Double roots of random Littlewood polynomials

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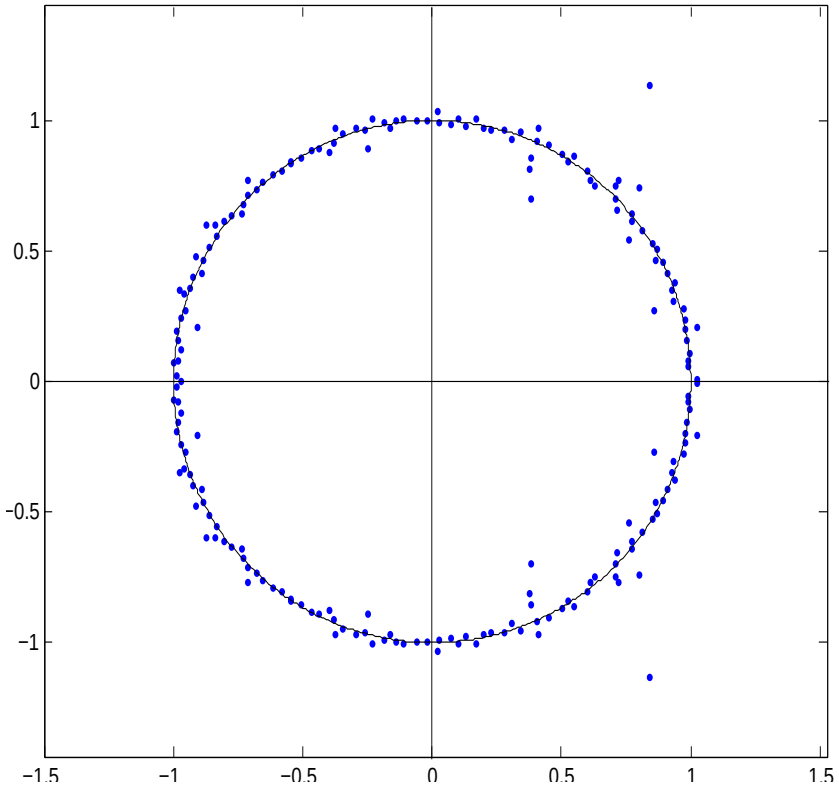
Joint with Ohad Feldheim (IMA), Ron Peled (Tel Aviv) and Ofer Zeitouni (Weizmann and NYU).

- Random Littlewood Polynomial of degree n:

$$P(z) = \xi_0 + \xi_1 z + \xi_2 z^2 + \cdots + \xi_n z^n,$$

ξ_i's are i.i.d. with $\mathbb{P}(\xi_0 = -1) = \mathbb{P}(\xi_0 = +1) = \frac{1}{2}$.

Roots of a random Littlewood polynomial of degree 200



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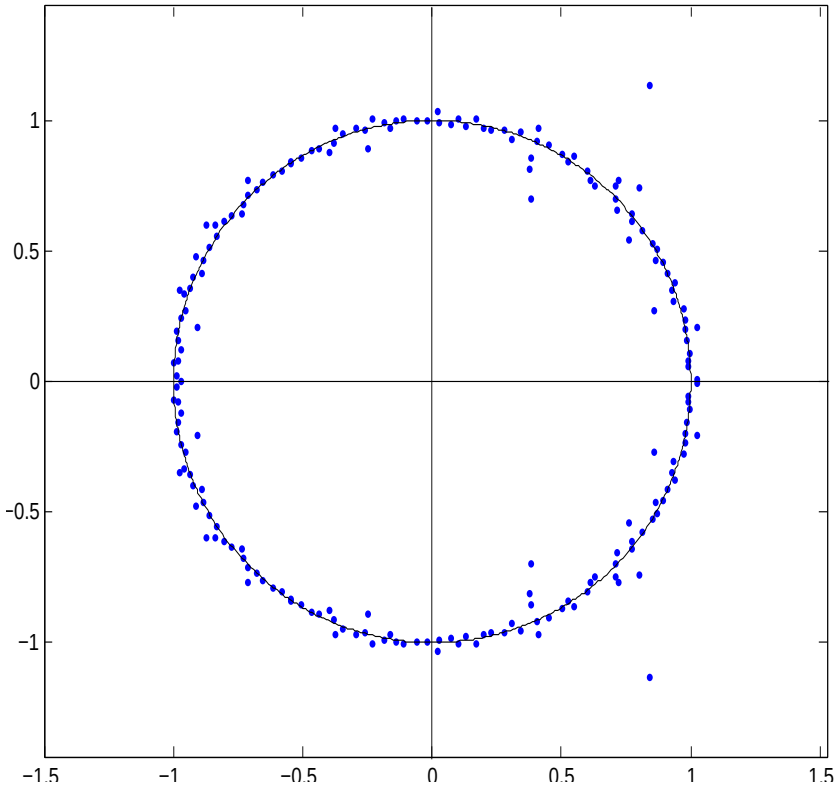

- The limiting distribution of the roots is uniform on the unit circle.
- The number of real roots is $\approx \frac{2}{\pi} \log n$.

Double roots

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- Intuition: The double root probability should go to zero as $n \to \infty$ since the roots of random polynomials tend to repel each other.

- Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0 = \prod_{i=1}^{n}(z - z_i)$ and $T : \mathbb{C}^n \to \mathbb{C}^n$ such that $(z_1, \ldots, z_n) \mapsto (a_0, a_1, \ldots, a_{n-1})$. Then T has Jacobian

$$\prod_{i<j} |z_i - z_j|^2.$$

Double roots

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- Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0 = \prod_{i=1}^{n}(z - z_i)$ and $T : \mathbb{C}^n \to \mathbb{C}^n$ such that $(z_1, \ldots, z_n) \mapsto (a_0, a_1, \ldots, a_{n-1})$. Then T has Jacobian
$$\prod_{i<j} |z_i - z_j|^2.$$
- Another related question (will not be addressed in this talk): what is the minimum separation between the roots of P in the complex plane?

P = random Littlewood polynomial of degree n.

Theorem (Peled-S.-Zeitouni'14)

$$\mathbb{P}(P \text{ has a double root}) = \begin{cases} \frac{8\sqrt{3}}{\pi n^2} + o(n^{-2}) & \text{if} \quad 4 \mid (n+1) \\ o(n^{-2}) & \text{otherwise.} \end{cases}$$

Random polynomials with (biased) ±1 coefficients.

- Let ξ_i be i.i.d. supported on $\{-1, 1\}$ such that

$$\max_{x \in \{-1,1\}} \mathbb{P}(\xi_0 = x) < \frac{1}{\sqrt{2}} = 0.707...$$

Theorem

$$\mathbb{P}(P \text{ has a double root}) = \mathbb{P}(P \text{ has a double root at } \pm 1) + o(n^{-2}).$$

$\mathbb{P}(P \text{ has a double root at } \pm 1)$

- When $\mathbb{P}(\xi_0 = -1) = \mathbb{P}(\xi_0 = +1) = \frac{1}{2}$:

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- Heuristics:

$$\mathbb{P}(P(1) = 0, P'(1) = 0) \approx \mathbb{P}\Big(P_{Gau}(1) \in [-\tfrac{1}{2}, \tfrac{1}{2}], P'_{Gau}(1) \in [-\tfrac{1}{2}, \tfrac{1}{2}]\Big)$$
$$\asymp \frac{1}{\sqrt{\mathrm{var}(P(1))}\sqrt{\mathrm{var}(P'(1))}}$$
$$\asymp \frac{1}{n^{1/2}} \frac{1}{n^{3/2}}.$$

$\mathbb{P}(P \text{ has a double root at } \pm 1)$

- When $\mathbb{P}(\xi_0 = -1) = \mathbb{P}(\xi_0 = +1) = \frac{1}{2}$:

$$\mathbb{P}(P \text{ has a double root at } \pm 1) = \begin{cases} \frac{8\sqrt{3}}{\pi n^2} + o(n^{-2}) & \text{if } \quad 4 \mid (n+1) \\ 0 & \text{otherwise.} \end{cases}$$

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$$\asymp \frac{1}{n^{1/2}} \frac{1}{n^{3/2}}.$$

- The above estimate can be proved by comparing the characteristic functions of the random vector $(P(1), P'(1))$ and its Gaussianized version.

Some basic algebraic terminologies

- algebraic numbers = roots of integral polynomial
- algebraic integers = roots of monic integral polynomial
- degree of an algebraic number = the degree of its minimal polynomial.

Better error term

Theorem

Suppose $\xi_i \in \{-1, 1\}$ *i.i.d. such that* $\max_{x \in \{-1,1\}} \mathbb{P}(\xi_0 = x) < \frac{1}{\sqrt{2}}$. *Then*

$$\mathbb{P}(P \text{ has a double root}) = \mathbb{P}(P \text{ has a double root at } \pm 1) + o(n^{-2}).$$

- ± 1 are the only roots of unity of degree 1.
- The same proof gives

  $\mathbb{P}(P$ has a double root$) =$

  $\mathbb{P}(P$ has a double root at some root of unity of degree $\leq d) + o(n^{-2d})$.

Let ξ_i be i.i.d. integer-valued random variables.

Theorem (Feldheim-Peled-S-Zeitouni' 15+)

If ξ_0 has bounded support and if

$$\max_{x \in \mathbb{Z}} \mathbb{P}(\xi_0 = x) \leq \tfrac{1}{2},$$

then

$$\mathbb{P}(P \text{ has a double root}) = \mathbb{P}(P \text{ has a double root at } 0, \pm 1) + o(n^{-2}),$$

and if $\mathbb{P}(\xi_0 = 0) = 0$, then

$$\mathbb{P}(P \text{ has a double root}) = O(n^{-2}).$$

Related results : I

- Kozma-Zeitouni '13 : Let P and Q be two independent random Littlewood polynomials of degree n. Then the probability that P and Q have a common root is

$$\underbrace{\mathbb{P}(P \text{ and } Q \text{ have a common root at } \pm 1)}_{O(n^{-1})} + o(n^{-1}).$$

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- Main step:

$$\begin{aligned}
&\sum_{\alpha \in \mathbb{C}, \deg(\alpha) \geq D} \mathbb{P}(P(\alpha) = Q(\alpha) = 0) \\
&= \sum_{\alpha \in \mathbb{C}, \deg(\alpha) \geq D} \mathbb{P}(P(\alpha) = 0)\mathbb{P}(Q(\alpha) = 0) \\
&\leq \sup_{\alpha \in \mathbb{C}, \deg(\alpha) \geq D} \mathbb{P}(P(\alpha) = 0) \sum_{\alpha} \mathbb{P}(Q(\alpha) = 0) \\
&= n \sup_{\alpha \in \mathbb{C}, \deg(\alpha) \geq D} \mathbb{P}(P(\alpha) = 0),
\end{aligned}$$

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&= n \sup_{\alpha \in \mathbb{C}, \deg(\alpha) \geq D} \mathbb{P}(P(\alpha) = 0), \\
&\leq n \cdot O(n^{-\frac{D}{2}}),
\end{aligned}$$

by inverse Littlewood-Offord type bounds.

Related results : II

- Do-Nguyen-Vu' 14: Let ξ_i's be i.i.d of following two types.

discrete ξ_i uniform on $\{\pm 1, \pm 2, \ldots, \pm k\}$.

continuous ξ_i has p-integrable density with $p > 1$ and $\mathbb{E}|\xi_i|^{2+\epsilon} < \infty$.

Theorem

Given any $C > 0$, there exists $B > 0$ such that

$\mathbb{P}(P$ *has a pair of real roots* $\leq n^{-B}$ *apart*$) = \mathbb{P}(P$ *has a double root at* $\pm 1) + n^{-C}$.

- Their method does not readily extend to complex roots.

Sketch of the proof

- Suppose $\xi_i \in \{-1, 1\}$ i.i.d. such that $\max_{x \in \{-1,1\}} \mathbb{P}(\xi_0 = x) < \frac{1}{\sqrt{2}}$. Then

$$\mathbb{P}(P \text{ has a double root}) = \mathbb{P}(P \text{ has a double root at } \pm 1) + o(n^{-2}).$$

- We need to show

$$\mathbb{P}(P \text{ has a double root at some } \alpha \text{ with } \deg(\alpha) \geq 2) = o(n^{-2}).$$

Lemma

Given $B > 0$ *there exists* $C > 0$ *such that*

$$\mathbb{P}\big(\exists \alpha \text{ with } \deg(\alpha) \geq C \log n \text{ such that } P \text{ has a double root at } \alpha\big) = O(n^{-B}).$$

Lemma

Given $B > 0$ there exists $C > 0$ such that

$$\mathbb{P}\big(\exists \alpha \text{ with } \deg(\alpha) \geq C \log n \text{ such that } P \text{ has a double root at } \alpha\big) = O(n^{-B}).$$

- **Proof of Lemma.** Suppose P has a double root at α with $\deg(\alpha) = d$.

  Let P_1 = minimal poly. of α in $\mathbb{Z}[x]$. Then

  $$P(z) = P_1(z)^2 P_2(z) \cdots P_m(z)$$

  where $P_i \in \mathbb{Z}[x]$ are irreducible and $P_1(\alpha) = 0$ with $\deg(P_1) = d \geq C \log n$.

- Thus $P(\pm 2)$ is divisible by the integer $P_1(\pm 2)^2$.

Claim: $\max\big(P_1(-2), P_1(2)\big)$ *has to be large if* d *is large*

- $|P_1(2)| = \prod_{i=1}^{d} |\alpha_i - 2|$ and $|P_1(-2)| = \prod_{i=1}^{d} |\alpha_i + 2|$, where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are conjugates of α.
- Fact: The number of roots of P outside the ball $\{z : |z| \le 3/2\}$ is $O(1)$, deterministically. [Jensen's formula]
- Hence $\#\{i : |\alpha_i| > 3/2\} = O(1)$.
-

$$
\begin{aligned}
|P_1(-2)P_1(2)| &= \prod_{i=1}^{d} |\alpha_i + 2||\alpha_i - 2| \\
&= \prod_{i:|\alpha_i|\le 3/2} |\alpha_i^2 - 4| \prod_{i:|\alpha_i|>3/2} |\alpha_i + 2||\alpha_i - 2| \\
&= \left(\tfrac{7}{4}\right)^{d-O(1)} \prod_{i:|\alpha_i|>3/2} |\alpha_i + 2||\alpha_i - 2|
\end{aligned}
$$

Claim contd.

- Let $\mathcal{E} := \{P \text{ has a root inside } B(\pm 2, n^{-K})\}$.
- Fact: For $K > 0$ big enough, $\mathbb{P}(\mathcal{E}) = O(n^{-B})$.
- On $\mathcal{E}^c$,

$$|P_1(-2)P_1(2)| = \left(\frac{7}{4}\right)^{d-O(1)} \prod_{i:|\alpha_i|>3/2} |\alpha_i + 2||\alpha_i - 2|$$
$$= \left(\frac{7}{4}\right)^{d-O(1)} n^{-2K\cdot O(1)} \geq e^{\Omega(d)} n^{-O(1)}.$$

- Thus, on $\mathcal{E}^c$, when $d \geq C \log n$,

$$\max\left(P_1(-2), P_1(2)\right) \geq n^{C_1}.$$

Proof of the lemma

- $$\mathbb{P}(P \text{ has a double root } \alpha \text{ with } \deg(\alpha) \geq C \log n)$$
$$\leq \sum_{k \geq n^{C_1}} \mathbb{P}(P(\pm 2) \text{ is divisible by } k^2) + \mathbb{P}(\mathcal{E})$$

Proof of the lemma

- $$\mathbb{P}(P \text{ has a double root } \alpha \text{ with } \deg(\alpha) \geq C \log n) \leq \sum_{k \geq n^{C_1}} \mathbb{P}(P(\pm 2) \text{ is divisible by } k^2) + \mathbb{P}(\mathcal{E})$$

- Choose r satisfying $2^r \leq k^2 < 2^{r+1}$. Note that $|\sum_{j=0}^{r-1} \xi_j 2^j| < 2^r/2$.

$$\mathbb{P}\left(P(2) \bmod k^2 = 0\right) \leq \max_{m \in \mathbb{Z}} \mathbb{P}\left(\sum_{j=0}^{r-1} \xi_j 2^j \bmod k^2 = m\right) = \max_{m \in \mathbb{Z}} \mathbb{P}\left(\sum_{j=0}^{r-1} \xi_j 2^j = m\right).$$

- The mapping $(a_0, \ldots, a_{r-1}) \mapsto \sum_{j=0}^{r-1} a_j 2^j$ is one-to-one on $\{-1, 1\}^r$.

- $$\max_{m \in \mathbb{Z}} \mathbb{P}\left(\sum_{j=0}^{r-1} \xi_j 2^j = m\right) \leq \left(\max_{x \in \{-1,1\}} \mathbb{P}(\xi_0 = x)\right)^r := \left(\frac{1}{\sqrt{2}} - \delta\right)^r.$$

$$\leq \left(\frac{1}{\sqrt{2}} - \delta\right)^{\frac{2 \log k}{\log 2}} = k^{-\gamma}, \quad \text{for some } \gamma > 1.$$

P cannot have low degree roots far away from the unit circle

Lemma

For any algebraic integer $\alpha \neq 0$,

$$\mathbb{P}(\alpha \text{ is a root of } P) \leq e^{-c\left|1-|\alpha|\right|n}.$$

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- **Fact.**

$$\#\Big\{\text{algebraic integers with algebraic degree} = O(\log n)$$
$$\text{such that all of their conjugates are inside } \{\tfrac{1}{2} < |z| < 2\}\Big\} \leq e^{C(\log n)^2}.$$

- A simple union bound implies that

$$\mathbb{P}\Big(P(\alpha) = 0 \text{ for some } \alpha \text{ with } \deg(\alpha) = O(\log n),$$
$$\text{and } \underbrace{|1-|\alpha||}_{\text{or any conjugate of } \alpha} = \Omega((\log n)^3/n)\Big) = O(n^{-3}).$$

- Let $\alpha = \exp(\frac{2\pi i}{k})$ and let $d := \deg(\alpha)$.

Lemma

$$\mathbb{P}(\alpha \text{ is a double root of } P) \leq \left(\frac{cn}{k}\right)^{-2d}.$$

- Example: When $\alpha = \sqrt{-1}$, then $k = 4$ and $d = 2$. So,

$$\mathbb{P}(P \text{ has a double root at } \sqrt{-1}) = O(n^{-4}).$$

- If $d \geq 2$ and $k \ll n$, then

$$\mathbb{P}(\alpha \text{ is a double root of } P) = O(n^{-4(1+o(1))}).$$

Counting low degree roots of unity

- Let α be kth primitive root of unity with $\deg(\alpha) = d$.
- The minimal polynomial of α is given by the kth cyclotomic polynomial

$$\Phi_k(x) := \prod_{\substack{1 \le j \le k, \\ \gcd(j,k)=1}} (1 - e^{2\pi i j/k}) \in \mathbb{Z}[x]$$

- The number of cyclotomic polynomials of degree $\le d$ is $O(d \log\log d)$.

Low degree roots of unity can not be a double root of P

The probability that some roots of unity of degree between 2 and $O(\log n)$ is a double root of P is at most

$$O(\log n \log\log\log n) \times O(n^{-4(1+o(1))}) = o(n^{-2}).$$

α = algebraic integer.

- High algebraic degree, i.e., $\deg(\alpha) \geq C \log n$. Done.
- Low algebraic degree, i.e., $\deg(\alpha) \leq C \log n$.

  Case I. Roots far off unit circle. α has a conjugate root β such that $|1 - |\beta|| = \Omega\left(\frac{(\log n)^3}{n}\right)$. Done.

  Case II. Roots of unity (of $\deg \geq 2$). Done.

  Case III. Intermediate case. α is NOT a root of unity and ALL conjugates of α are within distance $O\left(\frac{(\log n)^3}{n}\right)$ from the unit circle. Needs to be done.

Intermediate case does not arise!

- If α is an algebraic integer having minimal polynomial

$$g(z) = \prod_{\beta \in C(\alpha)} (z - \beta) \qquad [\text{monic}]$$

- The Mahler measure of α :

$$M(\alpha) := \prod_{\substack{\beta \in C(\alpha) \\ |\beta| \geq 1}} |\beta| \quad \geq 1.$$

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- $M(\alpha) = 1 \Leftrightarrow \quad \alpha = 0 \quad$ OR $\quad \alpha$ is a root of unity.

- Lehmer's conjecture (1933): there exists an absolute constant $\mu > 1$ such that

$$\text{either} \quad M(\alpha) = 1 \quad \text{or} \quad M(\alpha) \geq \mu.$$

A partial result on Lehmer's conjecture

- Dobrowolski' 79:

$$M(\alpha) = 1 \text{ or } M(\alpha) \geq \exp\left(c\left(\frac{\log\log d}{\log d}\right)^3\right) \forall \text{ alg. integer } \alpha \text{ with } \deg(\alpha) = d.$$

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- If $\alpha \neq 0$ or α is not a root of unity, then

$$\exists \beta \in C(\alpha) \text{ such that } |\beta| \geq 1 + \frac{c}{d}\left(\frac{\log\log d}{\log d}\right)^3.$$

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  $$\exists \beta \in C(\alpha) \text{ such that } |\beta| \geq 1 + \frac{c}{d}\left(\frac{\log\log d}{\log d}\right)^3.$$

- If $\alpha \neq 0$ or α is not a root of unity and $\deg(\alpha) \leq C \log n$, then there exists a conjugate β of α which is at least

  $$c(\log n)^{-1}(\log\log n)^{-3} \gg \frac{(\log n)^3}{n}$$

  away from the unit circle.

- Hence intermediate case does not arise.

Difficulties in the general case

- A key ingredient in the earlier proof of lemma concerning high algebraic degree:

  When $\xi_i \in \{-1, +1\}$ with $\max_{x \in \{-1,+1\}} \mathbb{P}(\xi_i = x) = \frac{1}{\sqrt{2}} - \delta$, then

$$\max_{m \in \mathbb{Z}} \mathbb{P}\left(P(\pm 2) = m\right) \leq \left(\max_{x \in \{-1,+1\}} \mathbb{P}(\xi_0 = x) \right)^{n+1} \leq \left(\frac{1}{\sqrt{2}} \right)^{n(1+\epsilon)}.$$

- Let ξ_i be i.i.d. $\mathbb{Z}$-valued with $\max_{x\in\mathbb{Z}} \mathbb{P}(\xi_0 = x) \le \frac{1}{2}$, then

$$\max_{m\in\mathbb{Z}} \mathbb{P}\left(P(\pm 2) = m\right) \le \left(\frac{1}{\sqrt{2}}\right)^{n(1+\epsilon)}.$$

- Example: If $\mathbb{P}(\xi_0 = 0) = \frac{1}{2}, \mathbb{P}(\xi_0 = -1) = \frac{1}{4}$ and $\mathbb{P}(\xi_0 = 2) = \frac{1}{4}$, then one can check

$$\mathbb{P}(P(2) = 0) \ge 0.6^n$$

- The proof of the result is rather long, but it starts with ...

ξ_0 is integer-valued $+ \max_x \mathbb{P}(\xi_0 = x) \le \frac{1}{2} \Leftrightarrow \xi_0$ is a mixture of Bernoulli

$$\text{mixture of Bernoulli} \ : \quad \sum_i t_i \left(\tfrac{1}{2}\delta_{a_i} + \tfrac{1}{2}\delta_{b_i}\right),$$

where $t_i \ge 0, \quad \sum_i t_i = 1, \quad a_i \ne b_i \in \mathbb{Z}$.

Difficulties in the general case: II

- The roots of P are now algebraic numbers instead of algebraic integers. Let $a_d z^d + \ldots + a_1 z + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of α.

- The Mahler measure of α :

$$M(\alpha) := |a_d| \prod_{\substack{\beta \in C(\alpha) \\ |\beta| \geq 1}} |\beta| \quad \geq 1.$$

- Lehmer's conjecture trivially holds for the non-monic case $|a_d| > 1$

A replacement for Dobrowolski's result

- Using a result of Dubickas'99, we can show that

Theorem

Let $\epsilon > 0$ and a_0 be a non-zero integer. Then, for all d sufficiently large, the number of polynomials $f \in \mathbb{Z}[x]$ of degree d with leading coefficient a_0 such that

$$\text{max modulus of the roots of } f < 1 + \frac{\epsilon \log d}{|a_0| d}$$

is less than $\exp((|a_0|d)^{2/3+\epsilon})$.

- The number of all possible polynomial factors in the 'intermediate' case $o(n^{\epsilon})$.

An open question

- **Conjecture.** The probability that a random Littlewood polynomial of degree n is reducible goes to zero as $n \to \infty$.

- Konyagin (1999): If the coefficient distribution is uniform on $\{0, 1\}$, then whp any irreducible factor of the random polynomial P has degree at least $cn/\log n$.

Thank you!