Dynamics of Dendric Subshifts

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Dyadisc 7, Valparaíso

December, 2024

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Symbolic dynamics

- The **full-shift** $\mathcal{A}^{\mathbb{Z}}$ consists of **two-sided** infinite words
$$x = \dots x_{-2}x_{-1}.x_0x_1x_2\dots$$
- Substitutions $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ are assumed to be **non-erasing**, *i.e.*, $\tau(a) \neq \epsilon$ for all $a \in \mathcal{A}$.
- Set $\mathcal{L}(X) = \{\text{words occurring in any of point } X\}$ and, for $n \geq 0$, $\mathcal{L}_n(X) = \mathcal{L}(X) \cap \mathcal{A}^n$.
- We consider only **minimal subshifts**, *i.e.*, those for which
$$\mathrm{orbit}_S(x) = \{S^n x : n \in \mathbb{Z}\} \text{ is dense in } X,\ \forall x \in X.$$

Symbolic dynamics

- The **complexity function** of X is defined as $p_X(n) = \#\mathcal{L}_n(X)$.

- Its growth is a **conjugacy invariant**, and one of the simplest and most useful tools to study subshifts.

Symbolic dynamics

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- Its growth is a **conjugacy invariant**, and one of the simplest and most useful tools to study subshifts.

- The **Rauzy graph** $G_n(X)$ of X, $n \geq 1$, is the *directed, labeled* graph with vertex set $\mathcal{L}_n(X)$ and edges $au \xrightarrow{aub} ub$, where $aub \in \mathcal{L}_{n+1}(X)$, $a, b \in \mathcal{A}$, and $u \in \mathcal{A}^n$.

Symbolic dynamics

- **Extension graphs** provide a local view of the evolution of Rauzy graphs.
- Consider two copies $\mathcal{A}_L$ and $\mathcal{A}_R$ of the alphabet $\mathcal{A}$.

Definition. The **extension graph** $\Gamma_X(u)$ of $u \in \mathcal{L}(X)$ is the triple

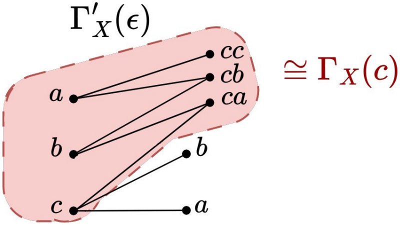
$$(L_X(u), R_X(u), E_X(u))$$

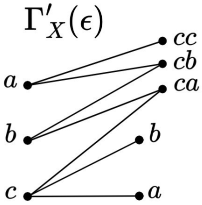
representing a (undirected and unlabeled) bipartite graph given by

$$E_X(u) = \{(a_L, b_R) : aub \in \mathcal{L}(X)\},$$

$L_X(u) = \{a_L : au \in \mathcal{L}(X)\}$, and $R_X(u) = \{b_R : ub \in \mathcal{L}(X)\}$.

Symbolic dynamics





Extension graphs

Prop. (Tijdeman '99, Andrieu-Cassaigne '22) Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and μ be one of its ergodic measures. Assume that the vector

$$\vec{f} = (\mu([a]) : a \in \mathcal{A}) \in \mathbb{R}^{\mathcal{A}}$$

has **rationally independent** coordinates. Then:

- $p_X(n) \geq (\#\mathcal{A} - 1)n + 1$ for all $n \geq 0$.
- If the previous bound is an equality for all $n \geq 0$, then all the graphs $\Gamma_X(u)$ are trees.

Extension graphs

Prop. Let Z be a compact, metrizable and **connected** space, and let $T\colon Z \to Z$ be a bijection. Consider a cover $\mathcal{P} = (P_a : a \in \mathcal{A})$ of Z such that, for all $a \in \mathcal{A}$,

- P_a is closed, **connected**, and has non-empty interior.
- $\mathrm{int}(P_a \cap P_b) = \emptyset$ for all $b \neq a$.
- $T|_{\mathrm{int}(P_a)}$ has a unique continuous extension to P_a.

Then, in the symbolic coding

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} : \exists z \in Z, \forall n \in \mathbb{Z}, T^n z \in P_{x_n}\} \subseteq \mathcal{A}^{\mathbb{Z}},$$

every extension graph $\Gamma_X(u)$ is connected.

- Ex. Arnoux-Rauzy substitutive subshifts, hypercubic billiard words, and interval exchange transformations (IETs).

Dendric subshifts

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift. We say that X

- has a **connected language** if $\Gamma_X(u)$ is *connected* for all $u \in \mathcal{L}(X)$.
- is **dendric** if $\Gamma_X(u)$ is a tree for all $u \in \mathcal{L}(X)$.
- is **eventually dendric** if $\Gamma_X(u)$ is a tree for all but finitely many $u \in \mathcal{L}(X)$.

Prop. (Balková *et al.* '08) Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift with a connected language. Then:

- $p_X(n) \geq (\#\mathcal{A} - 1)n + 1$ for all $n \geq 0$.
- X is dendric iff $p_X(n) = (\#\mathcal{A} - 1)n + 1$ for all $n \geq 0$.
- X is eventually dendric iff $(p_X(n)/n : n \geq 0)$ is bounded.

Dendric subshifts

Def. A **return word** to $W \subseteq \mathcal{L}_n(X)$ is a word $x_i x_{i+1} \dots x_{j-1}$, where $x \in X$ and i, j mark consecutive occurrences of words from W.

$$W = \{\mathrm{ab}, \mathrm{bc}\}$$

$$x = \dots \mathrm{a\,a\,b\,a\,c\,c\,a\,b\,a\,c\,c\,b\,c\,a\,a\,b\,a\,c\,c\,b\,c\,a\,a\,b\,a\,c\,c\,b\,c} \dots$$

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$$x = \dots \mathrm{a}|\mathrm{a\,b\,a\,c\,c}|\mathrm{a\,b\,a\,c\,c}|\mathrm{b\,c\,a}|\mathrm{a\,b\,a\,c\,c}|\mathrm{b\,c\,a}|\mathrm{a\,b\,a\,c\,c}|\mathrm{b\,c} \dots$$

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▷ The **coding substitution** $\tau_W : \mathcal{A}_W^* \to \mathcal{A}^*$ maps $\mathcal{A}_W$ bijectively onto

$$\{x_i x_{i+1} \dots x_{j+n-1} : x, i, j \text{ as above}\}.$$

▷ The **coding subshift** $X_W = \{y \in \mathcal{A}_W^* : \tau_W(y) \in X\}$.

$$W = \{\underline{\mathrm{ab}}, \underline{\mathrm{bc}}\}$$

$$x = \dots \mathrm{a}|\overbrace{\mathrm{a\,b\,a\,c\,c}}^{A_{\mathrm{ab}}}|\overbrace{\mathrm{a\,b\,a\,c\,c}}^{A_{\mathrm{bc}}}|\overbrace{\mathrm{b\,c\,a}}^{B_{\mathrm{ab}}}|\overbrace{\mathrm{a\,b\,a\,c\,c}}^{A_{\mathrm{bc}}}|\overbrace{\mathrm{b\,c\,a}}^{B_{\mathrm{ab}}}|\overbrace{\mathrm{a\,b\,a\,c\,c}}^{A_{\mathrm{bc}}}|\mathrm{b\,c} \dots$$

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$$y = \dots \; A_{\mathrm{ab}} \; A_{\mathrm{bc}} \; B_{\mathrm{ab}} \; A_{\mathrm{bc}} \; B_{\mathrm{ab}} \; A_{\mathrm{bc}} \; \dots$$

$$x = \dots \mathrm{a}|\underbrace{\mathrm{a\,b\,a\,c\,c}}|\underbrace{\mathrm{a\,b\,a\,c\,c}}|\underbrace{\mathrm{b\,c\,a}}|\underbrace{\mathrm{a\,b\,a\,c\,c}}|\underbrace{\mathrm{b\,c\,a}}|\underbrace{\mathrm{a\,b\,a\,c\,c}}|\mathrm{b\,c} \dots$$

$$\tau_W \begin{cases} A_{\mathrm{ab}} & \mathrm{a\,b\,a\,c\,c} \\ A_{\mathrm{bc}} \Rightarrow & \mathrm{a\,b\,a\,c\,c} \\ B_{\mathrm{ab}} & \mathrm{b\,c\,a} \end{cases}$$

Dendric subshifts

- We consider $\mathcal{A}^*$ to be a subset of $\mathbb{F}_{\mathcal{A}}$, the free group over $\mathcal{A}$.
- Any substitution $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ extends to a map

$$\hat{\tau}\colon \mathbb{F}_{\mathcal{A}} \to \mathbb{F}_{\mathcal{B}}$$

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Thm. (Berthé *et al.* '15, Gheeraert *et al.* '24)
A subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is dendric **iff**, for all $w \in \mathcal{L}(X)$, the extension

$$\hat{\tau}_w\colon \mathbb{F}_{\mathcal{A}_w} \to \mathbb{F}_{\mathcal{A}}$$

of the coding substitution τ_w to $\mathbb{F}_{\mathcal{A}}$ is an isomorphism.

Obs. $\hat{\tau}_w$ is an isomorphism iff it is onto and $\#\mathcal{A}_w = \#\mathcal{A}$.

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Obs. $\hat{\tau}_w$ is an isomorphism iff it is onto and $\#\mathcal{A}_w = \#\mathcal{A}$.

- The connectedness of $\mathcal{L}(X)$ implies that $\hat{\tau}_w$ is onto.
- The word-complexity constrains $\#\mathcal{A}_w$.

Dendric subshifts

- Any eventually dendric subshift is Kakutani equivalent to a dendric subshift (Gheeraert *et al.* '15).
- Dendricity is preserved under maximal bifix-decoding (Berthé *et al.*). Eventual dendricity is preserved by k-to-one extensions (E-Leroy).
- Description of the dimension group (Cecchi *et al.* '19).
- In an eventually dendric subshift, there are at most $\#\mathcal{A}/2$ different ergodic measures (Damron & Fickenscher '20).
- Ev. dendricity is preserved by dynamical conjugacy (Dolce & Perrin '21).

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Dendric subshifts generalize Sturmian subshifts, Arnoux-Rauzy subshifts and IETs, while maintaining core properties.

Our work: Continue exploring the dynamical properties of dendric subshifts.

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1. Dendric subshifts
2. Dynamical factors
3. S-adic structure
4. Final remarks

Preservation by dynamical morphisms

A continuous map $\pi\colon X \to Y$ between subshifts is called a **factor map** if $\pi(Sx) = S\pi(x)$ for all $x \in X$.

If moreover π is bijective, then π is a **conjugation**.

Thm. (Dolce & Perrin '21) Any minimal subshift that is **conjugate** to an eventually dendric subshift is itself eventually dendric.

- As with SFTs and finite top. rank systems, the question about factors is more difficult (and, probably, rewarding).

Question A. Is every *subshift factor* of an eventually dendric subshift also an eventually dendric subshift?

Preservation by dynamical morphisms

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- As with SFTs and finite top. rank systems, the question about factors is more difficult (and, probably, rewarding).

Question A. Is every *subshift factor* of an eventually dendric subshift also an eventually dendric subshift? **Yes!**

Theorem A. (E-Leroy) Every *subshift factor* of a subshift with an eventually connected language has the same property.

Cor. Since linear complexity is inherited by factors, Question A has an affirmative answer.

Dynamical factors

Let us recall:

Thm. (Berthé *et al.* '15, Gheeraert *et al.* '24)
A subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is dendric **iff**, for all $w \in \mathcal{L}(X)$, the extension

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Thm. (E-Leroy) A subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ has a connected language **iff**, for all $u \in \mathcal{L}(X)$ and $u' \in \mathcal{L}(X_u)$, the extension

$$\hat{\tau}_{u'} : \mathbb{F}_{\mathcal{A}_{u'}} \to \mathbb{F}_{\mathcal{A}_u}$$

of the coding substitution $\tau_{u'}$ to $\mathbb{F}_{\mathcal{A}_{u'}}$ is onto.

Obs. The theorem fails if only the surjectivity of $\hat{\tau}_u$ is considered.

Dynamical factors

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of the coding substitution $\tau_{u'}$ to $\mathbb{F}_{\mathcal{A}_{u'}}$ is onto.

Obs. $\hat{\tau}_{u'}$ is onto iff it is a composition of *elementary* and *letter-to-letter* substitutions.
In general, $\hat{\tau}_{u'}$ onto for some $u' \in \mathcal{L}(X_u) \setminus \{\epsilon\}$ implies $\Gamma_X(u)$ connected.

Sketch of the proof

Fix a factor map $X \xrightarrow{\pi} Y$ between subshifts, where X is **ev. connected**.

Prop. (E.-Leroy) Let $v \in \mathcal{L}(Y)$ and $v' \in \mathcal{L}(Y_v)$.
Then, there exist $W \subseteq \mathcal{L}_n(X)$, $W' \subseteq \mathcal{L}_m(X_W)$, and **letter-to-letter** substitutions ρ and ρ' s.t. the following diagram commute:

$$\begin{tikzcd}
\mathcal{A}_{W'}^* \arrow[r, "\rho'"] \arrow[d, "\tau_{W'}"'] & \mathcal{B}_{v'}^* \arrow[d, "\tau_{v'}"] \\
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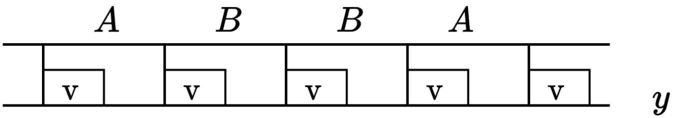
Sketch of the proof

Fix a factor map $X \overset{\pi}{\to} Y$ between subshifts, where X is **ev. connected**.

$$X \xrightarrow{\pi} Y$$

$$\pi(x)_i = \phi(x_{[i,i+r]}) \qquad \mathrm{v} \in \mathcal{L}(y)$$



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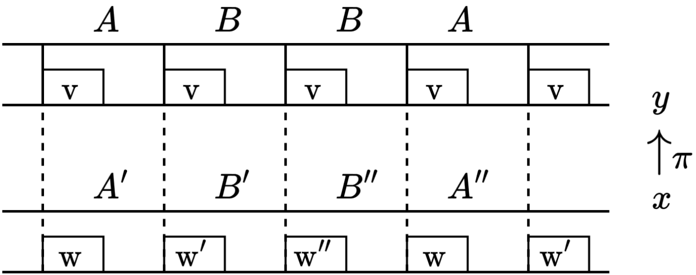

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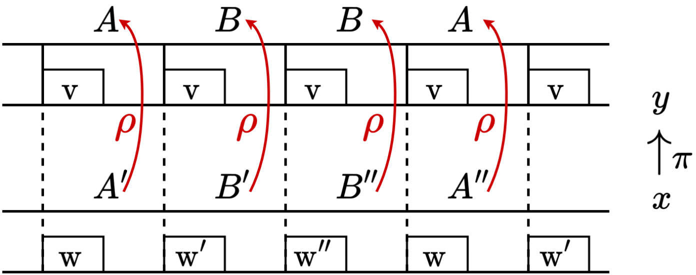

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- Goal: show that $\hat{\tau}_{W'}$ extends to an onto substitution between the corresponding free groups.

Sketch of the proof

$$\begin{tikzcd}
\mathbb{F}_{\mathcal{A}_{W'}} \arrow[r, "\hat{\rho}'"] \arrow[d, "\hat{\tau}_{W'}"'] & \mathbb{F}_{\mathcal{B}_{v'}} \arrow[d, "\hat{\tau}_{v'}"] \\
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\end{tikzcd}$$

Lemma. (E-Leroy) If v is long enough, then:

- $\Gamma_{X_W}(u)$ is connected for all $u \in \mathcal{L}(X_W) \setminus \{\epsilon\}$.
- $\{a_R : aw \in W\}$ intersects every connected component of $\Gamma_{X_W}(\epsilon)$.

These two things imply that $\hat{\tau}_{W'}$ is onto.

- Therefore, $\hat{\rho} \circ \hat{\tau}_{W'}$ is onto;
- thus $\hat{\rho}'$ is onto;
- so $\Gamma_Y(v)$ is connected.

Index

S-adic formalism

...1 0 0 0 1 0 0 1 0 0...

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$$\theta_1 : \begin{array}{l} 1 \to 10 \\ 0 \to 0 \end{array}$$

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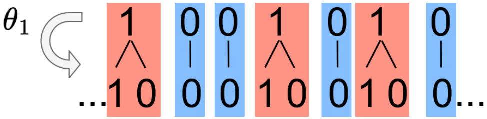

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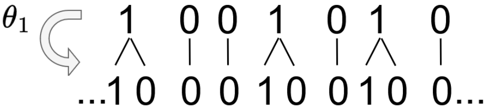

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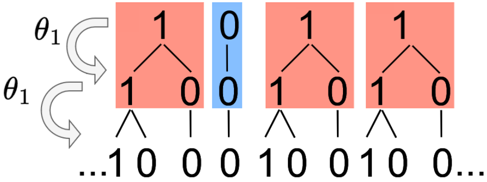

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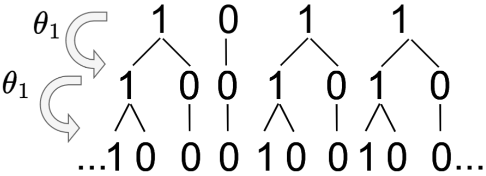

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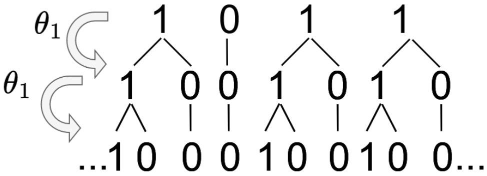

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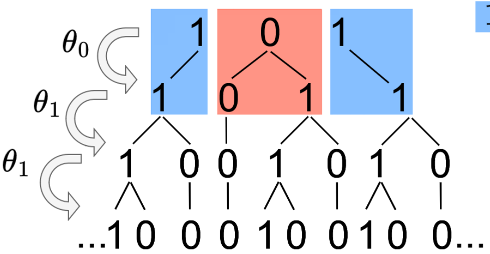

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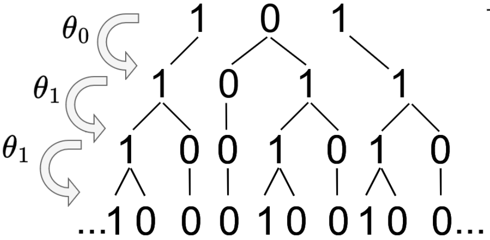

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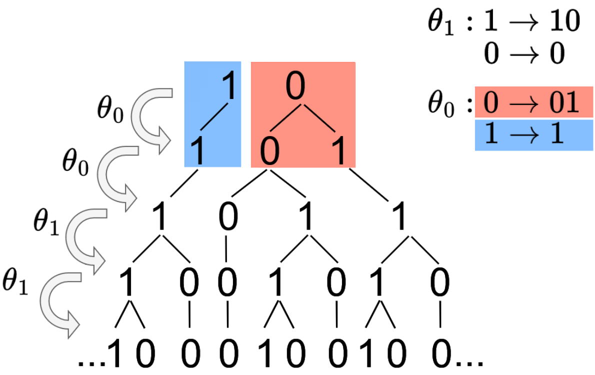

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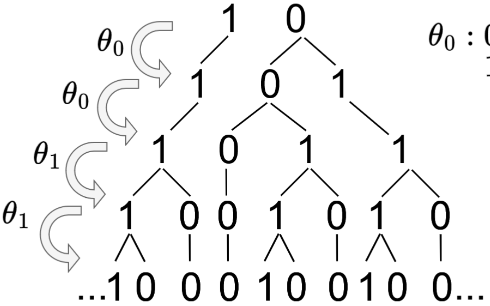

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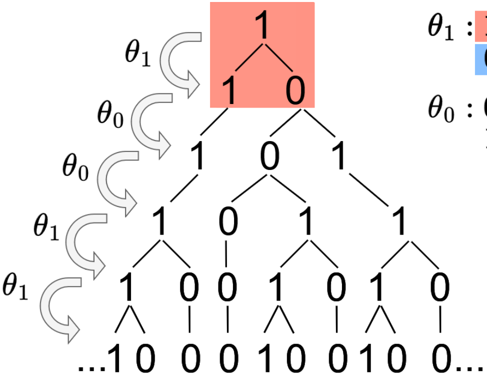

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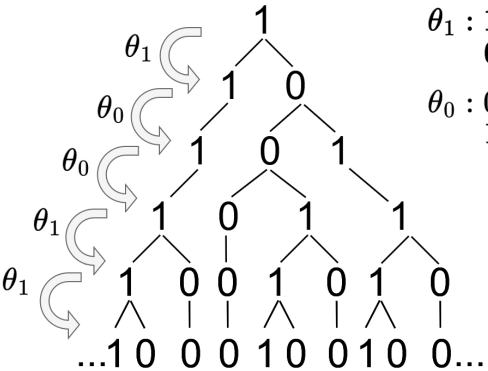

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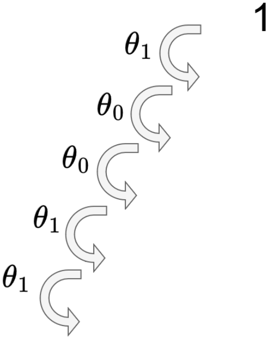

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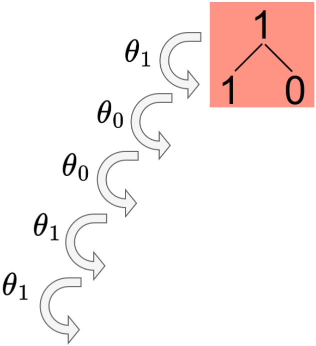

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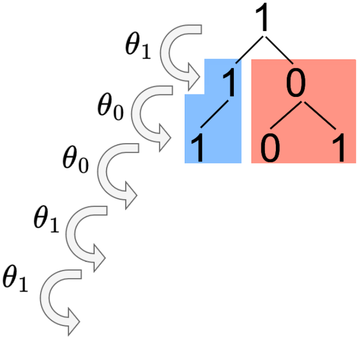

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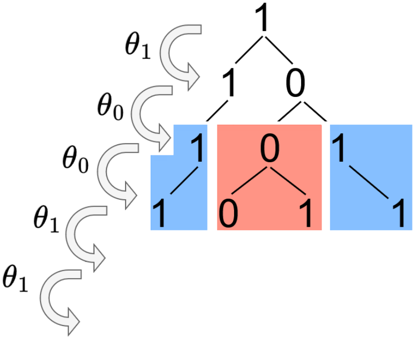

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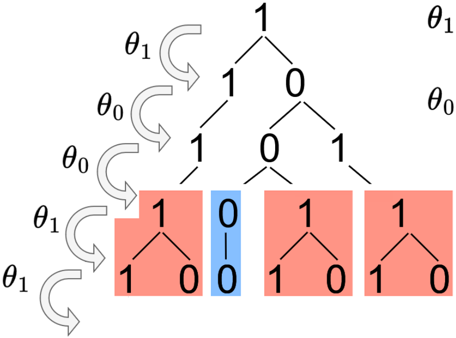

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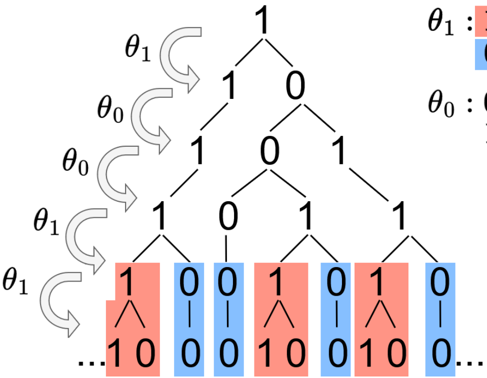

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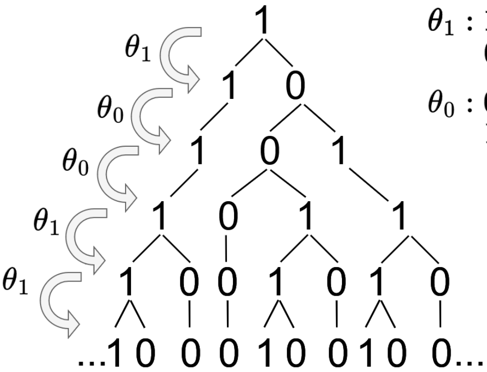

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S-adic formalism

- A **directive sequence** is a sequence of substitutions $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$ having the form

$$\mathcal{A}_0^* \xleftarrow{\tau_0} \mathcal{A}_1^* \xleftarrow{\tau_1} \mathcal{A}_2^* \xleftarrow{\tau_2} \mathcal{A}_3^* \xleftarrow{\tau_3} \mathcal{A}_4^* \xleftarrow{\tau_4} \dots$$

We write $\tau_{n,m} = \tau_n \circ \tau_{n+1} \circ \dots \circ \tau_{m-1}$ for $m > n \geq 0$.

- Words of the form $\tau_{0,n}(a)$, $a \in \mathcal{A}_n$, are the ***n*-blocks**.

Definition. The **$\boldsymbol{\mathcal{S}}$-adic subshift** $X_{\boldsymbol{\tau}}$ generated by $\boldsymbol{\tau}$ is defined as

$$\{x \in \mathcal{A}_0^{\mathbb{Z}} : \text{all finite subwords of } x \text{ occur in some } n\text{-block}\}.$$

- $\boldsymbol{\tau}$ is **primitive** if $\forall n \geq 0$, $\exists m > n$ s.t.:

every $b \in \mathcal{A}_n$ occurs in $\tau_{n,m}(a)$ for every $a \in \mathcal{A}_m$.

- *In the context of this talk: $X_{\boldsymbol{\tau}}$ is minimal iff $\boldsymbol{\tau}$ is primitive.*

S-adic structures

- The S-adic framework has permitted the unification of a variety of results and techniques in the field.
- To access the S-adic tool set, it is necessary to first translate the properties defining a system into a directive sequence.

Problem. Find an S-adic description for the class of dendric subshifts.

S-adic structures

- The S-adic framework has permitted the unification of a variety of results and techniques in the field.
- To access the S-adic tool set, it is necessary to first translate the properties defining a system into a directive sequence.

Problem. Find an S-adic description for the class of dendric subshifts.

A minimal subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is:

- Arnoux-Rauzy iff $X = X_{\boldsymbol{\tau}}$ for some primitive $\boldsymbol{\tau} \in \{\mathrm{AR}_a : a \in \mathcal{A}\}^{\mathbb{N}}$.
- (the coding of) an IET iff $X = X_{\boldsymbol{\tau}}$ where $\boldsymbol{\tau}$ reads an infinite path in some Rauzy class.

Question. Is there an S-adic **graph** description for dendric subshifts?

A graph characterizing dendricity

Thm. (Gheeraert & Leroy '22) For every $d \geq 2$, there exists a directed, labeled multigraph $\mathscr{G}_d$ s.t.:

- A minimal subshift $X \subseteq \{1, \ldots, d\}^{\mathbb{Z}}$ is dendric iff there exists a infinite path
$$v_0 \xrightarrow{\tau_0} v_1 \xrightarrow{\tau_1} v_2 \xrightarrow{\tau_2} \ldots$$
in $\mathscr{G}_d$ such that $\boldsymbol{\tau} = (\tau_n : n \geq 0)$ is a primitive directive sequence generating X.
- $\mathscr{G}_d$ has a finite vertex set and an **infinite** edge set.
- The edges of $\mathscr{G}_d$ are labeled with **dendric** return substitutions.

Observations:

- An infinite number of different labels.
- No concrete description of dendric return substitutions.

The transfer of dynamical data to combinatorial data is ***incomplete****.*

A graph characterizing dendricity

Observations:

- An infinite number of different labels.
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The transfer to combinatorial data is ***incomplete****.*

Thm. (Berthé *et al.* '15) Any dendric return substitution is a composition of elementary substitutions.

- **Our approach:** elementary substitutions instead of return substitutions.

The graph structure theorem

Thm. (E.-Leroy) For every $d \geq 2$, there exists a directed graph $\mathscr{G}_d$ s.t.:

- A minimal subshift X is dendric **iff** there exists a infinite path $(v_n \xrightarrow{\tau_n} v_{n+1} : n \geq 0)$ in $\mathscr{G}_d$ such that $\boldsymbol{\tau} = (\tau_n : n \geq 0)$ is a primitive and proper directive sequence generating X.
- $\mathscr{G}_d$ has a finite vertex set and a **finite** edge set.
- The edges of $\mathscr{G}_d$ are labeled with substitutions $\boldsymbol{\theta^{t}_{C,b}}$.

- Vertices are triples $(\Gamma(\epsilon), \mathcal{H}^L, \mathcal{H}^R)$.

Dendric codings

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be dendric.

- Given a substitution $\theta^{\mathsf{p}}_{a,b}$, set

$$Y^{\mathsf{p}}_{a,b} = \{y \in \mathcal{A}^{\mathbb{Z}} : \theta^{\mathsf{p}}_{a,b}(y) \in X\}.$$

Obs. $Y^{\mathsf{p}}_{a,b}$ is non-empty iff ba is the only left-extension of a in X.

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Obs. $Y^{\mathsf{p}}_{a,b}$ is non-empty iff ba is the only left-extension of a in X.

▷ Extension graphs in the coding $Y^{\mathsf{p}}_{a,b}$ can be computed from those in X.

Prop. (E-Leroy) If $v \in \mathcal{L}(Y^{\mathsf{p}}_{a,b}) \setminus \{\varepsilon\}$ and $u = \theta^{\mathsf{p}}_{a,b}(v)$, then $\Gamma_{Y^{\mathsf{p}}_{a,b}}(v)$ is obtained from $\Gamma_X(u)$ as follows:

1. Merge the vertex $\pi_R(a)$ with $\pi_L(b)$, if both appear.
2. Add the edges $(\pi_L(c), \pi_R(a))$ of $\Gamma_X(ub)$.

Dendric codings

We need information to control the previous process.

- For $x \in X$, let

$$E^R(x_{(-\infty,0)}) = \{y_0 : y_{(-\infty,0)} = x_{(-\infty,0)}, y \in Y\} \subseteq \mathcal{A}.$$

- Consider the **hypergraph** $\mathcal{H}_X^R = (\mathcal{A}, F_X^R)$, where

$$F_X^R = \{E^R(x) : x \in X\}.$$

Prop. (E-Leroy) The subshift $Y_{a,b}^{\mathrm{p}}$ is dendric iff:

1. The subhypergraph induced in $\mathcal{H}_X^R$ by $\{a, b\}$ is connected.
2. $\pi_L(b)$ is the unique neighbor of $\pi_R(a)$ in $\Gamma_X(\epsilon)$.

Obs. For any dendric X, we can always choose a, b satisfying 1. and 2.

The graph structure theorem

- The last proposition enables the simulation of the coding construction procedure using a **finite graph**.

The graph structure theorem

- The last proposition enables the simulation of the coding construction procedure using a **finite graph**.
- Let
$$\theta^{\mathrm{t}}_{C,b} = \theta^{\mathrm{t}}_{c_1,b} \circ \cdots \circ \theta^{\mathrm{t}}_{c_\ell,b}$$
for $C = \{c_1, \ldots, c_\ell\} \subseteq \mathcal{A}$.

Thm. (E.-Leroy) For every $d \geq 2$, there exists a directed graph $\mathscr{G}_d$ s.t.:

- A minimal subshift X is dendric **iff** there exists a infinite path $(v_n \xrightarrow{\tau_n} v_{n+1} : n \geq 0)$ in $\mathscr{G}_d$ such that $\boldsymbol{\tau} = (\tau_n : n \geq 0)$ is a primitive and proper directive sequence generating X.
- $\mathscr{G}_d$ has a finite vertex set and a **finite** edge set.
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- Vertices are triples $(\Gamma(\epsilon), \mathcal{H}^L, \mathcal{H}^R)$.

S-adic structure: particular cases

Arnoux-Rauzy. If $\mathcal{A}_d = \{1, \ldots, d\}$ and $X \subseteq \mathcal{A}_d^{\mathbb{Z}}$ is Arnoux-Rauzy:

- Our representations coincide with the usual S-adic expansions, which have the form

$$\boldsymbol{\tau} = (\theta^{\mathrm{t}_n}_{\mathcal{A}\setminus\{b_n\}, b_n} : n \geq 0), \quad \text{with } b_n \in \mathcal{A}_d \text{ and } \mathrm{t}_n \in \{\mathrm{p}, \mathrm{s}\}.$$

- One shape for $\Gamma_X(\epsilon)$.
- $\mathcal{H}_X^L$ and $\mathcal{H}_X^R$ have one hyperedge: $\mathcal{A}_d$.

IET. If X is (the coding of) an IET.

- $\Gamma_X(\epsilon)$ is a «planar» tree.
- $\mathcal{H}_X^R$ is a path, with $\#e = 2$, $\forall e \in H_X^R$.
- Our representations are accelerations of the Rauzy induction.

S-adic structure: Cassaigne's algorithm

Def. (Cassaigne's substitutions)

$$\sigma_0 : \begin{cases} 0 & \mapsto 0 \\ 1 & \mapsto 02 \\ 2 & \mapsto 1 \end{cases} \qquad \sigma_1 : \begin{cases} 0 & \mapsto 1 \\ 1 & \mapsto 02 \\ 2 & \mapsto 2 \end{cases}$$

Denoting π_{ijk} the substitution $0 \mapsto i$, $1 \mapsto j$, $2 \mapsto k$, we have:

$$\sigma_0 = \pi_{021} \circ \theta^{\mathsf{p}}_{\{1\},0} \quad \sigma_0^2 = \theta^{\mathsf{p}}_{\{1,2\},0}$$

$$\sigma_1 = \pi_{102} \circ \theta^{\mathsf{s}}_{\{1\},2} \quad \sigma_1^2 = \theta^{\mathsf{s}}_{\{0,1\},2}$$

Thm. (Cassaigne *et al.* '21). The subshift $X_{\boldsymbol{\tau}}$ is dendric for all primitive $\boldsymbol{\tau} = (\tau_n)_{n \geq 0} \in \{\sigma_0, \sigma_1\}^{\mathbb{N}}$.

- The structure of $X_{\boldsymbol{\tau}}$ given by our theorem matches $\boldsymbol{\tau}$, up to letter-permuting substitutions.

Index

Final remarks and future work

New research directions have been unlocked.

1. Theorem on factors:
   - Description of intermediate substitutions $\tau_{W'}$?
2. Applications of the S-adic structure.
   - To study ergodic measures?
   - To design multidimensional continued fraction algorithms?
   - For geometric realizations?

   Similar S-adic structure for the connected language case.