# Factors of substitutive spaces
and
# the column semigroup

**Dyadisc 7: Brazilian, Chilean and French Interplay for Symbolic Dynamics**

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# Roadmap



- Discussion of terminology and basic notions of substitutions
- Introduction of $S_\theta$ and some applications (automorphisms, etc.)
- Factor problem: almost automorphic case
- Factor problem: bijective case

# Our terminology



Standard symbolic dynamics assumptions:

- $\mathcal{A}$ = finite set of *symbols*, called *alphabet*; $\mathcal{A}^+ = \bigcup_{\ell \geq 1} \mathcal{A}^\ell$ = set of all nonempty *words*
- the phase space (*shift space*) is a closed set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ of sequences of symbols
  - its *language* $\mathcal{L}_X = \{x|_{[m,n]} : m < n \in \mathbb{Z}\}$ is the set of all finite subwords seen in some $x \in X$
- group action by translations (*shift*): $\sigma(\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots) = \ldots x_{-1}x_0.x_1x_2x_3\ldots$
- we describe topological and dynamical properties combinatorially, e.g.:
  - transitivity and minimality can be described by *recurrence* of words,
  - equivariant maps are described locally as *sliding block codes* $\mathcal{A}_X^{[-m,n]} \longrightarrow \mathcal{A}_Y$; if $m = n = 0$, we just call them *codes* (code factor, code automorphism, etc.)

**Definition.** *(Substitution)*

Given an alphabet $\mathcal{A}$, a *substitution* is a function $\theta\colon \mathcal{A} \longrightarrow \mathcal{A}^+$. It extends to a function $\mathcal{A}^+ \longrightarrow \mathcal{A}^+$ by concatenation:

$$\theta(w_1 \dots w_r) = \theta(w_1)\dots\theta(w_r).$$

If there is some $\ell \geqslant 1$ such that, for all $a \in \mathcal{A}$, we have $|\theta^k(a)| = \ell$, we say $\theta$ is of *constant length* $\ell$.

A *legal word* for $\theta$ is a subword of some $\theta^k(a)$, for some $k \geqslant 1, a \in \mathcal{A}$. The *substitutive shift space* generated by $\theta$ is the set of all $x \in \mathcal{A}^{\mathbb{Z}}$ whose subwords are all legal:

$$X_\theta := \{x \in X : (\forall m \leqslant n \in \mathbb{Z})(\exists k \geqslant 1, a \in \mathcal{A}) : x|_{[m,n]} \sqsubseteq \theta^k(a)\}.$$

# Example: Thue–Morse



*Thue–Morse substitution* $\theta_{\mathrm{TM}}\colon \mathcal{A} \longrightarrow \mathcal{A}^2$:

$$\begin{aligned} \theta_{\mathrm{TM}}\colon 0 &\mapsto 0\ 1 \xrightarrow{\text{by concatenation}} 0\ 1\ 1\ 0 \mapsto 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1 \mapsto \cdots \\ 1 &\mapsto 1\ 0 \longrightarrow 1\ 0\ 0\ 1 \mapsto 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0 \mapsto \cdots \end{aligned}$$

The shift space $X_{\theta_{\mathrm{TM}}}$ is the orbit closure of the sequence obtained by «infinitely iterating $\theta_{\mathrm{TM}}$»:

$$\ldots 1001011010011001011 0.01101001100101101001\ldots$$

This is because of *primitivity* (our «sanity condition»): there is some $k$ such that for any $a \in \mathcal{A}$ the word $\theta^k(a)$ contains all letters of the alphabet. If $\theta$ is primitive, the corresponding shift space $X_\theta$ is *minimal*, i.e. all orbits are dense.

# The tiling factor and MEF



Under reasonable assumptions (e.g. primitivity), substitutive shifts are *recognizable*. We only need a weak version of this:

**Theorem.** (cor. of Mossé, 1992/1996)

Let $\theta\colon \mathcal{A} \longrightarrow \mathcal{A}^\ell$ be a constant length substitution. For any $k \geq 1$ and any $x \in X_\theta$, there exist unique $x^{(k)} \in X_\theta$ and $0 \leq n_k < \ell^k$ such that:

$$x = \sigma^{n_k} \circ \theta^k(x^{(k)}),$$

so, up to a shift of magnitude $n_k$, $x$ is a concatenation of words of the form $\theta^k(a), a \in \mathcal{A}$ (*$k$-th order supertiles*).

# The tiling factor and MEF



$$\begin{array}{ccccccccccccccccccccccc}
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & = & x \\
&&&&&&&&&&&&&&&&&&&&& \| \\
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & = & \sigma\circ\theta(x^{(1)}) \\
&&&&&&&&&&&&&&&&&&&&& \| \\
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & = & \sigma\circ\theta^2(x^{(2)}) \\
&&&&&&&&&&&&&&&&&&&&& \| \\
\dots & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & = & \sigma^5\circ\theta^3(x^{(3)})
\end{array}$$

The *tiling factor* $\pi_{\mathrm{tile}}\colon x \mapsto (n_1, n_2, n_3, \dots)$ maps $X_\theta$ onto an *odometer* $\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^k\mathbb{Z}$, and encodes the «placement information» for the supertiles.

**Theorem.** (MEF of a constant length substitution)

The maximal equicontinuous factor of $X_\theta$ is a finite extension of its tiling odometer, of the form $\mathbb{Z}_\ell \times \mathbb{Z}/h\mathbb{Z}$. The value $h$ is called the *height*, and $\theta$ is *pure* if $h = 1$.

# Constant-length substitutions and their columns



Thue–Morse substitution

$$\begin{array}{rcll} \theta_{\mathrm{TM}}\colon 0 & \mapsto & 0 & 1 \\ 1 & \mapsto & 1 & 0 \\ & & \downarrow & \downarrow \\ & & \mathrm{id}_{\mathcal{A}} & (0\,1) \end{array}$$

Period-doubling substitution:

$$\begin{array}{rcll} \theta_{\mathrm{PD}}\colon 0 & \mapsto & 0 & 1 \\ 1 & \mapsto & 0 & 0 \\ & & \downarrow & \downarrow \\ & & \mathrm{const}_0 & (0\,1) \end{array}$$

**Definition.** *(Column semigroup)*

For a constant-length substitution $\theta\colon \mathcal{A} \longrightarrow \mathcal{A}^{\ell}$, its $j$-th *column* $(0 \leq j < \ell)$ is the map $\theta_j\colon \mathcal{A} \longrightarrow \mathcal{A}$ that sends each $a$ to the $j$-th symbol of the word $\theta(a)$.

The *column semigroup* $S_\theta = \langle \theta_0, \ldots, \theta_{\ell-1} \rangle$ is the subsemigroup of $\mathcal{A}^{\mathcal{A}}$ generated by all columns.

- For $\theta_{\mathrm{TM}}$, the semigroup $S_{\theta_{\mathrm{TM}}} = \{\mathrm{id}_{\mathcal{A}}, (0\,1)\}$ is comprised only of bijections.
- For $\theta_{\mathrm{PD}}$, the semigroup $S_{\theta_{\mathrm{PD}}} = \{\mathrm{id}_{\mathcal{A}}, (0\,1), \mathrm{const}_0, \mathrm{const}_1\}$ contains constant functions.

Does this affect dynamical behaviour?

# The column number

*Full transformation monoid*: $(\mathcal{A}^{\mathcal{A}}, \circ)$. A map $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ is characterised by three objects:

- its *image* $\mathrm{im}(f)$ (whose cardinality is the *rank* of $f$),
- its *partition* $\mathcal{P}_f = \{f^{-1}[\{a\}] : a \in \mathcal{A}\}$,
- a bijection $\hat{f}\colon \mathcal{P}_f \longrightarrow \mathrm{im}(f)$.

**Definition.** *(Column number)*

For a pure constant length substitution $\theta$, its *column number* is the value

$$c_\theta = \min_{f \in S_\theta} \mathrm{rank}(f).$$

If $\mathrm{rank}(f) = c_\theta$, the set $\mathrm{im}(f) \subseteq \mathcal{A}$ is called a *minimal set*. We write $\mathcal{X}$ for the collection of all minimal sets.

# Dynamical properties encoded by $S_\theta$



**Theorem.** (Dekking; more by Barge–Kellendonk, Frettlöh–Sing, Coven–Quas–Yassawi?)

For a primitive, pure substitution $\theta$, any point $z \in \mathbb{Z}_\ell$ has at least $c_\theta$ preimages under the tiling factor $\pi_{\text{tile}}\colon X_\theta \longrightarrow \mathbb{Z}_\ell$, and there is a $G_\delta$-dense subset of $\mathbb{Z}_\ell$ whose elements have exactly $c_\theta$ preimages (*regular fibres*). Furthermore, the strongly connected components of the Cayley graph of $S_\theta$ determine which fibres are *irregular* and their cardinality, with the possible exception of the set of fixed points of $\theta$.

In particular, if $c_\theta = 1$, $X_\theta$ is an almost-1-1 extension of an odometer (and hence it is *almost automorphic*) and thus is a *Toeplitz shift*, meaning that there is a «quasi-periodic» structure on the sequences $x \in X_\theta$.

The opposite scenario is $c_\theta = |\mathcal{A}|$, where $S_\theta$ is a group. In this scenario, we say that $\theta$ is a *bijective substitution*, and all fibres are regular except for the fixed point orbit.

# Dynamical properties encoded by $S_\theta$



Remember that an *automorphism* of a shift space $X$ is an equivariant bijection $X \longrightarrow X$. These maps conform a group, $\mathrm{Aut}(X)$, which is a conjugacy invariant.

**Theorem.** (Lemanczyk–Mentzen, further details by others, e.g. B–Luz–Mañibo)

For a pure, primitive substitution $\theta$, a map $\Phi\colon \mathcal{A} \longrightarrow \mathcal{A}$ defines a code automorphism $\varphi$ of $X_\theta$ if, and only if, $\Phi \in \mathrm{cent}_{\mathcal{A}^{\mathcal{A}}}(S_\theta)$, i.e., $\Phi \circ f = f \circ \Phi$ for all $f \in S_\theta$.

If $\theta$ is bijective, then all automorphisms are codes (modulo a shift) and thus the automorphism group can be explicitly computed as:

$$\mathrm{Aut}(X_\theta) \cong \mathrm{cent}_{\mathcal{A}^{\mathcal{A}}}(S_\theta) \times \mathbb{Z}.$$

Similar observations can be made about *flip conjugacies* and their generalization in higher dimensions (*extended symmetries*), but those require more care.

# The problem: finding symbolic factors



**Central question**

Given a substitutive shift $(X_\theta, \sigma)$, which systems $(Y, T)$ are topological factors of $X_\theta$?

- Interesting because some dynamical properties «lift» through factor maps (e.g. if $\varphi\colon X \twoheadrightarrow Y$ is a surjection and $f$ is an eigenfunction of $Y$, then $f \circ \varphi$ is an eigenfunction of $X$),
- We limit ourselves to symbolic factors with interesting properties:
  - Symbolic factors of substitutive shifts are substitutive! (Müllner–Yassawi)
  - Bijective substitutions and Toeplitz shifts are well-studied subclasses, with properties of interest
  - Both classes can be characterized by their column number

# The almost automorphic case



**Question**

Given a substitutive shift $(X_\theta, \sigma)$, can one find a substitution $\eta$ with these properties?

- $X_\eta$ is Toeplitz (so $c_\eta = 1$),
- $X_\eta$ is a factor of $X_\theta$,
- $X_\eta$ and $X_\theta$ have the same MEF (in particular, $X_\eta$ cannot be finite).

**Example:** The following $\{0,1\}^{[0,1]} \longrightarrow \{0,1\}$ map defines a factor $\Phi\colon X_{\theta_{\mathrm{TM}}} \twoheadrightarrow X_{\theta_{\mathrm{PD}}}$:

$$\varphi\colon \begin{array}{ccc} 00 & \mapsto & 0 \\ 01 & \mapsto & 1 \\ 10 & \mapsto & 1 \\ 11 & \mapsto & 0 \end{array}$$

$$\begin{array}{cccccccccccccccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \downarrow & & & & \downarrow & & & & & \downarrow & & & & \downarrow & & \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

# A candidate for a factor



Let's start with a less ambitious goal: finding a *code* factor of $X_\theta$, for $\theta\colon \mathcal{A} \longrightarrow \mathcal{A}^\ell$.

**Proposition.**

If $A \in \mathcal{X}$ is a minimal set, and $f \in S_\theta$, then $f[A] \in \mathcal{X}$.

In particular, each column $\theta_j$ induces a map $\tilde{\theta}_j\colon \mathcal{X} \longrightarrow \mathcal{X}$. Taking these $\ell$ induced maps together we can build a new substitution using $\mathcal{X}$ as alphabet.

$$\theta\colon \begin{array}{rcl} a & \mapsto & b\ c \\ b & \mapsto & b\ d \\ c & \mapsto & c\ a \\ d & \mapsto & c\ b \end{array} \qquad \mathcal{X} = \{\underbrace{\{a,d\}}_{A}, \underbrace{\{b,c\}}_{B}\} \qquad \tilde{\theta}\colon \begin{array}{rcl} A & \mapsto & B\ B \\ B & \mapsto & \underset{\tilde{\theta}_1}{\underset{\downarrow}{B}}\ \underset{\tilde{\theta}_2}{\underset{\downarrow}{A}} \end{array}$$

It is not hard to prove that $\tilde{\theta}\colon \mathcal{X} \longrightarrow \mathcal{X}^\ell$ is of Toeplitz type (although it may be finite!).

# A candidate for a factor



In the previous example, actually $X_\theta \cong X_{\theta_{\mathrm{TM}}}$, and we easily see that $X_{\tilde{\theta}} \cong X_{\theta_{\mathrm{PD}}}$. Furthermore, $\mathcal{X}$ is a partition of $\mathcal{A} = \{a, b, c, d\}$, and thus we can easily define a code factor $\varphi\colon X_\theta \longrightarrow X_{\tilde{\theta}}$ via:

$$\Phi(x) = M \iff x \in M.$$

**Definition.**

The substitution $\tilde{\theta}\colon \mathcal{X} \longrightarrow \mathcal{X}^\ell$ is called the *synchronizing part* of $\theta$.

If $\mathcal{X}$ partitions $\mathcal{A}$, $\Phi$ is well-defined and $X_{\tilde{\theta}}$ is a factor of $X_\theta$. Not always the case, but:

**Theorem.** (Lemanczyk–Müllner)

There is a substitution $\eta$ on the alphabet $\{(a, M) : a \in M \in \mathcal{X}\}$ such that:

- $X_\eta$ is an almost-1-1 extension of $X_\theta$,
- $X_{\tilde{\theta}}$ is a Toeplitz factor of $X_\eta$.

Some problems with the synchronizing part approach:

- $\mathcal{X}$ may not be a partition, or might be trivial ($\mathcal{X} = \{\mathcal{A}\}$ in the bijective case),
- does not account for non-code factors (not letter to letter)

**Lemma.**
If there is a substitutive factor map $\varphi\colon X_\theta \longrightarrow X_\eta$ and $\theta$ is *strongly injective*, then its local function is of radius $\leqslant 1$, i.e. of the form $\Phi\colon \mathcal{A}^{[-1,1]} \longrightarrow \mathcal{A}$. Such a factor map becomes a code factor by changing the alphabet to $\mathcal{B} = \mathcal{A}^{[-1,1]}$ via a *higher block shift* (*collaring*).

The alphabet change «stores information» about the symbols adjacent to a given one. For $\theta_{\mathrm{TM}}$ and $\mathcal{B} = \mathcal{A}^{[0,1]}$, we have induced rules like

$$10 \mapsto 1001 \qquad \rightsquigarrow \qquad 1_0 \mapsto 1_0 0_0.$$

**Theorem.** (B–Kellendonk–Yassawi)

For a pure, primitive substitution $\theta$, let $\theta^{[-1,1]}$ be the substitution representing the higher block shift associated to the alphabet $\mathcal{B} = \mathcal{A}^{[-1,1]}$, and $\mathcal{X}^{[-1,1]}$ the corresponding collection of minimal sets. Let $\mathcal{P}$ be the finest partition of $\mathcal{B}$ such that every $M \in \mathcal{X}^{[-1,1]}$ is contained in some $P \in \mathcal{P}$, and define a map $\Phi\colon \mathcal{B} \longrightarrow \mathcal{P}$ by:

$$\Phi(x) = P \iff x \in P.$$

Then, for every $f \in S_\theta$ and $P \in \mathcal{P}$, the set $f[P]$ is contained in a unique $P' \in \mathcal{P}$. If we define the $\ell$ maps $\tilde{\theta}_j\colon \mathcal{P} \longrightarrow \mathcal{P}$ that send each $P$ to the unique $P'$ containing $\theta_j^{[-1,1]}[P]$, this defines a pure Toeplitz substitution $\tilde{\theta}\colon \mathcal{P} \longrightarrow \mathcal{P}^\ell$ with alphabet $\mathcal{P}$ such that the map $\varphi\colon X_\theta \twoheadrightarrow X_{\tilde{\theta}}, (x_n)_{n\in\mathbb{Z}} \mapsto (\varphi(x_n))_{n\in\mathbb{Z}}$ is a factor map, and any Toeplitz code factor of $X_\theta$ must also be a factor of $X_{\tilde{\theta}}$. In particular, if $|X_{\tilde{\theta}}| < \infty$, nontrivial Toeplitz factors of $X_\theta$ do not exist.

$$\begin{array}{llllllll} \theta\colon & a & \mapsto & a & b & b & d & b \\ & b & \mapsto & b & a & a & c & a \\ & c & \mapsto & c & d & d & b & c \\ & d & \mapsto & d & c & c & a & d \end{array} \quad \rightsquigarrow \quad \begin{array}{llllllll} \theta^{[0,1]}\colon & a_b & \mapsto & a_b & b_b & b_d & d_b & b_b \\ & a_c & \mapsto & a_b & b_b & b_d & d_b & b_c \\ & & \vdots & & & & & \\ & d_d & \mapsto & d_c & c_c & c_a & a_d & d_d \end{array}$$

Minimal sets:

$$\begin{aligned} \mathcal{X}^{[0,1]} = \ & \{\{a_a, b_b, c_c, d_d\}, \{a_b, b_a, c_d, d_c\}, \{a_c, b_d, c_a, d_b\}, \\ & \{a_d, b_c, c_a, d_b\}, \{a_b, b_a, c_c, d_d\}, \{a_a, b_b, c_d, d_d\}\} \end{aligned}$$

Induced (non-trivial) partition:

$$\mathcal{P} = \{\{a_a, a_b, b_a, b_b, c_c, c_d, d_c, d_d\}, \{a_c, a_d, b_c, b_d, c_a, d_b\}\} = \{P, Q\}$$

# An example



We see that $\theta_0^{[0,1]}[\mathcal{A}] = \{a_b, b_a, c_d, d_c\} \subseteq P$, so we define $\tilde{\theta}_0(P) = \tilde{\theta}_0(Q) = P$. Doing the same with the remaining columns, we get the substitution $\tilde{\theta}$ given by:

$$\tilde{\theta}\colon \begin{array}{lcl} P & \mapsto & P\ P\ Q\ Q\ P \\ Q & \mapsto & P\ P\ Q\ Q\ Q \end{array}$$

The shift $X_{\tilde{\theta}}$ is a factor of $X_\theta$ via the code sending $a_a, a_b, b_a, \dots, d_d$ to $P$ and $a_c, a_d, b_d, \dots, d_b$ to $Q$. This becomes a factor map $X_\theta \twoheadrightarrow X_{\tilde{\theta}}$ with memory set $[0,1]$:

$$\begin{array}{ccccccccccccccccccc} \dots & a & b & b & d & b & b & a & a & c & a & b & a & a & c & a & d & c & \dots \in X_\theta \\ & & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \downarrow & & & \\ \dots & P & P & Q & Q & P & P & P & Q & Q & P & P & P & Q & Q & Q & P & P & \dots \in X_{\tilde{\theta}} \end{array}$$

# Detour: an algebraic interpretation

$(S_\theta, \circ)$ is a subsemigroup of $(\mathcal{A}^{\mathcal{A}}, \circ)$. What if we look at its *ideals*?

**Definition.**
For a semigroup $(S, \cdot)$, a *left ideal* is a subset $\mathfrak{L} \subseteq S$ such that $S\mathfrak{L} \subseteq \mathfrak{L}$. *Right ideals* are defined dually, and *bilateral ideals* are those that are both left and right ideals.

A subset $M \subseteq \mathcal{A}$ defines a right ideal $\mathfrak{R}_M := \{f \in S_\theta : \mathrm{im}(f) \subseteq M\}$. In particular, if $M \in \mathcal{X}$, $\mathfrak{R}_M$ is a minimal right ideal (under inclusion).

If $\mathcal{P}$ is the partition that defines $\tilde{\theta}$, for any $f \in S_\theta$ and $P \in \mathcal{P}$ the set $f\mathfrak{R}_P$ is a subset of $\mathfrak{R}_Q$ for a unique $Q \in \mathfrak{R}_P$. Thus, we may interpret the columns $\tilde{\theta}_j$ of $\tilde{\theta}$ in terms of the action of $S_\theta$ over its collection of right ideals.

**Question:** Would this approach make it easier to study the dual problem of bijective factors?

# The bijective case



**Question**

Given a substitutive shift $(X_\theta, \sigma)$, can one find a substitution $\eta$ with these properties?

- $X_\eta$ is bijective,
- $X_\eta$ is a factor of $X_\theta$,
- $X_\eta$ is an almost-1-1 extension of $X_\theta$ (so $c_\eta = c_\theta$).

If this were the case, $X_\theta$ would be dynamically «very similar» to a bijective substitution, which can have interesting topological implications (e.g. for the automorphism group).

Once again, we simplify the problem by looking into code factors first.

# Not really a detour, after all

Similar to how right ideals relate to $\mathrm{im}(f)$, left ideals in $S_\theta$ relate to partitions.

**Definition.**
Given a partition $\mathcal{P}$ of $\mathcal{A}$ and $f \in S_\theta$, define:

$$f^*\mathcal{P} := \{f^{-1}[P] : P \in \mathcal{P}\} \setminus \{\varnothing\}.$$

This is also a partition of $\mathcal{A}$. We say that $f$ preserves $\mathcal{P}$ if $f^*\mathcal{P} = \mathcal{P}$.

We note that $f^*\mathcal{P}_g = \mathcal{P}_{g\circ f}$. From here, it is not hard to see that any partition $\mathcal{P}$ defines a left ideal as follows:

$$\mathfrak{L}_{\mathcal{P}} := \{f \in S_\theta : \mathcal{P}_f \text{ is coarser than } \mathcal{P}\}.$$

# Preserved partitions and bijectivity



**Key observation:**

$f$ preserves $\mathcal{P} \iff$ the map $P \mapsto f^{-1}[P]$ is a bijection $\mathcal{P} \to \mathcal{P}$.

$\rightsquigarrow$ What if there is a $\mathcal{P}$ that is preserved by all $\theta_j$? See the example:

$$\begin{array}{lcccc} \theta\colon & a & \mapsto & b & c \\ & b & \mapsto & b & d \\ & c & \mapsto & c & a \\ & d & \mapsto & c & b \end{array}$$

with $\mathcal{P} = \{\{a, b\}, \{c, d\}\}$. Each $\theta_j$ induces a permutation of $\mathcal{P}$ via preimages:

$$\begin{aligned} \theta_0^{-1}[\{a,b\}] &= \{a,b\} \qquad & \theta_1^{-1}[\{a,b\}] &= \{c,d\} \\ \theta_0^{-1}[\{c,d\}] &= \{c,d\} \qquad & \theta_1^{-1}[\{c,d\}] &= \{a,b\} \end{aligned}$$

# Preserved partitions and bijectivity



Using $\mathcal{P} = \{P, Q\}$ as alphabet, where $P = \{a, b\}, Q = \{c, d\}$, we define $\eta\colon \mathcal{P} \longrightarrow \mathcal{P}^2$ by the equalities:

$$\eta_j^{-1}(U) = \theta_j^{-1}[U], U \in \mathcal{P} \quad \rightsquigarrow \quad \eta\colon \begin{array}{rcl} P & \mapsto & P\ Q \\ Q & \mapsto & Q\ P \end{array}$$

which is a bijective substitution! Furthermore, the map $\Phi\colon \mathcal{A} \longrightarrow \mathcal{P}$ given by $\Phi(u) = U \iff u \in U$ is well-defined, as $\mathcal{P}$ is a partition, and it induces a factor map $\varphi\colon X_\theta \twoheadrightarrow X_\eta$.

**Key question we should be asking right now:**

Where did this «good» partition come from?

**Possible answer:** $\mathcal{P} = \mathcal{P}_{\theta_0}$. But: why did we choose this one specifically?

# The importance of left ideals



**Lemma.**

If $S_\theta$ has exactly one minimal left ideal $\mathfrak{I}$ under inclusion, then this ideal is *principal*: $\mathfrak{I} = S_\theta^1 u$ for some $u \in S_\theta$. Furthermore, every $f \in S_\theta$ preserves the partition $\mathscr{P}_u$.

The image of $u$ has exactly $c_\theta$ elements, so $|\mathscr{P}_u| = c_\theta$ as well. Thus, the previous procedure creates a code factor of $X_\theta$ which is guaranteed to be bijective:

**Theorem.** (B–Kellendonk–Yassawi; *almost* the main theorem)

Let $\theta\colon \mathcal{A} \longrightarrow \mathcal{A}^\ell$ be a primitive, pure substitution with column number $c_\theta$. The following are equivalent:

- there exists a length $\ell$ bijective substitution $\eta$ on an alphabet of $c_\theta$ letters and a code factor map $\varphi\colon X_\theta \twoheadrightarrow X_\eta$ that sends the fixed points of $\theta$ to those of $\eta$,
- $S_\theta$ has a unique minimal left ideal.

# The main result (bijective version)



Once again, some problems have not been addressed:

- the restriction on the fixed points is bothersome,
- we haven't addressed factor maps which are non-codes.

The second is a non-issue:

**Lemma.**
For any $m, n \geqslant 0$, the semigroup $S_{\theta[-m,n]}$ has a unique minimal left ideal if, and only if, $S_\theta$ has a unique minimal left ideal.

For the first, given any rational element $\frac{p}{q} \in \mathbb{Z}_\ell$, one may explicitly find a substitution $\beta$ such that $X_\theta \cong X_\beta$ and the fixed points of $X_\theta$ are mapped to $\left(\pi_{\mathrm{tile}}^{(\beta)}\right)^{-1}\left[\frac{p}{q}\right]$. The substitution $\beta = \theta^{(+k)}$ is computed by a procedure similar to the higher block shift, called *$k$-sliding collaring*, where $k$ is a parameter completely determined by $p$ and $q$.

# The main result (bijective version)



**Theorem.** (B–Kellendonk–Yassawi)

Let $\theta\colon \mathcal{A} \longrightarrow \mathcal{A}^{\ell}$ be a primitive, pure substitution with column number $c_\theta$. The following are equivalent:

- $X_\theta$ is an almost-1-1 extension of an aperiodic, bijective, length $\ell$ substitutive shift space $X_\eta$, which is defined on an alphabet of exactly $c_\theta$ letters.
- There exist constants $n \geqslant 1$ and $0 \leqslant k < \ell^n$ such that the semigroup $S_\beta$ has a unique minimal left ideal, where $\beta = (\theta^n)^{(+k)}$. Moreover, there is an explicit upper bound for $n$ that is determined by the left Cayley graph of $S_\theta$.

The minimal left ideals of $S_\theta$ can be computed by using a graph search algorithm on its right Cayley graph. As there are guaranteed bounds for $k$ and $n$, we have found an algorithm to find almost-1-1 bijective factors or determine that there aren't any.

# Some final remarks



- results come from algebraic and combinatorial techniques, probably simple to generalize to $d$-dimensional constant-shape substitutions
- non-pure case ($h > 1$) can be handled similarly
- techniques can be adapted to more general cases, e.g. finding a bijective factor on less than $c_\theta$ letters
- $S_\theta$ and its ideals are inherently computable objects, so this provides algorithmic solutions to the studied problems
- semigroups are strange creatures

Thank you for your attention!