Activated Random Walks with bias: activity at low density

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Conference "Disordered models of mathematical physics"
Valparaíso, Chile — July 23, 2015

Activated Random Walks (ARW) — Quick presentation

Dynamics: Particles evolve in continuous time on $\mathbb{Z}^d$, and can be either

- active, in **state A**: move as (independent) random walks, at rate 1;
- passive (sleeping), in **state S**: do not move.

Two kinds of mutations/interactions happen:

- $A \to S$ at rate λ: each particle gets asleep at rate λ (independently);
- $A + S \to 2A$ immediately: active particles awake the others on same site.

NB. Mutations $A \to S$ are only effective when the particle A is alone
$\Rightarrow$ On each site, there is either nothing, one S, or any number of A particles.

Parameters:

- jump distribution $p(\cdot)$ on $\mathbb{Z}^d$
- sleeping rate $\lambda \in (0, \infty)$
- initial configuration of A particles (finite support, or i.i.d. in general).

Behaviors of interest:

- **fixation**: in any finite box, activity vanishes eventually;
- **non-fixation**: in any finite box, activity goes on forever.

Motivations: 1. Phase transition

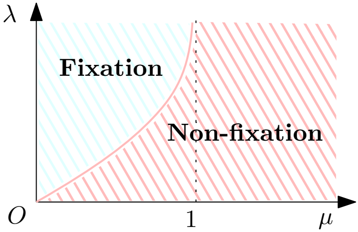
Let μ denote the initial density of particles (for i.i.d. initial configuration).

A **phase transition** is *expected* to happen: $\exists \mu_c(\lambda) \in (0,1)$ s.t.

- for $\mu < \mu_c(\lambda)$, a.s. fixation;
- for $\mu > \mu_c(\lambda)$, a.s. non-fixation.

Or also: for $\mu \geq 1$ then a.s. fixation and, for $\mu < 1$, $\exists \lambda_c(\mu) \in (0,\infty)$ s.t.

- for $\lambda > \lambda_c(\mu)$, a.s. fixation;
- for $\lambda < \lambda_c(\mu)$, a.s. non-fixation.



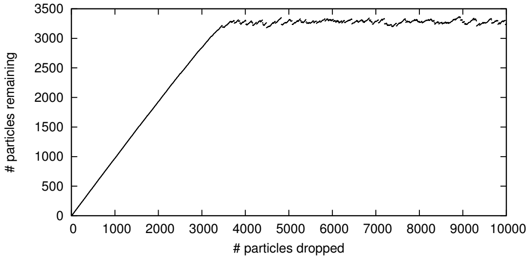
Existence of μ_c and λ_c follow by monotonicity in Diaconis-Fulton coupling. However, nontrivial bounds are the difficult part $\rightarrow$ cf. Leo Rolla's talk for an overview of known results.

Motivations: 2. Self-Organized Criticality (physics)

Ingredient of a toy example of **self-organized criticality**, i.e. a model that displays “critical behavior” (polynomial decay of correlations,...) spontaneously, without having to tune some parameter:
In a finite box,

- Drop a new particle at random,
- Stabilize the configuration by running the dynamics inside the box and by freezing particles that exit,

and repeat.



$\hookrightarrow$ Dynamics reach a stationary regime which, extended to infinite volume, should satisfy SOC.

Case of a biased random walk – Main result

Assume the jump distribution $p(\cdot)$ has a **bias**: for the simple random walk X with jump distribution $p(\cdot)$, for some direction ℓ, $X_n \cdot \ell \to +\infty$, a.s..

For $\lambda > 0$, $\boldsymbol{v} \in \mathbb{R}^d \setminus \{0\}$, if $T_{\boldsymbol{v}}$ is the time spent by X in $\{x \in \mathbb{Z}^d : x \cdot \boldsymbol{v} \leq 0\}$,

let
$$F_{\boldsymbol{v}}(\lambda) = E\left[\frac{1}{(1+\lambda)^{T_{\boldsymbol{v}}}}\right]$$
$$= P(\text{a walk killed at rate } \lambda \text{ in } \{x \cdot \boldsymbol{v} \leq 0\} \text{ survives forever})$$

NB. If $\boldsymbol{v} \cdot \ell > 0$, then $0 < F_{\boldsymbol{v}}(\lambda) \longrightarrow 1$ as $\lambda \to 0^+$.

Theorem (Taggi, 2014)

- *Assume* $d = 1$. $\mu > 1 - F_1(\lambda) \Rightarrow$ *non-fixation a.s.*
- *Assume* $d \geq 2$. $\mu F_{\boldsymbol{v}}(\lambda) > \mathbb{P}(\eta_0(0) = 0) \Rightarrow$ *non-fixation a.s.*

Theorem (Rolla-T., 2015)

- *Assume* $d \geq 2$. $\mu > 1 - F_{\boldsymbol{v}}(\lambda) \Rightarrow$ *non-fixation a.s.*

$\hookrightarrow$ for all d, for all $\mu < 1$, non-fixation happens for small λ.

Plan

Our proof is based on a mixed use of **particle-wise** and **site-wise** viewpoints.

1. Definition: particle-wise vs. site-wise
2. Definition: particle fixation vs. site fixation
3. A non-fixation condition *(particle-wise + site-wise argument)*
4. Proof of the result *(site-wise argument)*
5. Existence of the particle-wise construction.

Site-wise vs. particle-wise

The **site-wise viewpoint** attaches randomness to *sites*: from finite initial configuration, ("Diaconis-Fulton" construction)

- each **site** contains a random stack of i.i.d. *instructions* ("jump to y", or "sleep"), and a Poisson clock;
- when clock rings at a site, apply the top instruction to a particle there;
- clock runs at speed proportional to number of particles present at the site (as if each particle reads an instruction at rate 1).

$\hookrightarrow$ we don't distinguish particles at a site, and get $\eta_t(x) \in \{0, S, 1, 2, \dots\}$.
Crucial properties: abelianness and monotonicity.

The **particle-wise viewpoint** attaches randomness to *particles*:

- each **particle** (x,i) (i-th particle starting at x) has a "life plan" $(X_t^{x,i})_{t\geq 0}$ (that is a continuous-time RW), and a Poisson clock with rate λ;
- particles move according to their life plan,
- when the clock of a particle rings, if it is alone then its gets asleep, and in this case its clock stops;
- when a particle is awoken, its clock resumes ticking.

$\hookrightarrow$ we get a whole family of paths $(Y_t^{x,i})_{t\geq 0}$, which carries more information.
Properties: Not the above, but a control on the effect of adding one particle.

Site fixation vs. particle fixation

Definition

Site fixation occurs when, at each site, there is eventually no active particle.
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Example of use. Assume particles fixate a.s., then

$$\begin{aligned}
\mu &= \mathbb{E}[\#\text{ particles initially at } 0] \\
&= \mathbb{E}[\#\text{ sites where a particle initially at 0 settles}] \\
&= \sum_v \mathbb{P}(\text{some particle initially at 0 settles at } v) \\
&= \sum_v \mathbb{P}(\text{some particle initially at } -v \text{ settles at } 0) \\
&= \mathbb{E}[\#\text{ particles settling at } 0] \le 1.
\end{aligned}$$

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Theorem (Amir–Gurel-Gurevich, 2012)

Site fixation implies particle fixation. Thus, they are equivalent. And $\mu_c \geq 1$.

(for i.i.d. initial conditions, 0-1 laws hold for site and particle fixation)

A non-fixation condition

• Direct technique for proving non-fixation: finding a strategy that makes arbitrarily many particles *visit precisely* the site o.
• In fact, making a positive density of particles *exit a box* is sufficient.

Consider an ARW with i.i.d. initial configuration.
For $n \in \mathbb{N}$, let $V_n = \{-n, \ldots, n\}^d$, denote $\mathbb{P}_{[V_n]}$ the law of the ARW restricted to V_n (i.e. particles freeze outside), M_n the number of particles exiting V_n.

Proposition

$\limsup_n \frac{\mathbb{E}_{[V_n]}[M_n]}{|V_n|} > 0 \quad \Rightarrow \quad$ *(particle) non-fixation, a.s.*

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Let $\widetilde{V}_n = V_{n-\log n}$. Then, if $\eta_0(x) \le K$ a.s. (to simplify)

$$\begin{aligned}\mathbb{E}[M_n] &\le \mu |V_n \setminus \widetilde{V}_n| + \sum_{x \in \widetilde{V}_n, i \in \mathbb{N}} \mathbb{P}(i \le \eta_0(x),\ \text{particle } Y^{x,i} \text{ exits } V_n) \\ &\le o(|V_n|) + |\widetilde{V}_n| K\, \mathbb{P}(\text{particle } Y^{0,1} \text{ reaches distance } \log n)\end{aligned}$$

$$\mathbb{P}(Y^{0,1} \text{ does not fixate}) = \lim_n \mathbb{P}(Y^{0,1} \text{ reaches dist. } \log n) \ge \limsup_n \frac{\mathbb{E}[M_n]}{|V_n|}$$

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$\rightsquigarrow$ Why do we have $\mathbb{E}[M_n] \ge \mathbb{E}_{[V_n]}[M_n]$?

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Let us color in Blue the particles that start in V_n, in Red the particles that start outside V_n, and modify the site-wise construction as follows:

- Blue and Red use *independent* stacks of instructions
  (but active particles awaken any sleepy particle, hence an interaction)
- When a Blue particle exits V_n, it becomes Red.

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Then monotonicity extends, in two steps:

• With only Blue particles at the beginning, *not* freezing Red particles outside of V_n anymore increases the number of used Blue instructions, and thus M_n: (denoting the law of this process by $\mathbb{P}_{V_n}$)

$$\mathbb{E}_{[V_n]}[M_n] \leq \mathbb{E}_{V_n}[M_n]$$

• Adding Red particles at the beginning still increases the number of used Blue instructions, and thus M_n:

$$\mathbb{E}_{V_n}[M_n] \leq \mathbb{E}[M_n]$$

Non-fixation for biased ARW on $\mathbb{Z}^d$

Let $\boldsymbol{v} \in \mathbb{R}^d$ and assume $\mu > 1 - F_{\boldsymbol{v}}(\lambda)$.

Consider ARW restricted to V_n (particles freeze outside), with site-wise construction. Let us devise a **toppling strategy** that throws a positive density of particles outside of V_n.

Preliminary step: levelling

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Label $V_n = \{x_1, \dots, x_r\}$ so that $x_1 \cdot \boldsymbol{v} \leq \cdots \leq x_r \cdot \boldsymbol{v}$.

Main step

For $i = 1, \dots, r$, if there is a particle in x_i, then topple it, and topple it again, and so on until either it exits V_n, falls asleep on $x_i + \{x : x \cdot \boldsymbol{v} \leq 0\}$ or reaches an empty site in $\{x_{i+1}, \dots, x_r\}$.

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The probability of the middle case is lower than $F_{\boldsymbol{v}}(\lambda)$, and otherwise the number of particles outside V_n or in $\{x_{i+1}, \dots, x_r\}$ increases by 1. Hence,

$$\mathbb{E}_{[V_n]}[M_n] \geq \mu |V_n| - (1 - F_{\boldsymbol{v}}(\lambda))|V_n|.$$

Construction of the infinite-volume particle-wise process

How to prove **existence** of the ARW with infinite initial condition?
$\rightarrow$ for the usual process $(\eta_t(\cdot))_{t\geq 0}$ on $\{0, S, 1, \dots\}^{\mathbb{Z}^d}$, the standard theory from particle systems adapt (cf. Liggett, and Andjel on Zero-Range-Process)
$\rightarrow$ for the fully-labeled system of walks, no standard reference. Also, we need to prove the existence of the previous particle-wise construction specifically. Let us sketch a probabilistic proof of existence.

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What do we need to prove? Consider the following:

- η_0 a finite initial configuration,
- $X = (X^{(x,i)}\,;\, x\in\mathbb{Z}^d, i\in\mathbb{N})$ a family of (putative) paths,
- $\mathscr{P} = (\mathscr{P}^{(x,i)}\,;\, x\in\mathbb{Z}^d, i\in\mathbb{N})$ a family of PP (clocks).

Let $\overline{\eta}_t(z;\eta_0,X,\mathscr{P})$ = set of the labels "(x,i)" of the particles at z at time t.
Choose a sequence of finite subset $W_n\uparrow\mathbb{Z}^d$.

For an infinite η_0, the particle-wise construction of the ARW from $(\eta_0,X,\mathscr{P})$ is **well-defined** if, for all $z\in\mathbb{Z}^d$, $T>0$, $w\in\mathbb{Z}^d$, the sequence

$$\overline{\eta}_{|_{[0,T]}}(z;\eta_0\cdot\mathbf{1}_{w+W_n},X,\mathscr{P}),\qquad n\in\mathbb{N},$$

is eventually constant, and its limit does not depend on $w\in\mathbb{Z}^d$.

Influence of a particle

Given η_0, X, $\mathscr{P}$, the particle (x,i) has an influence on $z \in \mathbb{Z}^d$ during $[0,t]$ if removing this particle changes the fully-labeled process $\overline{\eta}|_{[0,t]\times\{z\}}(\eta_0, X, \mathscr{P})$.

To prove well-definedness at z, we have to ensure that, for a *finite* number of n's, some site in $W_{n+1} \setminus W_n$ has an influence on z. The key is the following.

Lemma

Let $Z_t^{x,i}(\eta_0, X, \mathscr{P})$ be the set of sites influenced by (x,i) before t.
There is a branching r.w. $\widetilde{Z}$ on $\mathbb{Z}^d$ such that, for any given finite config. π,

$$Z_t^{x,i}(\pi, X, \mathscr{P}) \subset_{\text{st.}} x + \widetilde{Z}_t,$$

and $E[|\widetilde{Z}_t|] \le e^{ct}$.

Theorem

Assume $\sup_x \mathbb{E}[\eta_0(x)] < \infty$. Then the particle-wise ARW is a.s. well-defined.

Conclusion

Extensions of parts of the proof, of possible independent interest:

- The non-fixation condition naturally extends to amenable graphs (assuming $|\partial V_n| = o(|V_n|)$, positive density of exits $\Rightarrow$ non-fixation).
- The particle-wise construction extends to transitive graphs with a unimodular subgroup of automorphisms that preserves the jump distribution (needs **mass transport principle**).

Most striking open questions:

- in the symmetric case, non-fixation for some $\mu < 1$? (even when $d = 1$)
- in the biased case, fixation for some $\mu > 0$? *(for symmetric case, see Sidoravicius-Teixeira 2014)*