs defined by two forbidden induced subgraphs. We show that if G is a k-colourable $(2K_2, C_4)$-free graph, then $\mathscr{R}_{k+1}(G)$ is connected with diameter at most $4n$. Furthermore, we show that $\mathscr{R}_{k+1}(G)$ is connected for every k-colourable (P_5, C_4)-free graph G.

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References

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