Fluctuation bounds for interface free energies of spin glasses

Louis-Pierre Arguin
City University of New York
joint with
C. Newman (NYU), D. Stein (NYU) & J. Wehr (Arizona)

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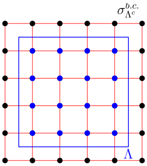
Edwards-Anderson Model and Random Field Ising Model

Consider $\Lambda \subset \mathbb{Z}^d$ a finite box and $E(\Lambda)$ the corresponding edges.

- Ising spin glass or EA model for $\sigma \in \{-1,+1\}^\Lambda$

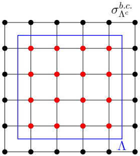
$$H_{\Lambda,\omega}(\sigma) = -\sum_{(x,y)\in E(\Lambda)} \omega_{xy}\sigma_x\sigma_y - \sum_{x\in\Lambda, y\in\Lambda^c} \omega_{xy}\sigma_x\sigma_y^{b.c.}$$



$\omega = (\omega_{xy}; (x,y) \in E(\mathbb{Z}^d))$ i.i.d. with continuous distribution $\mathbb{P}$ (Gaussian say)

- Random Field Ising Model (RFIM)

$$\widetilde{H}_{\Lambda,\omega}(\sigma) = -\sum_{(x,y)\in E(\Lambda)} J\sigma_x\sigma_y - \sum_{x\in\Lambda, y\in\Lambda^c} J\sigma_x\sigma_y^{b.c.} - \sum_{x\in\Lambda} \omega_x\sigma_x$$



$(\omega_x, x \in \mathbb{Z}^d)$ are i.i.d. random variables.

Variance bounds for the Free Energy

We want bounds on the fluctuations of the free energy:

$$F_\Lambda(\omega) = \log \sum_{\sigma \in \{-1,+1\}^\Lambda} \exp -\beta H_{\Lambda,\omega}(\sigma) \qquad \beta > 0$$

Theorem (Wehr-Aizenman '90, Chatterjee '09)

For the EA and RFIM model on $\mathbb{Z}^d$,

$$Var\ F_\Lambda(\omega) \geq C(\beta)\ |\Lambda|$$

Our goals

1. Study the fluctuations of free energy difference between b.c.
   Interface Free Energy
2. Understand the impact on the structure of the Gibbs states.

Interface Free Energy

Let $\Gamma, \Gamma' \in \mathcal{G}_\omega(\beta)$, two Gibbs states at disorder ω and inv. temp. β.
DLR equations

Interface free energy:

$$F_\Lambda(\omega, \Gamma, \Gamma') = \log \Gamma\Big(\exp \beta H_{\Lambda,\omega}(\sigma_\Lambda, \sigma_{\Lambda^c})\Big) - \log \Gamma'\Big(\exp \beta H_{\Lambda,\omega}(\sigma_\Lambda, \sigma_{\Lambda^c})\Big)$$

- By DLR, this reduces to

$$F_\Lambda(\omega, \Gamma, \Gamma') = \log \frac{\Gamma\Big(Z^{-1}_{\Lambda,\omega}(\beta, \sigma_{\Lambda^c})\Big)}{\Gamma'\Big(Z^{-1}_{\Lambda,\omega}(\beta, \sigma'_{\Lambda^c})\Big)}$$

- At $\beta = \infty$, the analogue is the difference of energies of σ and σ' in Λ ground states in $\mathbb{Z}^d$ at disorder ω.

The Random Field Ising Model

RFIM: The Aizenman-Wehr theorem

Theorem (Aizenman-Wehr '90)

In $d = 2$, for all $\beta > 0$, there is only one Gibbs state of the RFIM:

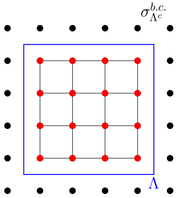
$$\boxed{\#\mathcal{G}_\omega(\beta) = 1 \quad \omega\text{-}a.s.}$$

Rounding of phase transition

- The proof is based on the argument by Imry & Ma '85.
- In $d \geq 3$, RFIM exhibits a phase transition (Imbrie '85, Bricmont & Kupiainen '87).

Variance bounds and Gibbs States

In $d = 2$, absence of phase transition in AW is shown by contradiction:



The r.v. $|F_\Lambda|/|\partial\Lambda|$ is bounded

$|F_\Lambda(\omega, \Gamma, \Gamma')| \leq C|\partial\Lambda|$ ω-a.s. for any Γ, Γ'

The r.v. $F_\Lambda/|\Lambda|^{1/2}$ is unbounded

- A martingale CLT argument shows that

$$\liminf_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{E}\left[\exp t \frac{F_\Lambda}{|\Lambda|^{1/2}}\right] \geq e^{Ct^2}$$

- If $\Gamma \neq \Gamma'$, AW shows that the fluctuations are non-trivial:

$$\mathrm{Var}\ \frac{F_\Lambda}{|\Lambda|^{1/2}} \geq C > 0\ .$$

The usefulness of the FKG inequality

For the RFIM, the interaction is ferromagnetic: FKG inequality

1. Existence of Γ_ω^+-state and Γ_ω^--state as weak limits of finite-volume Gibbs measure.
2. Domination of the states:

$$\Gamma_\omega^-\big(f(\sigma)\big) \leq \Gamma_\omega(f(\sigma)) \leq \Gamma_\omega^+\big(f(\sigma)\big)$$

   To prove uniqueness of the Gibbs state, it suffices to show

$$\Gamma_\omega^- = \Gamma_\omega^+$$

The absence of FKG is arguably
the biggest hurdle in the study of the EA model:

1. There is no known labeling between b.c. and states that is ω-independent.
2. There is no dominance between states.

The Edwards-Anderson Model

The EA model

Conjecture ($d = 2$)

At all $\beta > 0$, there is a unique Gibbs state almost surely.

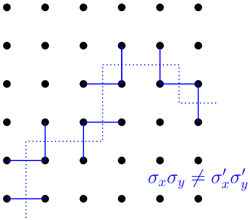
- Numerics strongly support this.
- Compare to ferromagnetic models where there are two pure states at low temperature: Rounding of phase transition.
- There could exist $\beta = \beta(\omega)$ with several Gibbs states.

Incongruent states

Some Gibbs states are more physically relevant.
$\Gamma, \Gamma' \in \mathcal{G}_\omega(\beta)$ are incongruent if

$$\liminf_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{(x,y) \in E(\Lambda)} 1_{\{|\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma_x \sigma_y)| > \delta\}} > 0$$

Positive density of edges with different edge-correlation function.



- Incongruent states correspond to states with non-trivial edge overlaps.
- For the SK model (mean-field Edwards-Anderson), there are an infinite number of incongruent states at low temperature.

The EA model

Conjecture ($d > 2$)

- *(Parisi) As in SK, there exists an infinite number of incongruent states at low temperature.*
  OR
- *(Fisher-Huse) At all $\beta > 0$, there are no incongruent states. At low temperature, there are two flip-related pure states.*

Our goal

Study the existence or non-existence of incongruent states at $d = 2$ and $d > 2$ by looking at its impact on the free energy fluctuations.

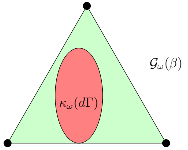
EA: choosing a Gibbs state

Recall that for EA

- there is no domination of (+) and (−) states.
- there might be more than one limit state for given b.c.

Way out: sample a state using metastate:

$$\boxed{\kappa_\omega(d\Gamma) \text{ prob. measure on } \mathcal{G}_\omega(\beta)}$$



such that

1. Coupling covariance: If $\omega_B = 0$ except on edges in a box B

$$\kappa_{\omega+\omega_B}(d\Gamma) = \kappa_\omega(d\, L_{\omega_B}\Gamma) \qquad \text{where } L_{\omega_B}\Gamma(\cdots) = \frac{\Gamma(\cdot \exp -\beta H_{B,\omega_B}(\sigma))}{\Gamma(\exp -\beta H_{B,\omega_B}(\sigma))}$$

2. Translation covariance: for any translation T, $\kappa_{T\omega}(d\Gamma) = \kappa_\omega(d\, T\Gamma)$.

The interface free energy $F_\Lambda(\omega, \Gamma, \Gamma')$ is now a r.v. over

$$\boxed{M = d\mathbb{P}(\omega)\ \kappa_\omega(d\Gamma) \times \kappa'_\omega(d\Gamma')}$$

This measure is translation-invariant.

Main Result

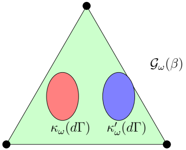
Consider κ_ω and κ'_ω two metastates on $\mathcal{G}_\omega$ and

$M = d\mathbb{P}(\omega)\, \kappa_\omega(d\Gamma) \times \kappa'_\omega(d\Gamma')$ Probability measure on the triplet $(\omega, \Gamma, \Gamma')$.

Assumption (Sufficient for existence of incongruent states)

There exists an edge (x, y) such that with positive $\mathbb{P}$-probability

$$\boxed{\kappa_\omega(\Gamma(\sigma_x \sigma_y)) \neq \kappa'_\omega(\Gamma'(\sigma_x \sigma_y))\ .}$$



Theorem (A-Newman-Stein-Wehr '14)

Under the above assumption, for all d, there exists $C > 0$ such that

$$\boxed{Var_M\big(F_\Lambda(\omega, \Gamma, \Gamma')\big) \geq C|\Lambda|}$$

Lower bound for the variance of the interface free energies in $\mathbb{Z}^d$

Possible directions

1. No incongruent states in $d = 2$ Prove that there is no incongruent states in $\mathbb{Z}^2$ as for RFIM.

$$\mathrm{Var}_M(F_\Lambda) = \underbrace{M\Big(\mathrm{Var}_M(F_\Lambda|\omega_\Lambda)\Big)}_{\text{Fluct. of b.c.}} + \underbrace{\mathrm{Var}_M\Big(M(F_\Lambda|\omega_\Lambda)\Big)}_{\text{Fluct. of couplings in }\Lambda}$$

2. No incongruent states in $d > 2$ Find a variance UPPER bound to get a contradiction

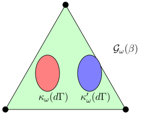
$$\boxed{\mathrm{Var}\left(\log \frac{Z^{per.}_{\Lambda,\omega}(\beta)}{Z^{anti}_{\Lambda,\omega}(\beta)}\right) \le C|\partial\Lambda|}$$

(Aizenman & Fisher, Newman & Stein, Contucci & Giardina)
Other than gauge-related ?

Picture of the proof

We want to show $\text{Var}_M\left(F_\Lambda\right) \geq C|\Lambda|$ for $M = d\mathbb{P}(\omega)\ \kappa_\omega(d\Gamma) \times \kappa'_\omega(d\Gamma')$ under

$$\kappa_\omega(\Gamma(\sigma_x\sigma_y)) \neq \kappa'_\omega(\Gamma'(\sigma_x\sigma_y)) \qquad \text{w.p.p.}$$



We have

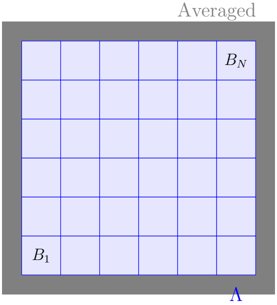
$$\text{Var}_M\left(F_\Lambda\right) \geq \underbrace{\text{Var}_M\Big(M\left(F_\Lambda|\omega_\Lambda\right)\Big)}_{\text{Fluct. of couplings in } \Lambda}$$

Divide Λ into equally sized blocks $B_1, \ldots B_k, \ldots, B_N$ where $N = c|\Lambda|$.

$$\text{Var}_M\left(F_\Lambda\right) \geq \sum_{k=1}^{N} \text{Var}_M\Big(M(F_\Lambda|\omega_{B_k})\Big)$$

To show

1. $\text{Var}_M\Big(M(F_\Lambda|\omega_{B_k})\Big) = \text{Var}_M\Big(M(F_\Lambda|\omega_{B_1})\Big)$
2. $\text{Var}_M\Big(M(F_\Lambda|\omega_{B_1})\Big) > c'$ for $c' > 0$ independent of Λ.



Picture of the proof

$$\mathrm{Var}\ M(F_\Lambda|\omega_B) = \frac{1}{2}\int d\mathbb{P}(\omega_B)\int d\mathbb{P}(\omega'_B)\ \Big\{M(F_\Lambda|\omega_B) - M(F_\Lambda|\omega'_B)\Big\}^2$$
$$= \frac{1}{2}\int d\mathbb{P}(\omega_B)\int d\mathbb{P}(\omega'_B)\left\{\int_{\omega'_B\to\omega_B}\nabla_B M(F_\Lambda|z_B)\cdot dz_B\right\}^2$$

Lemma

For any $(x,y)\in E(B)$

$$\boxed{\frac{\partial}{\partial\omega_{xy}} M\big(F_\Lambda|\omega_B\big) = \beta\ M\Big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)\Big|\omega_B\Big)\ a.s.}$$

Do not depend on Λ *AND Translation invariant*

Picture of the proof

$$\text{Var } M(F_\Lambda|\omega_B)$$

$$= \frac{\beta^2}{2} \int d\mathbb{P}(\omega_B) d\mathbb{P}(\omega'_B) \Big\{ \int_{\omega'_B \to \omega_B} \underbrace{\sum_{(x,y)\in E(B)} M\big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)\big| z_B\big)\ dz_{xy}}_{\nabla_B M(F_\Lambda|z_B)\cdot dz_B} \Big\}^2$$

The assumption implies that for B large enough

$$M\Big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)|\omega_B\Big) \overset{B \to \mathbb{Z}^d}{\longrightarrow} \kappa_\omega \times \kappa'_\omega \Big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)\Big) \neq 0 \text{ w.p.p.}$$

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1. No incongruent states in $d = 2$ Prove that there is no incongruent states in $\mathbb{Z}^2$ as for RFIM. Show

$$\mathrm{Var}_M(F_\Lambda) \geq C|\Lambda|$$

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Thank you!