Statistical properties of expanding circle maps with an indifferent fixed point

Irene Inoquio
Joint work with Eduardo Garibaldi

Universidad de La Serena, Chile
Dyadisc 7: Brazilian, Chilean and French Interplay for Symbolic Dynamics

December 9, 2024

Expanding circle maps with an indifferent fixed point

The dynamics is described by a continuous map $T : \mathbb{T} \to \mathbb{T}$ of the form

$$T(x) := x(1 + V(x)) \mod 1,$$

where:

- The phase space consists of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ endowed with the standard metric
$$d(x, y) = \min\{|x - y|, |x - y \pm 1|\}.$$
- $V : [0, +\infty) \to [0, +\infty)$ is continuous and increasing, $V(1) \in \mathbb{N}$.
For $\sigma \geq 0$,
$$\lim_{x \to 0} \frac{V(tx)}{V(x)} = t^{\sigma}, \text{ for all } t > 0.$$

⋆ When $\sigma > 0$, V *regularly varying with index* σ
⋆ When $\sigma = 0$, V is called *slowly varying*
(*E.* **Seneta (1976): Regularly varying functions**)

Prototype map: Manneville Pomeau map

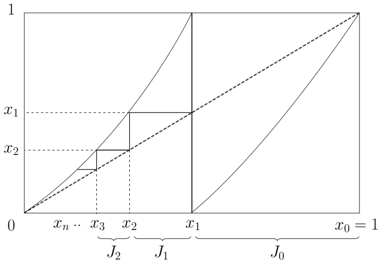

Figure: $T_s(x) = x(1 + x^s) \mod 1$

For a fixed $s \in (0, 1)$.
$T_s : \mathbb{R}/\mathbb{Z} = [0, 1) \to [0, 1)$,

$$T_s(x) = x(1 + x^s) \mod 1$$

Is non-uniformly expanding:
$T_s(0) = 0$, $DT_s(0) = 1$,
$DT_s(x) > 1$ for all $x \in \mathbb{R}/\mathbb{Z} \setminus \{0\}$.

Near to origin the dynamics of $T_s(x) = x(1 + x^s) \mod 1$ for $s \in (0, 1)$

Let $(x_n)_{n=0}^{+\infty} \subset \mathbb{T}$ be a sequence of points by defining $T(x_{n+1}) = x_n$, $n \geq 0$.

- ⋆

$$x_n \sim \frac{1}{n^{1/s}} \qquad |J_n| \sim \frac{1}{n^{1+1/s}}.$$

When the graph of T is less tangent to the diagonal at the indifferent fixed point

- For x small enough,

$$T_k(x) = x\big(1 + A_k(\log^k 1/x)^{-1}\big) \mod 1,$$

where $A_k > 0$ is a constant and $\log^k$ stands for the k-times composition of log.

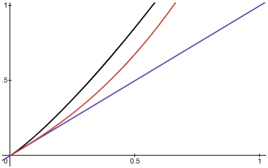

Figure: $T_k(x)$

(**Kloeckner: ETDS, 2020**): For $s \in (0, 1)$

$$T_s(x) = \begin{cases} 0, & x = 0 \\ x(1 + (1 - \log 2x)^{-s}), & x \in (0, 1/2), \\ 2x - 1, & x \in [1/2, 1]. \end{cases}$$

A quantitative version of the non-uniformly expanding property

$T(x) = x(1 + V(x)) \mod 1$ is expanding outside any subset of the form $[0, \epsilon)$, $0 < \epsilon < 1$. It follows that for all $x, y \in [\epsilon, 1)$ with $d(x, y) < \varrho_V$,

$$d(T(x), T(y)) \geq \lambda(\epsilon)\, d(x, y),$$

where $\lambda(\epsilon) := 1 + V(\epsilon) \to 1$ as $\epsilon \to 0$.

A quantitative version of the non-uniformly expanding property on the whole circle is provided by the following lemma.

Lemma

There exists a constant $\varrho_0 > 0$ such that for $x, y \in \mathbb{T}$ with $d(x, y) < \varrho_0$,

$$d(T(x), T(y)) \geq d(x, y) \left(1 + \frac{1}{2^{\sigma+2}} V(d(x, y))\right).$$

Garibaldi, I. 2020; I. Morris 2009.

Potentials defined on $\mathbb{T}$

We consider potentials $f : \mathbb{T} \to \mathbb{R}$ with a particular modulus of continuity ω: namely, potentials f such that

$$|f|_\omega := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(d(x, y))} < \infty.$$

where: $\omega : [0, +\infty) \to [0, +\infty)$ is continuous, non-decreasing, with $\omega(0) = 0$.

Potentials defined on $\mathbb{T}$

We consider potentials $f : \mathbb{T} \to \mathbb{R}$ with a particular modulus of continuity ω: namely, potentials f such that

$$|f|_\omega := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(d(x,y))} < \infty.$$

where: $\omega : [0, +\infty) \to [0, +\infty)$ is continuous, non-decreasing, with $\omega(0) = 0$.

Regularity beyond the usual Hölder modulus environment

1. $\omega(x) = x^\alpha$ (Hölder modulus of continuity)
2. $\omega(x) = (-\log x)^{-\beta}, \beta \geq 0$ (A class larger than Hölder continuous functions.)

Potentials defined on $\mathbb{T}$

We consider potentials $f : \mathbb{T} \to \mathbb{R}$ with a particular modulus of continuity ω: namely, potentials f such that

$$|f|_\omega := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(d(x, y))} < \infty.$$

where: $\omega : [0, +\infty) \to [0, +\infty)$ is continuous, non-decreasing, with $\omega(0) = 0$.

Regularity beyond the usual Hölder modulus environment

1. $\omega(x) = x^\alpha$ (Hölder modulus of continuity)

2. $\omega(x) = (-\log x)^{-\beta}, \beta \geq 0$ (A class larger than Hölder continuous functions.)

3. for $0 \leq \alpha < 1$ and $\beta \geq 0$ with $\alpha + \beta > 0$,

   $$\omega_{\alpha,\beta}(x) := \begin{cases} x^\alpha(-\log x)^{-\beta}, & 0 < x < x_0, \\ x_0^\alpha(-\log x_0)^{-\beta}, & x \geq x_0, \end{cases}$$

   where $x_0 = x_0(\alpha, \beta)$ is taken small enough so that $\omega_{\alpha,\beta}$ is concave.

4. $\omega(x) = (\log^k 1/x)^{-1}(\log 1/x)^{-1}(\log^2 1/x)^{-1}, \quad k \in \mathbb{Z}^+$ is defined in a small neighborhood of the origin so that is concave.

The transfer operator and Equilibrium states

Let $f \in \mathscr{C}_\omega(\mathbb{T})$ (real continuous map with modulus of continuity ω).
We define the *transfer operator* associated with f as

$$\mathscr{L}_f \phi(x) := \sum_{y \in T^{-1}(x)} e^{f(y)} \phi(y), \qquad \forall\, \phi \in C^0(\mathbb{T}).$$

We have that $\mathscr{L}_f$ is a bounded linear operator.

- For every $n \geq 1$ and $x \in \mathbb{T}$,

$$\mathscr{L}_f^n \phi(x) = \sum_{y \in T^{-n}(x)} e^{S_n f(y)} \phi(y),$$

  where

$$S_n f(x) := f(x) + f(T(x)) + \cdots f(T^{n-1}(x)).$$

- Let $\mathscr{L}_f^*$ denote the dual operator of $\mathscr{L}_f$, acting on the dual space of $C^0(\mathbb{T})$, as

$$\int \phi \, d(\mathscr{L}_f^* m) = \int \mathscr{L}_f \phi \, dm, \quad \forall\, \phi \in C^0(\mathbb{T}), \forall\, m \in \mathrm{Prob}(\mathbb{T}).$$

As is well known, if we can find

- a positive eigenfunction *h* for $\mathscr{L}_f$,
- and an eigenmeasure $d\nu$ for its dual $\mathscr{L}_f^*$
  (both corresponding to the same positive maximal eigenvalue χ), considering normalization,

  We expected that the probability :

$$d\mu = h\,d\nu$$

  be an **equilibrium state** *of the system.*

As is well known, if we can find

- a positive eigenfunction h for $\mathscr{L}_f$,
- and an eigenmeasure $d\nu$ for its dual $\mathscr{L}_f^*$
  (both corresponding to the same positive maximal eigenvalue χ), considering normalization,

  We expected that the probability :

$$d\mu = h\, d\nu$$

be an **equilibrium state** *of the system. That is,*

$$P(T, f) := \max_{m \in M(\mathbb{T}, T)} \left[h_m(T) + \int f\, dm \right] = h_\mu(T) + \int f\, d\mu.$$

We work with pairs of moduli of continuity (ω, Ω),

1. ω for the regularity of the potential,
2. Ω for the regularity of a such possible density.

Without inducing, a direct Ruelle-Peron-Frobenius theorem for a non-uniformly hyperbolic system was obtained in [**GI2022 Lett. Math. Phy.**].

Key property. T-compatibility of between moduli.

(ω, Ω) are T-compatible moduli of continuity

We say that Ω is T-compatible with respect to ω when there are positive constants ϱ_1 and C_1 such that, for any points x_0 and y_0 with $d(x_0, y_0) < \varrho_1$, there is a bijection among respective pre-orbits $\{x_k\}$ and $\{y_k\}$ fulfilling for all k

$$d(x_k, y_k) \leq d(x_0, y_0) < \varrho_1,$$

$$C_1 \sum_{j=1}^{k} \omega(d(x_j, y_j)) \leq \Omega(d(x_0, y_0)) - \Omega(d(x_k, y_k)).$$

(ω, Ω) are T-compatible moduli of continuity

We say that Ω is T-compatible with respect to ω when there are positive constants ϱ_1 and C_1 such that, for any points x_0 and y_0 with $d(x_0, y_0) < \varrho_1$, there is a bijection among respective pre-orbits $\{x_k\}$ and $\{y_k\}$ fulfilling for all k

$$d(x_k, y_k) \leq d(x_0, y_0) < \varrho_1,$$

$$C_1 \sum_{j=1}^{k} \omega(d(x_j, y_j)) \leq \Omega(d(x_0, y_0)) - \Omega(d(x_k, y_k)).$$

When the moduli are concave, T-compatibility may be ensured as follows.

(ω, Ω) are T-compatible moduli of continuity

We say that Ω is T-compatible with respect to ω when there are positive constants ϱ_1 and C_1 such that, for any points x_0 and y_0 with $d(x_0, y_0) < \varrho_1$, there is a bijection among respective pre-orbits $\{x_k\}$ and $\{y_k\}$ fulfilling for all k

$$d(x_k, y_k) \leq d(x_0, y_0) < \varrho_1,$$

$$C_1 \sum_{j=1}^{k} \omega(d(x_j, y_j)) \leq \Omega(d(x_0, y_0)) - \Omega(d(x_k, y_k)).$$

When the moduli are concave, T-compatibility may be ensured as follows.

Proposition (GI2022)

If $\liminf_{x \to 0} \dfrac{V(x)}{\omega(x)} (\Omega((1+c)x) - \Omega(x)) > 0$ for all $c > 0$ sufficiently small, then Ω **is T-compatible with respect to** ω.

For $T_s(x) = x(1 + x^s) \mod 1, \quad s \in (0, 1)$

- Let $\alpha \in (0, 1)$, $\beta \geq 0$ with $\alpha + \beta > 0$, consider

$$\omega_{\alpha,\beta}(x) := \begin{cases} x^{\alpha}(-\log x)^{-\beta}, & 0 < x < x_0, \\ x_0^{\alpha}(-\log x_0)^{-\beta}, & x \geq x_0, \end{cases} \tag{1}$$

where x_0 is taken small enough so that $\omega_{\alpha,\beta}$ is concave.

For $\alpha \in (s, 1)$, the modulus $\Omega(x) = \omega_{\alpha-s,\beta}(x)$ is T_s-compatible with $\omega_{\alpha,\beta}(x)$.

[1] This class in (1) was taken into account in the work of Kloeckner, *An optimal transportation approach to the decay of correlations for non-uniformly expanding maps*, ETDS, 2020.

For $T_s(x) = x(1 + x^s) \mod 1, \quad s \in (0, 1)$

- Let $\alpha \in (0, 1)$, $\beta \geq 0$ with $\alpha + \beta > 0$, consider

$$\omega_{\alpha,\beta}(x) := \begin{cases} x^\alpha(-\log x)^{-\beta}, & 0 < x < x_0, \\ x_0^\alpha(-\log x_0)^{-\beta}, & x \geq x_0, \end{cases} \tag{1}$$

where x_0 is taken small enough so that $\omega_{\alpha,\beta}$ is concave.

For $\alpha \in (s, 1)$, the modulus $\Omega(x) = \omega_{\alpha-s,\beta}(x)$ is T_s-compatible with $\omega_{\alpha,\beta}(x)$.
For $\beta = 0$ and $\alpha \in (s, 1)$, the modulus $\Omega(x) = x^{\alpha-s}$ is T_s-compatible with $\omega(x) = x^\alpha$.

1

[1] This class in (1) was taken into account in the work of Kloeckner, *An optimal transportation approach to the decay of correlations for non-uniformly expanding maps*, ETDS, 2020.

Example: Slowly varying scenario.

For $k \in \mathbb{Z}^+$, let

$$T_k(x) = x(1 + a_k(\log^k 1/x)^{-1}) \mod 1, \text{ for some } a_k > 0.$$

in a neighborhood of the origen

Consider the following moduli in a small neighborhood of the origin so that both are concave:

- $\Omega(x) = (\log^2 1/x)^{-1}$,
- $\omega_k(x) = (\log^k 1/x)^{-1}(\log 1/x)^{-1}(\log^2 1/x)^{-1}, \quad k \in \mathbb{Z}^+$.

Then $\Omega(x)$ is T_k-compatible with $\omega_k(x)$.

Ruelle-Perron-Frobenius Theorem (GI2022)

Let Ω be a T-**compatible modulus of continuity with respect to** ω. Suppose that $f \in \mathscr{C}_\omega(\mathbb{T})$.

1. There exists $\nu \in \text{Prob}(\mathbb{T})$ and a positive constant χ such that
$$\mathscr{L}_f^* \nu = \chi \nu.$$
2. The number χ is a simple eigenvalue and maximal eigenvalue of the operator $\mathscr{L}_f$ and there is a positive function $h \in \mathscr{C}_\Omega(\mathbb{T})$ such that
$$\mathscr{L}_f h = \chi h,$$
3. The measure $\mu := h\nu$ is a T-invariant probability such that
$$h_\mu(T) + \int f \, d\mu = \log \chi = P(T, f).$$
the measure μ is the unique Gibbs-equilibrium measure for f, that is: for every sufficiently small $r > 0$, there is a constant $K_r > 0$ such that, for $x \in \mathbb{T}$ and $n \geq 1$,
$$K_r^{-1} \leq \frac{\mu\big(B(x, n, r)\big)}{e^{S_n f(x) - nP(T,f)}} \leq K_r,$$
where $B(x, n, r) := \{y \in \mathbb{T} : d(T^j(x), T^j(y)) < r, \ 0 \leq j \leq n\}$.

Decay of correlations

- We provide a sufficient condition on the modulus Ω to guarantee spectral gap property, exponential decay of correlations and CLT

  (*Garibaldi, I. Nonlinearity 2024*)

What can we say about the zero-temperature formalism?

- Let $\beta > 0$ and $f \in \mathscr{C}_\omega(\mathbb{T})$. From the Ruelle-Perron-Frobenius Theorem with respect to the potential βf, we denote μ_β the unique Gibbs-equilibrium state associated with βf.

  For specific moduli of continuity ω and Ω. What can be said about the limits in the weak-star topology of $(\mu_\beta)_\beta$ as $\beta \to +\infty$? ($\beta = \frac{1}{T}$, T is the temperature)

  (*Garibaldi, I. Work in Progress 2024*)

Theorem (Exponential Decay of Correlations)

There exists $\rho \in (0,1)$ such that, given $\phi, \psi \in \mathscr{C}_\Omega(\mathbb{T})$, there is a positive constant $K = K(\phi, \psi)$ for which

$$\left| \int \phi \, \psi \circ T^n \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right| \leq K \rho^n \qquad \forall \, n \geq 1.$$

The transfer operator $\mathscr{L}_f$ acting on $\mathscr{C}_\Omega(\mathbb{T})$ has a property of **gap spectral** if:

There is $0 < r_0 < \chi = \text{spectral radius}$, such that the operator has a decomposition of the spectrum

$$spec(\mathscr{L}_f) = \{\chi\} \cup spec_0,$$

where $spec_0$ is contained in a ball $B(0, r_0)$.

As the spectral radius of $\mathscr{L}_{\tilde{f}}|_{\mathbb{C}_\Omega^\perp}$ is strictly smaller that 1. Then,

there are constants $\rho \in (0,1)$ and $K_0 > 0$ such that

$$\|\mathscr{L}_{\tilde{f}}^n \psi\|_\Omega \leq K_0 \, \|\psi\|_\Omega \, \rho^n$$

for all $\psi \in \mathbb{C}_\Omega^\perp$ and $n \geq 1$.

Therefore, for $\phi, \psi \in \mathscr{C}_\Omega(\mathbb{T})$, with $\int \psi \, d\mu = 0$, one has

$$\Big| \int \phi \; \psi \circ T^n \, d\mu \Big| \leq \|\phi\|_{L^1(\mu)} \, \|\mathscr{L}_{\tilde{f}}^n \psi\|_\infty \leq K_0 \, \|\phi\|_{L^1(\mu)} \, \|\psi\|_\Omega \, \rho^n.$$

Proposition (GI2024)
Let Ω be a T-compatible modulus of continuity with respect to ω. Suppose that Ω is **concave**. Given $n \geq 1$, $\phi \in \mathscr{C}_\Omega(\mathbb{T})$, and $x, y \in \mathbb{T}$ with $d(x,y) < \varrho_1$, for $\Gamma := \max\{2\kappa_f e^{2\kappa_f\,\Omega(1/2)}, \lceil \max h \,/\, \min h \rceil\}$ the following estimate holds

$$\left|\mathscr{L}_f^n\phi(x) - \mathscr{L}_f^n\phi(y)\right| \leq \Gamma\Big(|\phi|_\Omega\,\Omega\big(\theta(n)\,d(x,y)\big) + ||\phi||_\infty\,\Omega(d(x,y))\Big),$$

where $\theta(n) := \dfrac{1}{\chi^n}\left|\left|\mathscr{L}^n_{f-\log(1+V)}\mathbb{1}\right|\right|_\infty$. In particular, there exists a positive multiple $\tilde{\Gamma} = \tilde{\Gamma}(\varrho_1)$ of the constant Γ such that

$$\left|\mathscr{L}_{\tilde{f}}^n\phi\right|_\Omega \leq \tilde{\Gamma}\big(\tau(n)\,|\phi|_\Omega + ||\phi||_\infty\big),$$

with $\tau(n) := \sup\limits_{0<\mathrm{d}<1/2} \dfrac{\Omega\big(\theta(n)\,\mathrm{d}\big)}{\Omega(\mathrm{d})}$.

Sufficiently condition to guarantee spectral gap property

The additional attribute to be respected by a concave modulus Ω is the following limit

$$\lim_{x\to 0^+}\ \sup_{0<\mathrm{d}<1/2} \frac{\Omega\big(\mathrm{d}x\big)}{\Omega(\mathrm{d})} = 0.$$

Theorem ((GI2024) Spectral Gap Property)

Let Ω be a T-compatible modulus of continuity with respect to ω. Assume also that Ω is concave and

$$\lim_{x \to 0^+} \sup_{0<\mathrm{d}<1/2} \frac{\Omega(\mathrm{d}x)}{\Omega(\mathrm{d})} = 0.$$

Then, for any potential $f \in \mathscr{C}_\omega(\mathbb{T})$, the transfer operator $\mathscr{L}_f$, acting on $\mathscr{C}_\Omega(\mathbb{T})$ has the propery of gap spectral.

Canonical way to obtain pair of moduli dynamically compatible and satisfying $\lim_{x\to 0^+} \sup_{0<\mathrm{d}<1/2} \frac{\Omega(\mathrm{d}x)}{\Omega(\mathrm{d})} = 0$

Suppose that V, ω_0 and Ω_0 are nonnegative continuous functions, with ω_0 and Ω_0 non-decreasing. If the triple (V, ω_0, Ω_0) satisfies

$$\liminf_{x\to 0} \frac{V(x)}{\omega(x)}(\Omega((1+c)x) - \Omega(x)) > 0. \tag{2}$$

Then, for $s > 0$, the triple (V, ω_s, Ω_s), where

$$\omega_s(x) := x^s \omega_0(x)$$
$$\Omega_s(x) := x^s \Omega_0(x)$$

also satisfies condition (2) and Ω_s vanishes orderly:

$$\lim_{x\to 0^+} \sup_{0<\mathrm{d}<1/2} \frac{\Omega_s(\mathrm{d}x)}{\Omega_s(\mathrm{d})} = 0.$$

Examples

$T_s(x) = x(1 + x^s) \mod 1, \quad s > 0$

- $\omega(x) = x^p \Theta(x)$
- $\Omega(x) = x^m \Theta(x)$

Θ satisfying: for some m,

- $x \longrightarrow x^m \Theta(x)$ is concave, non-decreasing
- $\liminf_{x \to 0^+} x^{s-p+m} \left(\frac{\Theta((1+c)x)}{\Theta(x)} (1+c)^m - 1 \right) > 0.$

Given a potential $f \in \mathscr{C}_\omega(\mathbb{T})$, the transfer operator $\mathscr{L}_f$ acting on $\mathscr{C}_\Omega(\mathbb{T})$ satisfies a Ruelle-Perron-Frobenius theorem and has a gap spectral.

1. When $\Theta(x) = 1$, we recover Holder continuity:

   (Kloeckner 2020 ; Li and Rivera-Letelier 2014)

2. $\Theta(x) = 1 + |\log x|$, we deal with locally Hölder continuous.

Examples

Let

$$T_k(x) = x\big(1 + A_k(\log^k 1/x)^{-1}\big) \quad \text{mod } 1,\ \text{with } A_k > 0$$

- $\omega_0(x) = (\log^k 1/x)^{-1}(\log 1/x)^{-1}(\log^2 1/x)^{-2}$ and $\Omega_0(x) = (\log^2 1/x)^{-1}$

For any fixed $s \in (0,1)$, both

- $\omega_s(x) = x^s\omega_0(x)$ and $\Omega_s(x) = x^s\Omega_0(x)$

  are concave in a neighborhood of the origin, Then the following hold

Prop.

for any potential $f \in \mathscr{C}_{\omega_s}(\mathbb{T})$, there exists a unique associated Gibbs-equilibrium state μ which has exponential decay of correlations. with respect to the class $\mathscr{C}_{\Omega_s}(\mathbb{T})$.

For all $\beta > 0$, from the Ruelle-Perron-Frobenius Theorem with respect to the potential βf, with $f \in \mathscr{C}_\omega(\mathbb{T})$, we denote as:

- μ_β the unique Gibbs-equilibrium state associated with βf, as χ_β the spectral radius of the transfer operator $\mathscr{L}_{\beta f}$, which fulfills

$$\exp(P(T, \beta f) = \chi_\beta,$$

- $h_\beta \in \mathscr{C}_\Omega(\mathbb{T})$ the corresponding positive eigenfunction.

The parameter β represents the inverse of the temperature in Statistical Mechanics.

On the zero-temperature limit of Gibbs-equilibrium states for specific modulus of continuity ω, Ω.

So, **the goal** is to analyze the zero-temperature Gibbs measure, for specific modulus of continuity ω, Ω. What can be said about the limits in the weak-star topology of μ_β as $\beta \to +\infty$?

J. Brémont: Nonl.(2003): Topological mixing subshift of finite type and locally constant potentials.

On the zero-temperature limit of Gibbs-equilibrium states for specific modulus of continuity ω, Ω.

So, **the goal** is to analyze the zero-temperature Gibbs measure, for specific modulus of continuity ω, Ω. What can be said about the limits in the weak-star topology of μ_β as $\beta \to +\infty$?

J. Brémont: Nonl.(2003): Topological mixing subshift of finite type and locally constant potentials.

Contreras-Lopes-Thiuellen ETDS (2001): any accumulation measure (in the weak topology) of $\{\mu_\beta\}_\beta$ as $\beta \to \infty$ is a **maximizing measure** of *f*.

- The maximizing measure is such that attains the maximum:

$$m(f) := \max_{\mu \in M(\mathbb{T}, T)} \int f d\mu$$

Here, $m(f)$ is the ergodic maximizing value of *f*.

- **Expected result: (Large Deviation Principle when $\beta \to \infty$)**

  Let Ω be a T-compatible modulus of continuity with respect to the modulus ω. Let $f : \mathbb{T} \to \mathbb{R}$ be a function in $\mathscr{C}_\omega(\mathbb{T})$, **Suppose that f admits a unique maximizing measure** $\mu_{\max}$. Then for any dynamical ball $B(n, \epsilon, x)$ holds

$$\lim_{\beta \to +\infty} \frac{1}{\beta} \log \mu_\beta \Big(B(n, \epsilon, x) \Big) = - \inf_{y \in B(n,\epsilon,x)} I(y)$$

  Here, the function $I(y) = \sum_{n \geq 0} (U \circ T - U - (f - m(f)) \circ T^n(y)$ and U is any calibrated subacction for f.

  Where:

$$U: \text{ can be construct as acumulation point of } \frac{1}{\beta} \log h_\beta$$

$$\text{(calibrated subaction of } f \in \mathscr{C}_\omega(\mathbb{T}))$$

- Under what suitable conditions of (ω, Ω) we can prove that the existence of Large Deviation Principle for the family $\{\mu_\beta\}_\beta$, when $\beta \to +\infty$?

- We follow **Baraviera, A. O. Lopes and P. Thieullen: Stoch. Dynam.(2011)**, in the setting of subshift of finite type and Hölder continuous potentials, based on the dual shift and the existence of the Involution Kernel.

  Assuming the hypothesis that: **there exists a unique maximizing measure.**

  In order to have:

⋆ for any two calibrated subactions differ by a constant.

⋆ that deviation function $I(x)$ be well defined.

- We follow **Baraviera, A. O. Lopes and P. Thieullen: Stoch. Dynam.(2011)**, in the setting of subshift of finite type and Hölder continuous potentials, based on the dual shift and the existence of the Involution Kernel.

  Assuming the hypothesis that: **there exists a unique maximizing measure.**

  In order to have:

- ⋆ for any two calibrated subactions differ by a constant.
- ⋆ that deviation function $I(x)$ be well defined.

- The uniqueness of maximizing measure is a generic condition in several spaces of potentials:
  - ⋆ **Contreras, A. O. Lopes and Thiuellen: (2001) ETDS.**

Proposition (GI2024)

For a subadditive modulus of continuity ω, the space $\mathscr{C}_\omega(\mathbb{T})$ is dense in $(C^0(\mathbb{T}), \|\cdot\|_\infty)$.

The following proposition states that a (topologically) typical potential $f \in \mathscr{C}_\omega(\mathbb{T})$ has exactly one maximizing measure, provided that the modulus ω is assumed to be subadditive.

Density and Maximization

Proposition (GI2024)

For a subadditive modulus of continuity ω, the space $\mathscr{C}_\omega(\mathbb{T})$ is dense in $(C^0(\mathbb{T}), \|\cdot\|_\infty)$.

The following proposition states that a (topologically) typical potential $f \in \mathscr{C}_\omega(\mathbb{T})$ has exactly one maximizing measure, provided that the modulus ω is assumed to be subadditive.

Proposition: A generic potential $f \in \mathscr{C}_\omega$ admits a unique maximizing measure

Let ω be a subadditive modulus of continuity. Then, there is a residual set $\mathcal{R}_\omega$ in $\mathscr{C}_\omega(\mathbb{T})$ for the $||\cdot||_\omega$-topology such that every $f \in \mathcal{R}_\omega$ admits a unique maximizing measure, namely, the set

$$\left\{\mu \in M(\mathbb{T}, T) : \int f d\mu = m(f)\right\} \text{contains a unique measure.}$$

Having established due to that $(\mathscr{C}_\omega(\mathbb{T}), \|\cdot\|_\omega)$ is a **dense** Banach space in $C^0(\mathbb{T})$ which embeds continuously in $(C^0(\mathbb{T}), \|\cdot\|_\infty)$, this results follows immediately from **(Contreras, A. O. Lopes and Thiuellen (2001) ETDS).**