Quenched Voronoi percolation

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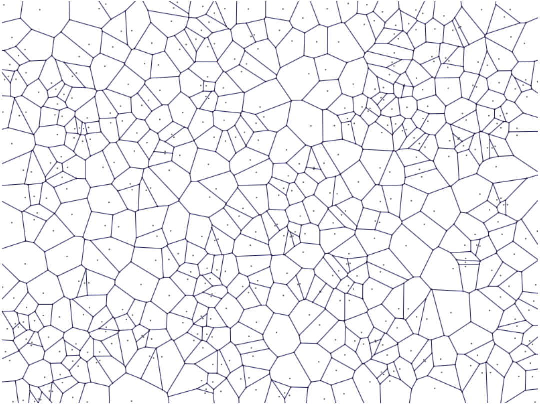
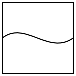
The problem

- Position n points in the unit square uniformly at random.
- Consider the Voronoi tessellation of $\eta = \{x_1, x_2, \ldots, x_n\}$.

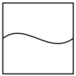


The problem

- Position n points in the unit square uniformly at random.
- Consider the Voronoi tessellation of $\eta = \{x_1, x_2, \ldots, x_n\}$.
- Toss a fair coin, once for each cell, to determine its colour.

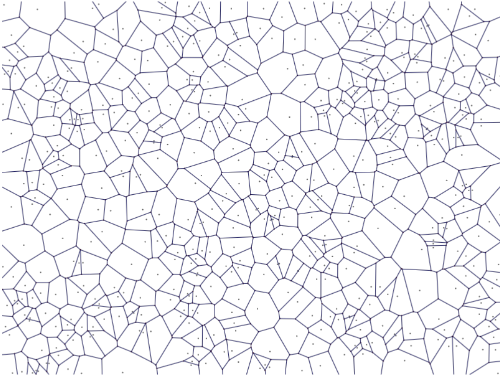
$\mathbb{P}\Big($  $\Big) = 1/2$ by symmetry.

Benjamini, Kalai and Schramm **conjectured** in 1999 that

$\mathbb{P}\Big($  $\Big| \, \eta \Big) \to 1/2$ as $n \to \infty$.

Theorem (A-Griffiths-Morris-Tassion '15+)

$$\forall \varepsilon > 0 \quad \mathbb{P}\left(\left| \mathbb{P}\left(\boxed{\sim} \,\middle|\, \eta \right) - 1/2 \right| > \varepsilon \right) \to 0 \quad \textit{as } n \to \infty.$$

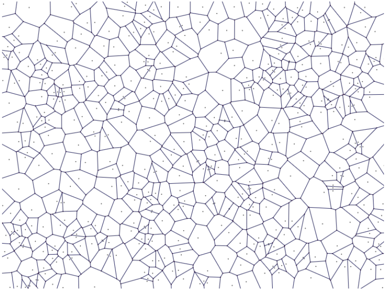


The density invariance conjecture

Conjecture (Benjamini-Schramm '98)

Let μ be some measure on the unit square comparable to Lebesgue measure. Place n points in the unit square according to μ. Then the limit

$$\lim_{n\to\infty} \mathbb{P}_\mu \Big(\quad \Big)$$  *exists and is independent of μ.*



Sensitivity to small perturbations

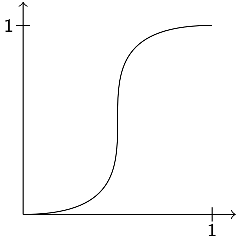
Boolean function: $f : \{0,1\}^n \to \{0,1\}$

Question: Can we tell the outcome of $f(\omega)$ by observing a slight perturbation ω^ε of ω?

Examples:
Dictatorship (yes),
Majority (yes),
Percolation crossings (no).

Why fair coin flips?

Many **monotone** Boolean functions present **sharp thresholds**.



Plot of the probability of success

$$\mathbb{P}_p(f = 1)$$

as a function of the bias p of the coin.

When sharp thresholds?

Monotone Boolean functions which do not depend on 'few' variables have sharp thresholds. Sharp thresholds occur when influences are 'small'.

Russo's formula: $$\frac{d}{dp}\mathbb{P}_p(f=1) = \sum_{i=1}^{n} \mathrm{Inf}_i^p(f).$$

The **influence** of bit $i \in \{1, 2, \ldots, n\}$ for $f : \{0,1\}^n \to \{0,1\}$ is

$$\mathrm{Inf}_i(f) := \mathbb{P}\big(f(\omega) \neq f(\sigma_i \omega)\big),$$

where $\sigma_i \omega$ is obtained from ω by flipping the value at position i.

Noise sensitivity of Boolean functions

Let ω^ε be obtained from $\omega \in \{0,1\}^n$ by flipping each bit with probab ε.

A sequence $(f_n)_{n\geq 1}$ of functions $f_n : \{0,1\}^n \to \{0,1\}$ is **noise sensitive** if

$$\forall \varepsilon > 0 \quad \mathbb{E}[f_n(\omega) f_n(\omega^\varepsilon)] - \mathbb{E}[f_n(\omega)]^2 \to 0 \quad \text{as } n \to \infty.$$

Theorem (Benjamini-Kalai-Schramm '99)

$$\sum_{i=1}^{n} \mathrm{Inf}_i(f_n)^2 \to 0 \quad \Rightarrow \quad (f_n)_{n\geq 1} \textit{ is noise sensitive.}$$

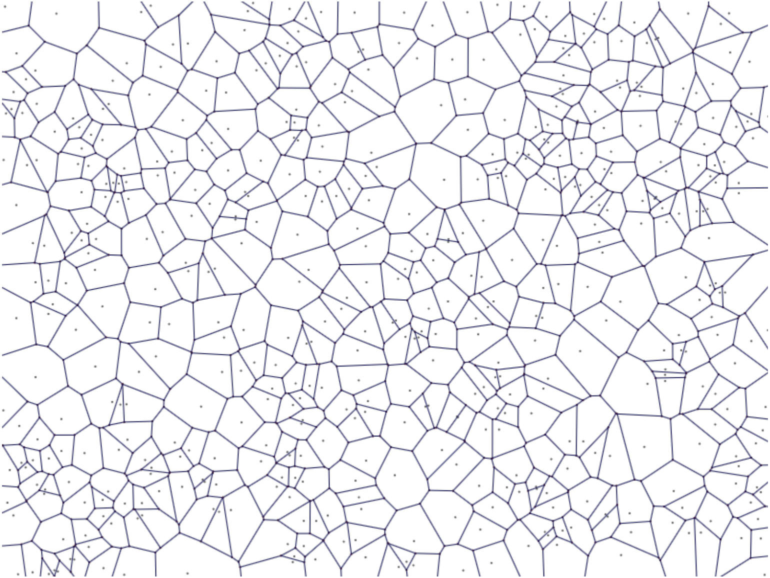
Previous work

Benjamini-Kalai-Schramm '99: The BKS-theorem, the 'algorithm method', bond percolation crossings are noise sensitive.

Schramm-Steif '10: Quantitative noise sensitivity and existence of exceptional times for dynamical percolation on the hexagonal lattice.

Garban-Pete-Schramm '10: Fourier spectrum for percolation crossings and existence of exceptional times for the square lattice.

A-Broman-Griffiths-Morris '14: Crossings in continuum percolation are noise sensitive.



Proof outline – Step 1

We will attempt a **martingale approach**, revealing the position of one point at the time, and estimating its effect on the crossing probability.

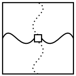
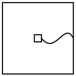
Let $\eta = \{x_1, x_2, \ldots, x_n\}$ be given. Then,

- $(X_k)_{k=1}^{n}$ with $X_k = \mathbb{P}($  $| \{x_1, x_2, \ldots, x_k\})$ is a martingale.
- The indicator of  is of the form $f_\eta : \{0,1\}^\eta \to \{0,1\}$.

Proposition

$\mathrm{Var}\Big(\mathbb{P}($  $|\eta)\Big) \leq \mathbb{E}\left[\sum_{i=1}^{n} \mathrm{Inf}_i(f_\eta)^2\right]$

Step 2 – The algorithm method

$\mathrm{Inf}_i(f_\eta) = \mathbb{P}\Big($  $\Big| \, \eta \Big) \leq \mathbb{P}\Big($  $\Big| \, \eta \Big)$

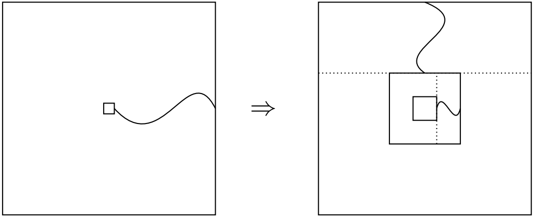
Four-arm probabilities are hard to estimate. One-arm probabilities are easier, but give too weak bounds.

The **algorithm method** will allow us to obtain a bound on the influences by estimating the **revealment** on an algorithm determining f_η.

An exploration algorithm may be used, and an upper bound on the revealment is obtained by the **one-arm event**.

Step 2 – One-arm estimate

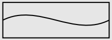
The probability of the one-arm event can be estimated by the existence of dual circuits in annuli.



It will suffice to estimate probabilities of the form $\mathbb{P}($  $|\eta)$.

Step 3 – Crossing probabilities

Proposition

$$\mathbb{P}\Big(\mathbb{P}(\ $$$$\ |\eta) \leq 1/2^k\Big) \leq (1-c)^k \quad \textit{for large } k.$$

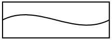
This shows that mass of $\mathbb{P}($  $|\eta)$ does not accumulate at 0 or 1.

Step 3 – Proof of Proposition

- $\mathbb{P}($  $) > c_0$, by Tassion '14+.
- Let X denote the maximal number of vertex disjoint monochromatic **vertical** crossings. Then, using colour-switching,

$\mathbb{P}($  $|\eta) = \sum_{k=1}^{\infty} (1/2)^k \, \mathbb{P}(X = k|\eta) = \mathbb{E}\left[2^{-X}\middle|\eta\right].$

- $\mathbb{P}(X \geq k) \leq \frac{1}{2}(1-c)^k$, using FKG- and BK-inequalities.

$\mathbb{P}\Big(\mathbb{P}($  $|\eta) \leq 1/2^k\Big) \leq \mathbb{P}\big(\mathbb{P}(X \geq k|\eta) \geq 1/2\big) \leq (1-c)^k.$

Proof summary

Step 1 Martingale approach to obtain a variance-influence relation.

Step 2 The algorithm method and one-arm estimate.

Step 3 First bound on quenched crossing probabilities.

Step 4 **Deal with boundary issues!**