Regularisation by noise in PDEs

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Regularisation by (Brownian) noise

Addition of noise has positive effects on the theory of the equation (in a pathwise sense)

→ ODEs:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + W_t$$

Many results: Veretennikov, Davie, Krylov-Röckner, Flandoli, Attanasio, Fedrizzi, Proske, Aryasova–Pilipenko,... Essentially bounded b. More precisely, Ladyzhenskaya-Prodi-Serrin (LPS) condition :

$$b \in L_t^q L_x^p \qquad \frac{d}{p} + \frac{2}{q} < 1.$$

→ Transport (or continuity) equation (Stratonovich integral):

$$\mathrm{d}_t u(t,x) + b(x)\cdot\nabla u(t,x)\mathrm{d}t = \nabla u(t,x)\circ \mathrm{d}W_t$$

good theory for L^∞ solutions and preservation of regularity. Flandoli–G.–Priola, Flandoli–Attanasio, Flandoli–Maurelli, Neves–Olivera.

→ Flandoli–Beck–G.–Maurelli: full LPS condition ($\leqslant 1$), new promising method of proof.

→ Stochastic vector advection equation (Flandoli–Maurelli–Neklyudov):

$$\mathrm{d}_t \boldsymbol{B} + \mathrm{curl}(\boldsymbol{v} \times \boldsymbol{B})\mathrm{d}t + \sigma \sum_{k=1}^{d} \mathrm{curl}(\mathbf{e}_k \times \boldsymbol{B}) \circ \mathrm{d}W_t^k = 0.$$

Noise avoid blow-up of $\|\boldsymbol{B}(t,\cdot)\|_{L_x^\infty}$ for $\boldsymbol{v} \in C^\alpha$ with $\alpha \in (0,1)$.

→ Non-linear PDEs with transport structure. Point vortices in 2d (Flandoli–G.–Priola), Vlasov–Poisson (Delarue–Flandoli–Vincenzi).

$$\mathrm{d}u(t,x) + u(t,x)\cdot\nabla u(t,x) = \sum_{k=1}^{N} \sigma_k(x)\cdot\nabla u(t,x) \circ \mathrm{d}W_t^k.$$

(Hypoelliptic) Noise helps to avoid collapse due to peculiar configurations.

→ Modulated non-linear Schrödinger equation in $d=1$. De Bouard–Debussche, Debussche–Tsutsumi.

$$\mathrm{d}_t\varphi(t,x) = i\Delta\varphi(t,x) \circ \mathrm{d}W_t + i|\varphi(t,x)|^{p-2}\varphi(t,x)\mathrm{d}t$$

Motivated by homogeneisation in optical wave–guides with dispersion management.

→ Averaging lemmas for kinetic equations. (Fedrizzi-Flandoli-Priola-Vovelle, Lions-Perthame-Souganidis, Gess–Souganidis)

Goal: provide a deterministic framework to discuss regularization by "perturbations/modulation" for the following model PDEs:

- **Transport equation:** $x \in \mathbb{R}^d$, $t \geqslant 0$, $w\colon \mathbb{R} \to \mathbb{R}^d$, $b\colon \mathbb{R}^d \to \mathbb{R}^d$

$$\partial_t u(t,x) + \dot{w}_t \cdot \nabla u(t,x) + b(x) \cdot \nabla u(t,x) = 0, \qquad u(0,\cdot) = u_0.$$

- **Non-linear Schrödinger equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$, $w\colon \mathbb{R} \to \mathbb{R}$

$$\partial_t \varphi(t,x) = i\Delta\varphi(t,x)\dot{w}_t + i|\varphi(t,x)|^{p-2}\varphi(t,x).$$

- **Korteweg–de Vries equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$, $w\colon \mathbb{R} \to \mathbb{R}$

$$\partial_t u(t,x) = \partial_x^3 u(t,x)\dot{w}_t + \partial_x (u(t,x))^2.$$

▷ By defining a suitable notion of "irregular" w we are able to show, in a quantitative way, that the more w is irregular the more some properties of these equations improves.

▷ The sample paths of Brownian motion or fractional Brownian motion and similar processes have almost surely this kind of irregularity.

[Joint work with Remi Catellier and Khalil Chouk]

Consider the linear transport PDE

$$\partial_t u(t,x) + \dot{w}_t \cdot \nabla u(t,x) = f(x), \qquad u(0,\cdot) = 0.$$

Solutions are given explicitly by

$$u(t,x) = \int_0^t f(x + w_s - w_t)\mathrm{d}s = T_t^w f(x - w_t)$$

where for any continuous function $w: [0,1] \to \mathbb{R}^d$ we define the **averaging operator**

$$T_t^w f(x) = \int_0^t f(x + w_s)\mathrm{d}s, \qquad T_{t,s}^w f = T_t^w f - T_s^w f$$

acting on functions (or distributions) $f: \mathbb{R}^d \to \mathbb{R}$.

Question: What is the relation between w, the regularity of f and that of $u(t,\cdot)$?

If w is smooth we do not expect anything special to happen and u to have the same regularity of f.

▷ $d=1$, $w_t=t$. Then if $F'(x)=f(x)$ we have $T_t^w f(x)=\int_0^t F'(x+s)\mathrm{d}s=F(x+t)-F(x)$ and $T^w\colon L^\infty\to\mathrm{Lip}$:

$$|T_t^w f(x)-T_t^w f(y)|\leqslant\|f\|_\infty|x-y|,\qquad |T_{t,s}^w f(x)|\leqslant\|f\|_\infty|t-s|$$

▷ Tao–Wright: if w "wiggles enough" then T_t^w maps L^q into $L^{q'}$ with $q'>q$.

▷ Davie: if w is a sample of BM then a.s. (the exceptional set depends on f)

$$|T_{t,s}^w f(x)-T_{t,s}^w f(y)|\leqslant C_w\|f\|_\infty|x-y|^{1-}|t-s|^{1/2-}$$

Problem: study the mapping properties of T^w with w sample path of a stochastic process.

Consider

$$Y_t^w(\xi) = \int_0^t e^{i\langle \xi, w_s\rangle} \mathrm{d}s$$

then $T_t^w f = \mathcal{F}^{-1}(Y_t^w \mathcal{F}(f))$. Mapping properties of T^w in $(H^s)_{s\in\mathbb{R}}$ spaces can be discussed in terms of Y^w:

$$\|T_{t,s}^w f\|_{H^s} = \left\|(1+\xi^2)^{s/2} Y_{t,s}^w(\xi) \mathcal{F}f(\xi)\right\|_{H_\xi^s}.$$

In our setting more convenient to look at the scale $(\mathcal{F}L^\alpha)_\alpha$:

$$\|f\|_{\mathcal{F}L^\alpha} = \int |f(\xi)|(1+\xi^2)^{\alpha/2} \mathrm{d}\xi$$

since $\mathcal{F}L^\alpha \subseteq C^\alpha$.

Definition 1 (Catellier–G.) *We say that w is (ρ,γ)–irregular if there exists a constant K such that for all $\xi \in \mathbb{R}^d$ and $0 \leqslant s \leqslant t \leqslant 1$:*

$$|Y_{t,s}^w(\xi)| \leqslant K(1+|\xi|)^{-\rho}|t-s|^\gamma.$$

Where we find irregularity?

▷ In $d = 1$ smooth functions are (ρ, γ) irregular for $\rho + \gamma = 1$. In particular if we insist on $\gamma > 1/2$ we have $\rho < 1/2$.

▷ Not easy to say if a function is irregular.

Theorem *The fBM of Hurst index H is ρ–irregular for any $\rho < 1/2H$.*

$\Rightarrow$ there exists functions of arbitrarily high irregularity and arbitrarily L^∞-near any given continuous function.

Lemma *An irregular function cannot be too regular.*

Proof. If $w \in C^\theta$ with $\alpha\theta + \gamma > 1$ and $\alpha \in [0, 1]$, using the Young integral, we find

$$|t - s| = |e^{ia}(t - s)| = \left| \int_s^t \underbrace{e^{ia - iaw_r}}_{C^{\alpha\theta}} \mathrm{d}_r \underbrace{Y_r^w(a)}_{C^\gamma} \right|$$

$$\leqslant C K_w (|t - s|^\gamma + |t - s|^{\alpha\theta + \gamma} |a|^\alpha) \|w\|_\theta (1 + |a|)^{-\rho} \rightarrow 0$$

if $t > s$ and $\alpha < \rho$. This implies that is not possible that $\theta > (1 - \gamma) / \rho$.

▷ For $d>1$ smooth functions are not irregular: if $|t-s|\ll 1$

$$\int_s^t e^{i\langle a,w_r\rangle}\mathrm{d}r \simeq \int_s^t e^{i\langle a,w'_s\rangle(t-s)}\mathrm{d}r \simeq (1+|\langle a,w'_s\rangle|)^{-1}\not\lesssim(1+|a|)^{-\rho}.$$

▷ If w is ρ–irregular and φ is a C^1 perturbation then $w+\varphi$ is at least $\rho-(1-\gamma)$ irregular since:

$$Y^{w+\varphi}_{t,s}(\xi) = \int_s^t e^{i\langle\xi,w_r+\varphi_r\rangle}\mathrm{d}r = \int_s^t e^{i\langle\xi,\varphi_r\rangle}\mathrm{d}_r Y^w_{s,r}(\xi)$$

and we can use Young integral estimates.

▷ If W is a fBM and Φ an adapted smooth perturbation then $W+\Phi$ is as irregular as W (via Girsanov theorem).

▷ Other results (see Catellier thesis): relation with intersection local times, irregularity for α-stable Levy processes, relation with local non-determinism.

Theorem *If w is ρ–irregular then*

$$T^w : H^s \to H^{s+\rho}$$

and

$$T^w : \mathcal{F}L^\alpha \to \mathcal{F}L^{\alpha+\rho}.$$

Proof. Indeed

$$\|T^w_{t,s} f\|_{\mathcal{F}L^{\alpha+\rho}} = \int \mathrm{d}\xi \,(1+|\xi|)^{\alpha+\rho} |Y^w_{t,s}(\xi)(\mathcal{F}f)(\xi)|$$

$$\leqslant K_w |t-s|^\gamma \int \mathrm{d}\xi\,(1+|\xi|)^\alpha |(\mathcal{F}f)(\xi)| = K_w |t-s|^\gamma \|f\|_{\mathcal{F}L^\alpha}.$$

Remark More difficult to understand the mapping properties in other spaces, for example Hölder spaces C^α. Only partial results available. Wide open problem.

▷ Consider the transport equation with a perturbation:

$$\partial_t u(t,x) + \dot{w}_t \cdot \nabla u(t,x) + b(x) \cdot \nabla u(t,x) = 0, \qquad u(0,\cdot) = u_0.$$

▷ In the Lipshitz case there is only one solution u given by the method of characteristics:

$$u(t,x) = u_0(\phi_t^{-1}(x))$$

where $\phi_t(x) = x_t$ is the flow of the ODE

$$\begin{cases} \dot{x}_t = b(x_t) + \dot{w}_t \\ x_0 = x \end{cases}$$

▷ Uniqueness of solutions is related to the uniqueness (and smothness) theory of the flow.

In order to exploit the averaging properties of w in the study of the ODE

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

we rewrite it in order to make the action of the averaging operator explicit: let $\theta_t = x_t - w_t$:

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s)\mathrm{d}s = \theta_0 + \int_0^t (\mathrm{d}_s G_s)(\theta_s)$$

where $G_s(x) = T_s^w b(x)$ so that $\mathrm{d}_s G_s(x) = f(w_s + x)$.

If we assume that G is C^γ in time $(\gamma > 1/2)$ with values in a space of regular enough functions we can study this equation as a Young type equation for $\theta \in C^\gamma$.

▷ **Non-linear Young integral:**

$$\int_0^t (\mathrm{d}_s G_s)(\theta_s) = \lim_{\Pi} \sum_i G_{t_{i+1},t_i}(\theta_{t_i})$$

This limit exists if $\theta \in C_t^\gamma$ and $G \in C_t^\gamma C_x^\nu$ with $\gamma(1+\nu) > 1$. The integral is in C_t^γ.

Theorem *The integral equation*

$$\theta_t = \theta_0 + \int_0^t (\mathrm{d}_s G_s)(\theta_s)$$

is well defined for $\theta \in C^\gamma$ *and* $G \in C_t^\gamma C_{x,\mathrm{loc}}^\nu$ *with* $(1+\nu)\gamma > 1$*.*

- *Existence of global solutions if* G *of linear growth.*
- *Uniqueness if* $G \in C_t^\gamma C_{x,\mathrm{loc}}^{\nu+1}$ *and differentiable flow.*
- *Smooth flow if* $G \in C_t^\gamma C_x^{\nu+k}$*.*

Theorem *The equation*

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

has a unique solution for w ρ*–irregular and* $b \in \mathcal{F}L^\alpha$ *for* $\alpha > 1 - \rho$*. In this case we can take* $\theta \in C^1$ *above and the condition for uniqueness (and Lipshitz flow) is* $G \in C_t^\gamma C_x^{3/2}$*.*

▷ Say that x is controlled by w if $\theta = x - w \in C^{\gamma}$. In this case we have

$$I_x(b) = \int_0^t b(x_s)\mathrm{d}s = \int_0^t (\mathrm{d}_s T_s^w b)(\theta_s)$$

and the r.h.s. is well defined as soon as $T^w b \in C_t^{\gamma} C_x^{\nu}$.

▷ If w is ρ irregular and $b \in \mathcal{F}L^{\alpha}$ then $T^w b \in C_t^{\gamma} \mathcal{F}L_x^{\alpha+\rho}$ so if $\alpha + \rho \geqslant \nu$ we have $T^w b \in C_t^{\gamma} C_x^{\nu}$.

In this case $I_x(b)$ can be extended by continuity to all $b \in \mathcal{F}L^{\alpha}$ and in particular we have given a meaning to

$$\int_0^t b(x_s)\mathrm{d}s$$

when b is a distribution *provided* x is controlled by a ρ-irregular path.

▷ For controlled paths the ODE

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

make sense even for certain distributions b as a Young equation for $\theta = x - w$.

(work of R. Catellier)

We want to give a meaning and study the uniqueness problem for the transport equation

$$(\partial_t + b(x)\cdot\nabla + \dot{w}_t\cdot\nabla)u(t,x) = 0$$

for $u \in L^\infty$ and $w \in C^\sigma$ with $\sigma > 1/3$ such that $(w, \mathbb{W})$ is a geometric σ-Hölder rough path such that w is ρ-irregular. For the moment only in the case $\operatorname{div} b = 0$.

▷ **Weak formulation:** We consider u as a distribution: $u_t(\varphi) = \int \mathrm{d}x\,\varphi(x)\,u(t,x)$ for all $\varphi \in L^1(\mathbb{R}^d)$. The integral formulation of the equation is

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla\cdot(b\varphi))\mathrm{d}r + \int_s^t u_r(\nabla\varphi)\mathrm{d}_r w_r$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $0 \leqslant s \leqslant t$.

We need to give a meaning to such an integral equation in order to discuss the regularization by noise phenomenon. (No way out!)

▷ Possible via the theory of **controlled rough paths** (G. 2004).

Let $(X, \mathbb{X})$ be a σ-Hölder rough path with $\sigma > 1/3$:

$$\mathbb{X}_{t,s} = \mathbb{X}_{t,u} + \mathbb{X}_{u,s} + (X_t - X_u) \otimes (X_u - X_s), \qquad |X_t - X_s| + |\mathbb{X}_{s,t}|^{1/2} = O(|t-s|^{\sigma})$$

▷ We say that $y \in C_t^{\sigma}$ is **controlled by $\boldsymbol{X}$** if there exists $y^X \in C_t^{\sigma}$ such that

$$y_t - y_s - y_s^X (X_t - X_s) =: y^{\sharp}_{s,t} = O(|t-s|^{2\sigma}).$$

▷ For a controlled path y we can define the integral against X by compensated Riemman sums:

$$I_t = \int_0^t y_s \mathrm{d}X_s := \lim_{\Pi} \sum_i y_{t_i}(X_{t_{i+1}} - X_{t_i}) + y_{t_i}^X \mathbb{X}_{t_{i+1}, t_i}$$

▷ This integral is the only function (up to constants) which has the following property

$$I_t - I_s = y_s (X_t - X_s) + y_s^X \mathbb{X}_{t,s} + O(|t-s|^{3\sigma}).$$

In particular, the integral is itself controlled by X and $I^X = y$.

Definition *We say that u is a function controlled by w if for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$u_t(\varphi) - u_s(\varphi) = u_s^w(\varphi)(w_t - w_s) + u_{t,s}^\sharp(\varphi)$$

where $u_\cdot^w(\varphi) \in C^\sigma$ and $|u_{t,s}^\sharp(\varphi)| \lesssim |t-s|^{2\sigma}$.

Definition *If u is controlled we say that it is a L^∞ solution of the rough transport equation (RTE) if*

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi))\mathrm{d}r + \int_s^t u_r(\nabla \varphi)\mathrm{d}_r w_r$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $0 \leqslant s \leqslant t$.

Remark: If $\sigma > 1/2$ we can just assume that $u_t(\nabla \varphi) \in C_t^\sigma$ so that the rough integral becomes a Young integral.

Equivalently, u is a solution to the RTE iff

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi))\mathrm{d}r + u_s(\nabla \varphi)(w_t - w_s) + u_s(\nabla^2 \varphi)\mathbb{W}_{t,s} + O(|t-s|^{3\sigma})$$

Lemma *If b is Lipshitz there exists a solution to the RTE given by $u(t,x) = u_0(\phi_t^{-1}(x))$.*

Theorem *Let $b \in \mathcal{F}L^\alpha$ for $\alpha > 0$ and $\alpha + \rho > 3/2$ and let w be ρ-irregular. Then there exists a unique solution to the RTE given by the method of characteristics.*

Proof. Approximate b by b_ε, then by the previous Lemma there exists a unique solution u_ε to the RTE. Analysis of the approximate flow ϕ_ε shows that this solution converges to a controlled solution u of the RTE with vectorfield b. Since ϕ is Lipschitz we can prove again uniqueness. □

Remark The above result is path-wise. In particular b can depend on w.

Remark If $b \in C^\alpha$, b deterministic and w is a fBm of Hurst index H then the uniqueness holds almost surely when $\alpha > 1 - 1/(2H)$ and $\alpha > 0$. This recovers the results of Flandoli–Gubinelli–Priola for the Brownian case but extend them well beyond the Brownian context.

(joint work with K. Chouk)

Two simple dispersive models with ρ-irregular modulation w:

- **Non-linear Schödinger equation:** $x \in \mathbb{T}, \mathbb{R}, \mathbb{R}^2$, $t \geqslant 0$

$$\partial_t \varphi(t,x) = i\Delta\varphi(t,x)\partial_t w_t + i|\varphi(t,x)|^{p-2}\varphi(t,x).$$

- **Korteweg–de Vries equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$

$$\partial_t u(t,x) = \partial_x^3 u(t,x)\partial_t w_t + \partial_x(u(t,x))^2.$$

To be compared to the non-modulated setting where $\partial_t w_t = 1$ and studied in the scale of $(H^s)_s$ spaces.

The equations are understood in the mild formulation

$$u(t) = U_t^w u(0) + \int_0^t U_t^w (U_s^w)^{-1} \partial_x (u(s))^2 \mathrm{d}s.$$

with $U_t^w = e^{i w_t \partial_x^3}$. (similarly for NLS). Here w can be an arbitrary continuous function.

Rewrite the mild formulation as ($U_t^w = e^{\partial_x^3 w_t}$)

$$v(t) = (U_t^w)^{-1} u(t) = u(0) + \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 \mathrm{d}s.$$

Theorem *Let*

$$X_t(\varphi) = X_t(\varphi, \varphi) = \int_0^t (U_s^w)^{-1} \partial_x (U_s^w \varphi)^2 \mathrm{d}s$$

If w is ρ irregular then $X \in C^\gamma \mathrm{Lip}_{\mathrm{loc}}(H^\alpha)$ for $\alpha > -\rho$ and $\rho > 3/4$.

For $v \in C^\gamma H^\alpha$ we can give a meaning to the non–linearity as a Young integral

$$\int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 \mathrm{d}s := \int_0^t (\mathrm{d}_s X_s)(v(s)) := \lim_{\Pi} \sum_i X_{t_{i+1}}(v(t_i)) - X_{t_i}(v(t_i))$$

The continuity of the Young integral implies that if $v_n \rightarrow v$ in $C^\gamma H^\alpha$ then

$$\int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 \mathrm{d}s = \lim_n \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v_n(s))^2 \mathrm{d}s$$

Theorem *The Young equation for $v \in C^\gamma H^\alpha$:*

$$v(t) = u(0) + \int_0^t (\mathrm{d}_s X_s)(v(s))$$

has local solutions for initial conditions in H^α with locally Lipshitz flow. Uniqueness in $C^\gamma H^\alpha$.

▷ Equivalent "differential" formulation:

$$v(t) - v(s) = X_{t,s}(v(s)) + O(|t-s|^{2\gamma}), \qquad v(0) = u_0$$

Regularization by modulation. In the non-modulated case it is known that there cannot be a continous flow for $\alpha \leqslant -1/2$ on $\mathbb{T}$ and $\alpha \leqslant -3/4$ on $\mathbb{R}$.

▷ Global solutions thanks to the L^2 conservation and smoothing for $\alpha > 0$ or an adaptation of the I-method for $-3/2 \leqslant \alpha < 0$ and $\alpha > -\rho/(3-2\gamma)$.

▷ **NLS:** 1d, global solutions for $\alpha \geqslant 0$ and $\rho > 1/2$. 2d, local solutions for $\alpha \geqslant 1/2$.

▷ Global solutions for 1d NLS with $\alpha > 0$ come from a smoothing effect of the non–linearity which is due to the irregularity of the driving function.

A different line of attack to the modulated Schrödinger equation comes from the application of the following Strichartz type estimate which can be proved under the same ρ-irregularity assumption.

Theorem *Let $T > 0$, $p \in (2, 5]$, $\rho > \min\left(\frac{3}{2} - \frac{2}{p}, 1\right)$ then there exists a finite constant $C_{w,T} > 0$ and $\gamma^\star(p) > 0$ such that the following inequality holds:*

$$\left\| \int_0^\cdot U_\cdot^w (U_s^w)^{-1} \psi_s \, d\, s \right\|_{L^p([0,T], L^{2p}(\mathbb{R}))} \leq C_w T^{\gamma^\star(p)} \| \psi \|_{L^1([0,T], L^2(\mathbb{R}))}$$

for all $\psi \in L^1([0, T], L^2(\mathbb{R}))$.

▷ In the deterministic case the Strichartz estimate does not have the factor of T in the critical case $p = 5$. This is a sign of a *mild* regularization effect of the noise.

Remark Similar path–wise statements (in w) holds true for averaging lemmas in kinetic equations with irregular perturbations (similar to the results of Lions–Perthame–Souganidis in the Brownian case).

As an application we obtain global well-posedness for the modulated NLS equation with generic power nonlinearity $i\,e$: $\mathcal{N}(\phi) = |\phi|^{\mu}\,\phi$: (Debussche–de Bouard, Debussche–Tsutsumi)

Theorem *Let $\mu \in (1,4]$, $p = \mu + 1$, $\rho > \min(1, 3/2 - \frac{2}{p})$ and $u^0 \in L^2(\mathbb{R})$ then there exists $T^\star > 0$ and a unique $u \in L^p([0,T], L^{2p}(\mathbb{R}))$ such that the following equality holds:*

$$u_t = U_t^w\, u^0 + i \int_0^t U_t^w (U_s^w)^{-1} (|u_s|^{\mu}\, u_s)\, d\, s$$

for all $t \in [0, T^\star]$. Moreover we have that $||\, u_t||_{L^2(\mathbb{R})} = ||\, u_0||_{L^2(\mathbb{R})}$ and then we have a global unique solution $u \in L^p_{loc}([0,+\infty), L^{2p}(\mathbb{R}))$ and $u \in C([0,+\infty), L^2(\mathbb{R}))$. If $u^0 \in H^1(\mathbb{R})$ then $u \in C([0,\infty), H^1(\mathbb{R}))$.

Thanks.