Phase transition for the dilute clock model

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A technical lemma

Finite graph $(V, \mathcal{E})$

$S :=$ discrete circumference with q points.

$$\text{For } q = 4: \qquad S = \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi\right\}$$

Spin configuration: $\sigma : V \to S$.

Define

$$S_{\mathrm{mod}} := \{|a| : a \in S\} \quad \left(= \left\{0, \frac{\pi}{2}, \pi\right\}, \quad \text{in our case}\right)$$

Edge configuration: $\omega : \mathcal{E} \to S_{\mathrm{mod}}$.

σ is **compatible** with ω if the gradient of σ is dominated by ω:

$$|\sigma_x - \sigma_y| \le \omega_{\langle xy \rangle}, \quad \langle xy \rangle \in \mathcal{E}.$$

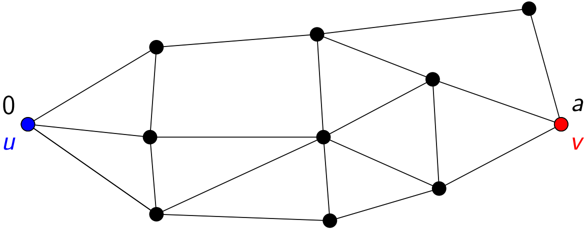
Fix an edge configuration ω, two vertices $u, v \in V$ an angle $a \in S$ and define

$$L_\omega(a) := \left|\{\sigma : \sigma_u = 0,\ \sigma_v = a,\ \sigma \text{ compatible with } \omega\}\right|$$

Combinatorial Lemma

For any edge configuration ω, it holds

$$L_\omega(a) \le L_\omega(0)$$



Combinatorial Lemma

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Application: symmetric random walks with uniform jumps

X_n symmetric random walk with delay:

$$\text{jump probabilities:} \quad p(x, x+z) = \tfrac{1}{3}, \quad \text{if } |z| \le 1.$$

Periodic random walk:

$$Y_n = (X_n)_{\mathrm{mod}\, k}.$$

Then, if $Y_0 = 0$, we have

$$P(Y_n = a) \le P(Y_n = 0), \qquad \text{for any } a \in \mathbb{Z}.$$

Here the graph is

$$V = \{0, \ldots, n\}, \quad \mathcal{E} = \{\langle i, i+1\rangle, i = 0, \ldots, n-1\},$$

$$x = n,\ y = 0, \qquad \omega \equiv 1.$$

ω gives the restriction of the random walk to jump to nearest neighbors or to stay.

The uniformity of the jump reduces de computation of the probabilities to the counting.

Extension to the non-homogeneous following case:

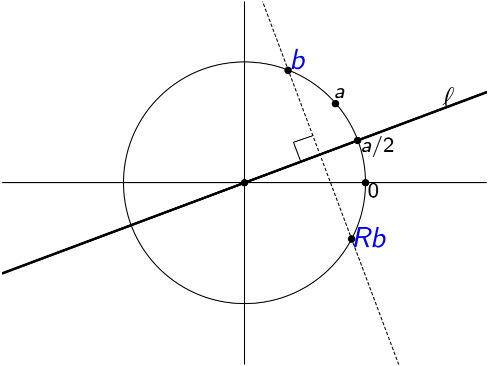
To each bond $\langle x, x+1 \rangle$ assign an "environment" $\omega_{\langle x,x+1 \rangle} \in \{0, 1, \dots\}$ such that the jump at time x is uniform in $\{z : |z| \le \omega_{\langle x,x+1 \rangle}\}$.

Proof of combinatorial lemma

We construct injection $F : L_\omega(a) \hookrightarrow L_\omega(0)$.

Fix $a \in S$ and let

$R : S \to S$ be the reflection with respect to line ℓ at angle $a/2$:



Clearly, $R(a) = 0$.

Transform $\sigma \in L_\omega(a)$ into $\sigma' \in L_\omega(0)$, iteratively:

1. Reflect the spin at v:

$$\sigma'_v \leftarrow Ra \quad (= 0).$$

2. New conflicting vertices? $|\sigma_y - \sigma_v| > \omega_{\langle yv \rangle}$?
3. If yes: Reflect new conflicting vertices y: set

$$\sigma'_y \leftarrow R\sigma_y$$

   Go to (2).
4. If no: $\sigma'_z \leftarrow \sigma_z$ for remaining spins z. END.

Show that the resulting configuration σ' belongs to $L_\omega(0)$, and that the map is injective. □

Back to the title of the talk

Goal

show phase transition in the dilute ferromagnetic nearest-neighbor q-state clock model in $\mathbb{Z}^d$, for every $q \geq 2$ and $d \geq 2$.

Method

Edwards-Sokal random-cluster representation of the clock model stochastically dominates a supercritical Bernoulli bond percolation probability (as in Potts)

Main tool

combinatorial lemma.

Dilute clock model

sites $\mathbb{Z}^d$

n.n. edges $\mathcal{E} := \{\langle xy \rangle : \|x - y\| = 1\}$

Disorder:

$$J = (J_{\langle xy \rangle} : \langle xy \rangle \in \mathcal{E}(\mathbb{Z}^d)) \tag{1}$$

iid Bernoulli(p).

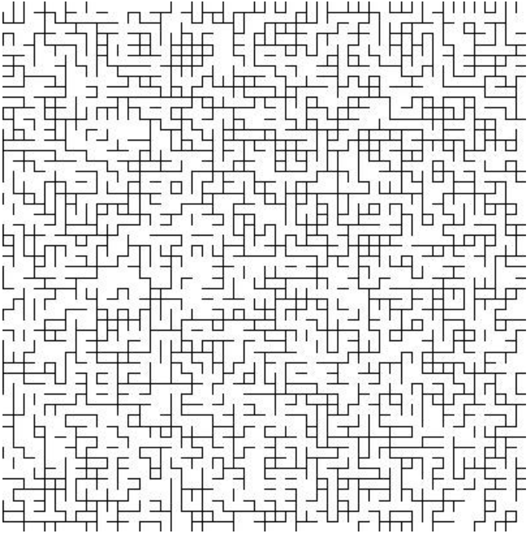
Random graph:

$$(\mathbb{Z}^d, J)$$

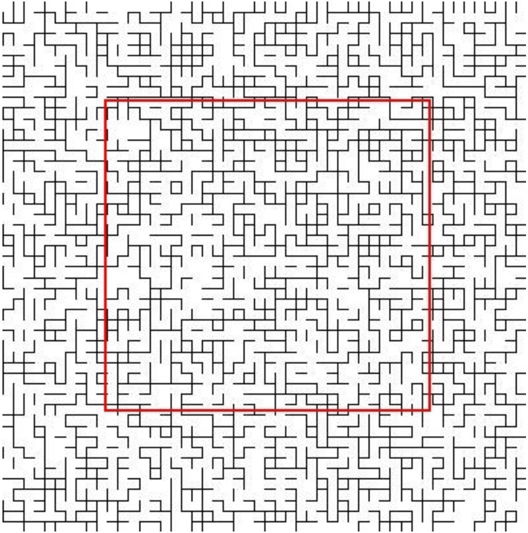
Spins in S (1-d circunference)

Spin configuration: $\sigma = (\sigma_x, x \in \mathbb{Z}^d)$, σ_x = spin at x

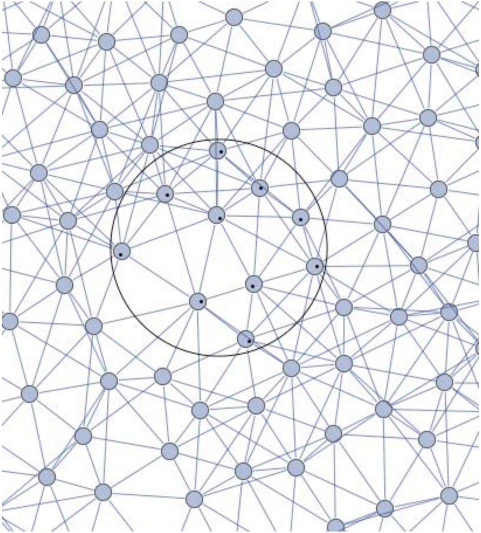
Bond percolation diluted graph



Bond percolation diluted graph, finite Λ



Other graphs: point processes



Hamiltonian:

$$H_{\Lambda,J}(\sigma) := \sum_{\substack{\langle xy\rangle \in \mathcal{E}(\mathbb{Z}^d) \\ \{x,y\}\cap\Lambda\neq\emptyset}} J_{\langle xy\rangle}\big(1-\cos(\sigma_x-\sigma_y)\big). \tag{2}$$

$q=2$ is Ising model

$q\to\infty$, XY model, continuum of spin angles.

Finite Λ specification:

$$\mu^{\omega}_{\Lambda,J}(\sigma) := \frac{1}{Z^{\omega}_{\Lambda,J}} e^{-\beta H_{\Lambda,J}(\sigma)} \mathbf{1}[\sigma \overset{\Lambda^c}{=} \omega]\,, \tag{3}$$

$\beta>0$ is inverse temperature, $Z^{\omega}_{\Lambda,J}$ normalizing constant.

Gibbs measure μ_J associated to J satisfies the DLR condition:

$$\mu_J f = \int_{S^{\mathbb{Z}^d}} \mu_J(d\omega)\, \mu^{\omega}_{\Lambda,J} f \tag{4}$$

for finite $\Lambda \subset \mathbb{Z}^d$ and local function $f : S^{\mathbb{Z}^d} \to \mathbb{R}$.

$\mathcal{G}_J$: Gibbs measures associates to J.

Since S is finite, $\mathcal{G}_J$ is not empty.

In case $|\mathcal{G}_J| > 1$: phase transition.

Background of the homogeneous clock

The homogeneous version:

$p = 1$ or, equivalently, $J_{\langle xy \rangle} \equiv 1$ for every $\langle xy \rangle$.

Low temperature: **Pirogov-Sinai** theory or **reflection positivity** as in **Frölich, Israel, Lieb and Simon** see also **Biskup** prove that, there exist at least q **different Gibbs measures**.

High temperature: **Dobrushin**, **van den Berg and Maes** show that there exists only one Gibbs measure.

Both Pirogov-Sinai theory and reflection positivity depend on **symmetry**, an assumption that breaks down for the properly dilute model $p < 1$.

Background for dilute Potts

Main tool is the **Fortuin-Kasteleyn random-cluster representation**, originally introduced for the **Ising** and the **Ashkin-Teller-Potts** models, and then generalized to arbitrary models by **Edwards and Sokal**.

Idea: relate non-uniqueness of Gibbs measure in the statistical-mechanical model to the existence of an infinite cluster in the random-cluster model: a percolation problem.

First applied by **Aizenman, Chayes, Chayes and Newman** to study the phase diagram of the **dilute Ising and Potts** models.

Our result

We fix p large enough to have percolation of the disorder J.

We derive a lower bound for the critical temperature: for every dimension d and every number q of spins there is $\beta_0 = \beta_0(q, d, p) > 0$ such that there are at least q Gibbs measures at inverse temperatures $\beta > \beta_0$, for almost all disorders J.

Main theorem

Let $p > p_c$, Bernoulli bond percolation critical probability in $\mathbb{Z}^d$.

Disorder $J \sim$ i.i.d. Bernoulli(p).

Then there is $\beta_0 = \beta_0(d, p, q) > 0$ such that $\beta > \beta_0$ implies

$$P_p(J : |\mathcal{G}_J| \geq q) = 1.$$

Comparison with other methods

For fixed d and p,

$$\beta_0(q, d, p) \sim q^2 \log q \quad \text{as} \quad q \to \infty$$

same as Pigorov-Sinai and reflection positivity in the 2-dimensional homogeneous case.

In particular,

$$\lim_{q \to \infty} \beta_0(q, d, p) = \infty$$

so not suitable to study the XY model ($q \to \infty$); (van Enter, Külske and Opoku).

For $d \geq 3$ and $p = 1$, reflection positivity gives β_0 independent of q, (Maes and Shlosman).

Clock model and random-cluster in a finite graph

non-oriented finite graph $(V, \mathcal{E})$

$U \subset V$ *boundary*.

For dilute clock in $\Lambda \subset \mathbb{Z}^d$,

$U = \partial\Lambda := \{y \in \mathbb{Z}^d \setminus \Lambda, \exists x \in \Lambda : \|x - y\|^2 = 1\}$,

vertex and edge sets are

$$V = \Lambda \cup \partial\Lambda \quad \text{and} \quad \mathcal{E} = \{\langle xy \rangle, \{x, y\} \not\subset \Lambda^c, \|x-y\| = 1, J_{\langle xy \rangle} = 1\}. \tag{5}$$

The clock model.

spins or angles:

$$S := \left\{ \frac{2\pi i}{q} : i = 0, \dots, q-1 \right\}. \tag{6}$$

$S \ni a$, b and c,

spin-configurations in S^V are denoted by σ.

Hamiltonian $H = H(V, \mathcal{E})$

$$H(\sigma) := \sum_{\langle xy \rangle \in \mathcal{E}} \big(1 - \cos(\sigma_x - \sigma_y)\big). \tag{7}$$

write $\sigma \overset{U}{=} a$ when $\sigma_x = a$ for all $x \in U$.

Clock probability $\mu = \mu(V, U, \mathcal{E}, \beta)$ with 0-boundary condition:

$$\mu(\sigma) := \frac{1}{Z} e^{-\beta H(\sigma)} \mathbf{1}[\sigma \overset{U}{=} 0], \tag{8}$$

Edge product measure.

Recall $S_{\mathrm{mod}} = \{|a| : a \in S\}$. $k = |S_{\mathrm{mod}}|$ where $k \approx q/2$.

Weights $W : S_{\mathrm{mod}} \to (0,1]$

$$W(a) := e^{-\beta(1-\cos a)} \tag{9}$$

$$0 = a_0 < a_1 < \cdots < a_k$$

Define θ probability on S_{mod}:

$$\theta(a_i) := W(a_i) - W(a_{i+1}), \quad i = 0, \ldots, k$$

with the convention $W(a_{k+1}) = 0$.

Edge product measure with marginals θ:

$$\boxed{\hat{\phi}(\omega) := \prod_{\langle xy \rangle \in \mathcal{E}} \theta(\omega_{\langle xy \rangle}), \qquad \omega \in (S_{\mathrm{mod}})^{\mathcal{E}}} \tag{10}$$

Spin-edge compatibility

$$\omega \sim \sigma \quad \Leftrightarrow \quad |\sigma_x - \sigma_y| \le \omega_{\langle xy \rangle} \text{ for every } \langle xy \rangle \in \mathcal{E}. \tag{11}$$

If $\omega \sim \sigma$:

$\omega_{\langle xy \rangle} = 0$ implies $\sigma_x = \sigma_y$;

$\omega_{\langle xy \rangle} = a_k$ imposes no restriction on σ_x and σ_y.

Clock measure as edge measure

We can write:

$$\boxed{\mu(\sigma) = \hat{\phi}(\omega : \sigma \sim \omega)}$$

Random-cluster probability $\phi = \phi(V, U, \mathcal{E}, \beta)$ on $\mathcal{I}^{\mathcal{E}}$

edge-configuration has weight proportional to the number of compatible vertex-configurations:

$$\boxed{\phi(\omega) := \frac{1}{Z} \left|\{\sigma : \omega \sim \sigma,\, \sigma \stackrel{U}{=} 0\}\right| \hat{\phi}(\omega).} \tag{12}$$

Spin-product measure uniform on S^V with 0 at U:

$$\boxed{\hat{\mu}(\sigma) := \frac{1}{q^{|V \setminus U|}} \mathbf{1}[\sigma \stackrel{U}{=} 0]}. \tag{13}$$

Joint edge-vertex probability

$Q = Q(V, U, \mathcal{E}, \beta)$ on $\mathcal{I}^{\mathcal{E}} \times S^V$.

$$\boxed{Q(\omega, \sigma) := \frac{1}{Z'} \, \mathbf{1}[\omega \sim \sigma] \; \hat{\phi}(\omega) \; \hat{\mu}(\sigma),} \tag{14}$$

(product probability $\hat{\phi} \times \hat{\mu}$ conditioned to compatibility.)

Edwards-Sokal representation theorem

The measures ϕ and μ are respectively the first and second marginals of Q.

Q is a coupling between clock μ and random-cluster ϕ.

Conditional distribution of σ given ω is uniform on compatible set

$$Q(\sigma \mid \omega) = \frac{Q(\omega, \sigma)}{\sum_{\sigma'} Q(\omega, \sigma')} = \frac{\hat{\mu}(\sigma)\, \mathbf{1}[\omega \sim \sigma]}{\hat{\mu}\big(\sigma' : \omega \sim \sigma'\big)}. \tag{15}$$

Sampling a clock configuration:

First choose $\omega \sim \phi$, then choose σ uniformly among those compatible with ω and satisfy the boundary restriction:

$$\mu(\sigma) = \sum_{\omega \in \mathcal{I}^{\mathcal{E}}} \frac{\hat{\mu}(\sigma)\, \mathbf{1}[\omega \sim \sigma]}{\hat{\mu}\big(\sigma' : \omega \sim \sigma'\big)}\, \phi(\omega). \tag{16}$$

Percolation

Denote $x \overset{\omega}{\longleftrightarrow} y$ if there is a path between x and y with $\omega = 0$: an ω-open path.

Connection to boundary and "magnetization"

If x is connected to the boundary by a ω-open path, then $\sigma_x = 0$:

$$\{(\omega, \sigma) : x \overset{\omega}{\longleftrightarrow} U\} \subset \{(\omega, \sigma) : \sigma_x = 0\}$$

Hence,

$$\begin{aligned} &\mu(\sigma : \sigma_x = a) \\ &\quad = \phi\big(\omega : x \overset{\omega}{\longleftrightarrow} U\big)\mathbf{1}[a = 0] \; + \; Q\big((\omega, \sigma) : \sigma_x = a,\, x \overset{\omega}{\not\longleftrightarrow} U\big). \end{aligned}$$

Positive correlations lemma

For any $x \in V$ and spin $a \neq 0$,

$$\mu(\sigma : \sigma_x = 0) \;\geq\; \mu(\sigma : \sigma_x = a) + \phi\big(\omega : x \stackrel{\omega}{\longleftrightarrow} U\big). \qquad (17)$$

Stochastic domination theorem

Let B^ρ Bernoulli product measure on $\{0,1\}^{\mathcal{E}}$ with parameter ρ.

Theorem *For any $\rho \in [0,1)$ there exists $\beta_0 = \beta_0(\rho) > 0$ such that if $\beta \geq \beta_0$, let $\omega \sim \phi$. Then,*

B_ρ is stochastically dominated by $1 - \omega$

β_0 is independent of the graph $(V, \mathcal{E})$ and the boundary U.

This means that if B_ρ has an infinite cluster, then there is an infinite path of zeroes in ω.

Recall combinatorial Lemma

For every $x \in V$, $a \in S$ and $\omega \in \mathcal{I}^{\mathcal{E}}$,

$$\left|\{\sigma : \omega \sim \sigma, \sigma \overset{U}{=} 0, \sigma_x = a\}\right| \;\leq\; \left|\{\sigma : \omega \sim \sigma, \sigma \overset{U}{=} 0, \sigma_x = 0\}\right|.$$

Equivalently,

$$\hat{\mu}(\sigma : \sigma_x = a,\, \omega \sim \sigma) \leq \hat{\mu}(\sigma : \sigma_x = 0,\, \omega \sim \sigma). \qquad (18)$$

Comparison with diluted Potts with q spins

Potts Hamiltonian: $\sum_{\langle xy \rangle} J_{\langle xy \rangle} \mathbf{1}[\sigma_x \neq \sigma_y]$,

Random-cluster on $\{0,1\}^{\mathcal{E}}$;

If $\omega \sim \sigma$ and

$\omega_{\langle xy \rangle} = 1$ then $\sigma_x = \sigma_y$

$\omega_{\langle xy\rangle} = 0$ then no restriction on $\sigma_x - \sigma_y$

Component of the graph $(V, \{\langle xy\rangle : \omega_{\langle xy\rangle} = 1\})$ is an ω-cluster.
Then $\omega \sim \sigma$ implies that σ is constant on ω-clusters.

Hence, for Potts model, simpler combinatorial term:

$$|\{\sigma : \omega \sim \sigma, \sigma \stackrel{U}{=} 0\}| = q^{\text{number of } \omega\text{-clusters}}. \tag{19}$$

Proof of main Theorem: Phase co-existence.

Disorder $J \sim \text{Bernoulli}(p)$.

Take $p > p_c$, the critical value for bond percolation in $\mathbb{Z}^d$.

$$P_p(J : x \stackrel{J}{\longleftrightarrow} \infty) > 0$$

Take $\rho \in (0,1)$ such that $p\rho > p_c$

Let J' be an independently sampled P_ρ-disorder.

Denote by JJ' vertices open for J and J', (JJ' is a $P_{p\rho}$-disorder)

Let $\mathcal{X} \subset \{0,1\}^{\mathcal{E}(\mathbb{Z}^d)}$ be the set of disorders J such that there is an infinite JJ'-open cluster with probability 1:

$$\mathcal{X} := \{J : P_\rho(J' : \text{ there is an infinite } JJ'\text{-open cluster}) = 1\}. \tag{20}$$

Hence, for each $J \in \mathcal{X}$, there exists a vertex $x \in \mathbb{Z}^d$ belonging to an infinite JJ'-open cluster with positive P_ρ-probability:

$$P_\rho\big(J' : x \stackrel{JJ'}{\longleftrightarrow} \infty\big) > 0. \tag{21}$$

Let $\beta_0 = \beta_0(\rho)$ be as in the statement of the domination Theorem.

Fix a $J \in \mathcal{X}$ and a vertex x satisfying (21). Since $1 - \omega \geq J'$ by Holley and (17),

$$\mu^0_{\Lambda_n,J}(\sigma : \sigma_x = 0) \;\geq\; \mu^0_{\Lambda_n,J}(\sigma : \sigma_x = a) + P_\rho\big(J' : x \xleftrightarrow{JJ'} \infty\big), \quad a \neq 0.$$

Conclude that any weak limit as $n \to \infty$ will satisfy

$$\mu^0_J(\sigma : \sigma_x = 0) > \mu^0_J(\sigma : \sigma_x = a), \qquad a \neq 0. \tag{22}$$

By the rotational symmetry,

$$\mu^b_J(\sigma : \sigma_x = b) > \mu^b_J(\sigma : \sigma_x = a), \qquad a \neq b,$$

and therefore the q-Gibbs measures μ^b_J, $b \in S$, must be different. □

Proof of positive correlations.

For any spin $a \neq 0$,

$$\mu(\sigma : \sigma_x = a) = \sum_{\omega :\, x \overset{\omega}{\not\longleftrightarrow} U} \frac{\hat{\mu}(\sigma : \sigma_x = a, \omega \sim \sigma)}{\hat{\mu}(\sigma : \omega \sim \sigma)} \, \phi(\omega)$$

$$\leq \sum_{\omega :\, x \overset{\omega}{\not\longleftrightarrow} U} \frac{\hat{\mu}(\sigma : \sigma_x = 0, \omega \sim \sigma)}{\hat{\mu}(\sigma : \omega \sim \sigma)} \, \phi(\omega)$$

$$= \; Q\big((\omega, \sigma) : \sigma_x = 0, \, x \overset{\omega}{\not\longleftrightarrow} U\big),$$

$$\leq \; Q\big((\omega, \sigma) : \sigma_x = 0, \, x \overset{\omega}{\not\longleftrightarrow} U\big) + \phi(\omega : x \overset{\omega}{\longleftrightarrow} U)$$

$$= \mu(\sigma : \sigma_x = 0) \tag{23}$$

where the first inequality is the combinatorial lemma. □

Proof of stochastic domination.

Holley's inequality says:

Global domination $B_\rho \leq_{st} \phi$ follows from local domination:

$$\rho \leq \phi\big(\omega : 1-\omega_{\langle xy\rangle} = 1 \big| \, \omega : \omega \overset{\mathcal{E}\setminus\langle xy\rangle}{=} \omega'\big) =: \alpha(\langle xy\rangle, \omega'), \qquad (24)$$

for each edge $\langle xy\rangle \in \mathcal{E}, \;\; \omega' \in \mathcal{I}^{\mathcal{E}}$.

So, we need to prove (24).

For $t \in S_{\mathrm{mod}}$ define $t_{\langle xy\rangle}\omega' \in \mathcal{I}^{\mathcal{E}}$ by

$$(t_{\langle xy\rangle}\omega')_{\langle xy\rangle} = t \qquad \text{and} \qquad t_{\langle xy\rangle}\omega' \overset{\mathcal{E}\setminus\langle xy\rangle}{=} \omega'.$$

Omitting the dependence of α on $(\langle xy\rangle, \omega')$ in the notation, and recall $a_0 = 0$ to get

$$\alpha = \frac{\phi(0_{\langle xy\rangle}\omega')}{\sum_t \phi(t_{\langle xy\rangle}\omega')} = \frac{\theta(0)|\{\sigma : 0_{\langle xy\rangle}\omega' \sim \sigma, \sigma \overset{U}{=} 0\}|}{\sum_t \theta(t)|\{\sigma : t_{\langle xy\rangle}\omega' \sim \sigma, \sigma \overset{U}{=} 0\}|}, \qquad (25)$$

where the sums are on $\{t : t \in S_{\mathrm{mod}}\}$. Hence,

$$\alpha^{-1} = \sum_{t \in S_{\mathrm{mod}}} \frac{\theta(t)}{\theta(0)} \frac{|\{\sigma : t_{\langle xy\rangle}\omega' \sim \sigma, \sigma \overset{U}{=} 0\}|}{|\{\sigma : 0_{\langle xy\rangle}\omega' \sim \sigma, \sigma \overset{U}{=} 0\}|}. \qquad (26)$$

For $t \in S_{\mathrm{mod}}$, let

$$K(t) := |\{(a,b) \in S \times S : |a-b| = t\}|. \qquad (27)$$

(depends on β).

Identifying all points in U and calling $\tilde{\omega}$ the resulting configuration in the new graph and observing that

$$|\{\sigma : \sigma \sim \tilde{\omega}, \sigma_y = 0, \sigma_x = a, \sigma \stackrel{U}{=} 0\}| = \frac{1}{q}|\{\sigma : \sigma \sim \tilde{\omega}, \sigma_y = 0, \sigma_x = a\}|$$

we can get rid of $\sigma \stackrel{U}{=} 0$ by changing the graph, so that

$$\alpha^{-1} = \sum_t \sum_{a \le t} \frac{\theta(t)}{\theta(0)} \frac{K(t)}{K(0)} \frac{|\{\sigma : \sigma \sim \tilde{\omega}, \sigma_y = 0, \sigma_x = a\}|}{|\{\sigma : \sigma \sim \tilde{\omega}, \sigma_y = 0, \sigma_x = 0\}|}. \quad (28)$$

By the combinatorial Lemma,

$$\alpha^{-1} \le \sum_t \sum_{a \le t} \frac{\theta(t)}{\theta(0)} \frac{K(t)}{K(0)} =: (\gamma(\beta))^{-1}. \quad (29)$$

We conclude that $B_\rho \leq_{st} \phi$ if β satisfies

$$\rho \leq \gamma(\beta) \tag{30}$$

The function γ is increasing in β.

$\lim_{\beta\to\infty} \gamma(\beta) = 1$.

$\lim_{\beta\downarrow 0} \gamma(\beta) = 0$. See figure.

$\gamma^{-1} : (0,1) \to (0,\infty)$ is well defined.

Hence,

if $\beta_0 = \gamma^{-1}(\rho)$, then local domination holds for all $\beta \geq \beta_0$. $\square$

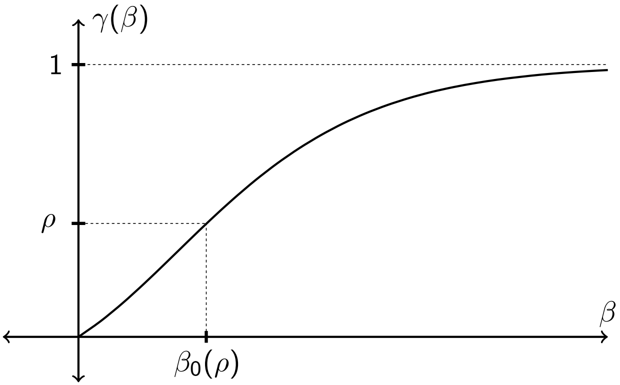


Figure: The graph of γ when $q = 4$.

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