Decidability of the isomorphism problem between constant-shape substitutions

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Joint work with Julien Leroy

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Let $\mathcal{A}$ be a finite alphabet and consider in $\mathcal{A}^{\mathbb{Z}^d}$ the product topology.

The shift action is defined on $\mathcal{A}^{\mathbb{Z}^d}$ as:

$$\forall x \in \mathcal{A}^{\mathbb{Z}^d}, \forall \boldsymbol{n}, \boldsymbol{k} \in \mathbb{Z}^d, (S^{\boldsymbol{n}} x)_{\boldsymbol{k}} = x_{\boldsymbol{n}+\boldsymbol{k}}.$$

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Is there an **isomorphism** between two given subshifts $X \subseteq \mathcal{A}^{\mathbb{Z}^d}, Y \subseteq \mathcal{B}^{\mathbb{Z}^d}$?
Isomorphism: A continuous bijective map $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ s.t.

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In the case ϕ is only surjective, we say that ϕ is a **factor map**.

Subshifts of finite type (1-D)

Take $\mathcal{F} \subseteq \mathcal{A}^*$ finite.

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Note that subshifts of finite type are given by **finite data**.

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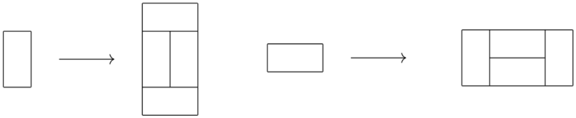
$$\mathcal{L}(\zeta) = \{\mathtt{p} \in \mathcal{A}^* \colon (\exists n \in \mathbb{N})(\exists a \in \mathcal{A})\, \mathtt{p} \sqsubseteq \zeta^n(a)\}$$

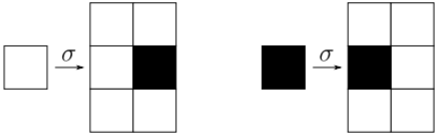
$$X_\zeta = \{x \in \mathcal{A}^{\mathbb{Z}} \colon (\forall \mathtt{p} \sqsubseteq x), \mathtt{p} \in \mathcal{L}(\zeta)\}.$$

The multidimensional case (square and block substitutions)

The table substitution



Rectangular substitutions



Example given by T. Fernique and V. Lutfalla

Non-linearly recurrent

Constant-shape substitutions (Introduced in 2023 by C.).

- Let $L \in M(d, \mathbb{Z})$ be an expansion matrix, i.e. L is invertible, $\|L\| > 1$, $\|L^{-1}\| < 1$.
- Let $F \Subset \mathbb{Z}^d$ be a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$ (containing $\mathbf{0}$).
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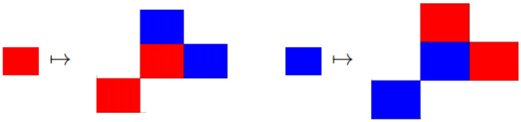
Ex.: $L = 2 \cdot Id_{\mathbb{R}^2}$, $F = \{(0,0),(1,0),(0,1),(-1,-1)\}$.



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For any $n > 0$ the n-th iteration is defined as $\zeta^n : \mathcal{A} \to \mathcal{A}^{F_n}$, with $F_{n+1} = L(F_n) + F_1$.

Example of iterations of a substitution:

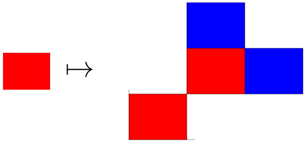

Figure: First iteration of a substitution.

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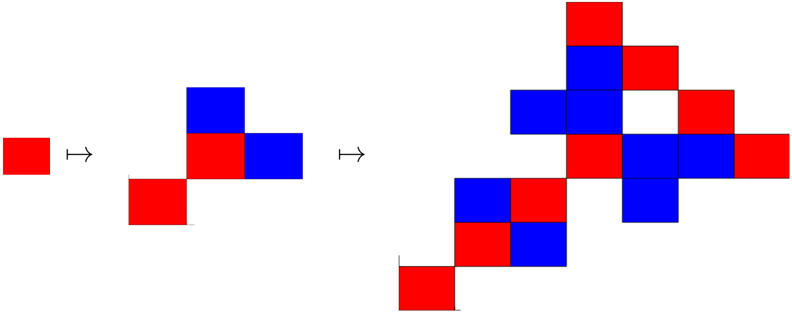

Figure: First and second iterations of a substitution.

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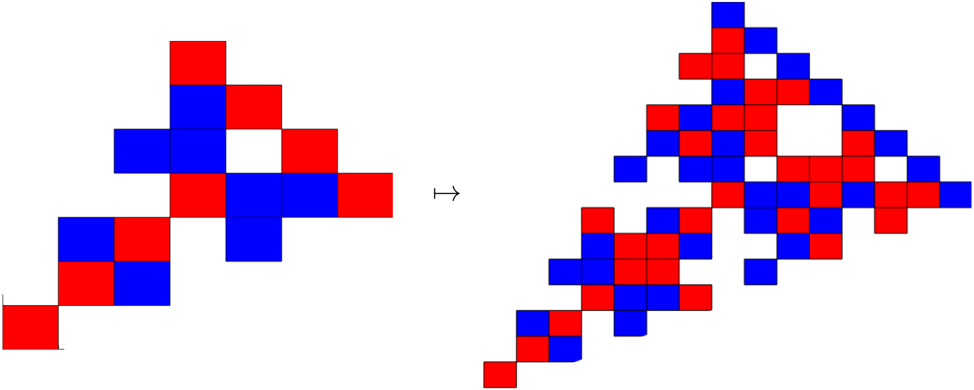

Figure: Second and third iterations of a substitution.

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The language of the substitution is defined as

$$\mathcal{L}(\zeta) = \{\mathtt{p} \colon \exists n > 0, \exists a \in \mathcal{A},\ \mathtt{p} \sqsubseteq \zeta^n(a)\}.$$

If the language is well defined*, then the language uniquely determined a subshift $X_\zeta \subseteq \mathcal{A}^{\mathbb{Z}^d}$ as the set of all elements $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that any pattern occurring in x is in $\mathcal{L}(\zeta)$.

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Ex.: $L = \times 2$, $F = \{0, 3\}$.

$$\begin{aligned} \sigma : \quad 0 &\mapsto 0\text{--}1 \\ 1 &\mapsto 1\text{--}0. \end{aligned}$$

The language of σ does not define a subshift (there is no words of length 2!).

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The following are equivalent:

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3. *The sequence of finite sets $(F_n)_{n>0}$ is Følner, i.e., for all $\mathbf{n} \in \mathbb{Z}^d$ we have*

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Remark

By a work of J.-Y. Lee, R. V. Moody, and B. Solomyak, these substitutive subshifts are always uniquely ergodic.

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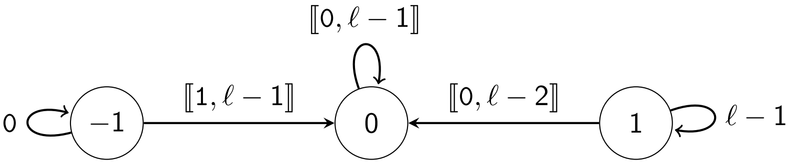

Figure: The labeled graph for the classical one-dimensional case for a substitution of length ℓ.

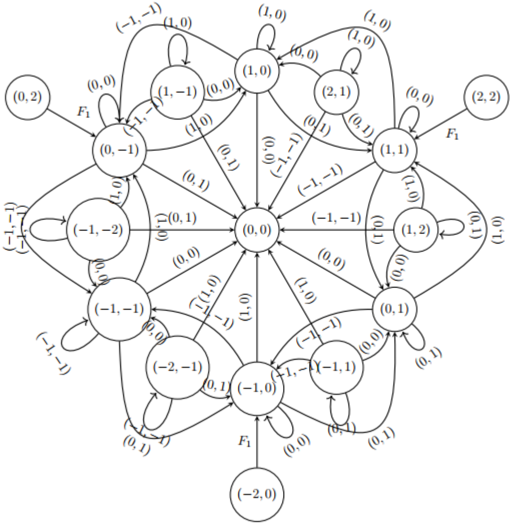

Figure: The labeled graph for the triangular Thue-Morse. Outgoing edges from vertex $(0,0)$ are not represented since they are self-loops.

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The substitution is primitive if and only if the substitutive subshift is minimal, i.e., $\forall x \in X_\zeta$, $\overline{\mathrm{Orb}(x, \mathbb{Z}^d)} = X_\zeta$

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Proposition (C., J. Leroy (2024))

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$$p_{X_\zeta}(n) \leq K \cdot n^{-\frac{\log(\det(L))}{\log(\|L^{-1}\|)}},$$

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C. W. Hansen (2000, Dissertation Thesis): Upper bound of the complexity function for block substitutions.

Decidability of the isomorphism problem

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Let ζ_1, ζ_2 be two aperiodic primitive constant-shape substitution with the same expansion matrix L. It is decidable to know whether there exists a factor map between $(X_{\zeta_1}, S, \mathbb{Z}^d)$ and $(X_{\zeta_2}, S, \mathbb{Z}^d)$.

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I. Fagnot (1997): Constant-length case, using the first-order logic framework of Presburger arithmetic.

F. Durand, J. Leroy (2022): General case for minimal substitutive subshifts.

- Non constant-length case: Return words and dill maps.
- Constant-length case: Automata theory and decidability of the first-order logic structure $\langle \mathbb{N}, +, V_k, = \rangle$.

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There exists a **computable** *aperiodic primitive constant-shape substitution $\tilde{\zeta}$ with support G_1 such that $(X_\zeta, S, \mathbb{Z}^d)$ and $(X_{\tilde{\zeta}}, S, \mathbb{Z}^d)$ are topologically conjugate.*

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We can assume that ζ_1 and ζ_2 have the same support.

$$\sigma: \quad 0 \mapsto \begin{matrix} 2 & 3 \\ 0 & 1 \end{matrix} \qquad 1 \mapsto \begin{matrix} 5 & 6 \\ 7 & 4 \end{matrix} \qquad 2 \mapsto \begin{matrix} 6 & 4 \\ 7 & 5 \end{matrix} \qquad 3 \mapsto \begin{matrix} 8 & 0 \\ 0 & 8 \end{matrix}$$

$$4 \mapsto \begin{matrix} 8 & 6 \\ 0 & 1 \end{matrix} \qquad 5 \mapsto \begin{matrix} 10 & 11 \\ 12 & 9 \end{matrix} \qquad 6 \mapsto \begin{matrix} 2 & 4 \\ 0 & 8 \end{matrix} \qquad 7 \mapsto \begin{matrix} 5 & 0 \\ 7 & 5 \end{matrix}$$

$$8 \mapsto \begin{matrix} 14 & 11 \\ 8 & 13 \end{matrix} \qquad 9 \mapsto \begin{matrix} 0 & 10 \\ 8 & 13 \end{matrix} \qquad 10 \mapsto \begin{matrix} 14 & 9 \\ 8 & 0 \end{matrix} \qquad 11 \mapsto \begin{matrix} 0 & 8 \\ 8 & 0 \end{matrix}$$

$$12 \mapsto \begin{matrix} 15 & 8 \\ 12 & 15 \end{matrix} \qquad 13 \mapsto \begin{matrix} 15 & 10 \\ 12 & 9 \end{matrix} \qquad 14 \mapsto \begin{matrix} 10 & 9 \\ 12 & 15 \end{matrix} \qquad 15 \mapsto \begin{matrix} 6 & 3 \\ 7 & 4 \end{matrix}$$

Figure: A square substitution conjugate to the triangular Thue-Morse substitution.

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Theorem (M. Curtis, G. Hedlund, R. Lyndon (1969))

If $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ *is a factor map, there exists* $r > 0$ *and a map* $\Phi : \mathcal{L}_{B(\mathbf{0},r)}(X) \to \mathcal{A}_Y$ *such that for all* $\mathbf{n} \in \mathbb{Z}^d$, $\phi(x)_{\mathbf{n}} = \Phi(x|_{\mathbf{n}+B(\mathbf{0},r)\cap\mathbb{Z}^d})$.

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1. *The factor map ψ is a sliding block code of radius $\bar{r} + R_{\zeta_2}$, where R_{ζ_2} is a* **recognizability radius** *for ζ_2, and $\bar{r}$ is a constant depending only on L and F which is invariant under iteration.*

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2. *There exists a* **computable** *integer $n > 0$ and $\boldsymbol{f} \in F_n$ such that $S^{\boldsymbol{f}}\psi\zeta_1^n = \zeta_2^n\psi$.*

Step 2. Characterization of factor maps

Theorem (C., J. Leroy (2024))

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B. Host, F. Parreau (1989): Measurable factors between constant-length substitutions.
V. Salo, I. Törmä (2015): Factor maps between constant-length and Pisot substitutions.
F. Durand, J. Leroy (2022): Factor maps between minimal substitutive subshifts.
C. (2023): Measurable factors between constant-shape substitutions.

Step 3. Computability of the recognizability radius

Proposition (C. (2023), C., J. Leroy (2024))

Let ζ be an aperiodic primitive constant-shape substitution with expansion matrix L and $x \in X_\zeta$ be a fixed point of ζ. There is a **computable** $R > 0$ *such that for all* $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$,

$$x|_{[B(L_\zeta(\boldsymbol{i}),R)\cap\mathbb{Z}^d]} = x|_{[B(\boldsymbol{j},R)\cap\mathbb{Z}^d]} \implies (\exists \boldsymbol{k} \in \mathbb{Z}^d)((\boldsymbol{j} = L_\zeta(\boldsymbol{k})) \wedge (x_{\boldsymbol{i}} = x_{\boldsymbol{k}})).$$

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B. Mossé (1992): Proof of the recognizability property for 1-D aperiodic primitive substitutions.

F. Durand, J. Leroy (2017): Computability of the constant of recognizability.

M.-P. Béal, D. Perrin, A. Restivo (2023): Recognizability of morphisms.

$L = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$$0 \mapsto \begin{matrix} 0 & \\ 0 & 2 \\ & 1 \end{matrix}, \qquad 1 \mapsto \begin{matrix} 0 & \\ 1 & 2 \\ & 1 \end{matrix}, \qquad 2 \mapsto \begin{matrix} 0 & \\ 2 & 2 \\ & 1 \end{matrix}.$$

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With the proof of the computability, it can be proved that

$$R_2 = 10^{10^{10^{10^{579}}}}$$

is a recognizability radius.

Step 4. Coalescence

Corollary

Let ζ be an aperiodic primitive constant-shape substitution. Then $(X_\zeta, S, \mathbb{Z}^d)$ is coalescent, i.e., any factor map between X_ζ to itself is invertible.

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Corollary

If the factorization problem is decidable, then the isomorphism problem also is.

Final step. Factorization problem

Proposition

Let $\zeta_1 : \mathcal{A} \to \mathcal{A}^{F_1}, \zeta_2 : \mathcal{B} \to \mathcal{B}^{F_1}$ be two aperiodic primitive constant-shape substitutions with the same expansion matrix L and the same support F_1. Let $\Phi : \mathcal{A}^{[B(\mathbf{0},r)\cap\mathbb{Z}^d]} \to \mathcal{B}$ be a block map of radius r and let ϕ be the associated factor map from $\mathcal{A}^{\mathbb{Z}^d}$ to $\phi(\mathcal{A}^{\mathbb{Z}^d})$.

- *If there exists $\mathbf{f} \in \mathbb{Z}^d$ such that $S^{\mathbf{f}}\phi\zeta_1(x) = \zeta_2\phi(x)$ for all $x \in X_{\zeta_1}$, then ϕ is a factor map from X_{ζ_1} to X_{ζ_2}.*

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- *If there exists* $\mathbf{f} \in \mathbb{Z}^d$ *such that* $S^{\mathbf{f}}\phi\zeta_1(x) = \zeta_2\phi(x)$ *for all* $x \in X_{\zeta_1}$*, then* ϕ *is a factor map from* X_{ζ_1} *to* X_{ζ_2}*.*
- *it is decidable whether* $S^{\mathbf{f}}\phi\zeta_1(x) = \zeta_2\phi(x)$ *for all* $x \in X_{\zeta_1}$*.*

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Corollary (C., J. Leroy (2024))

The isomorphism problem between aperiodic primitive substitutions with the same matrix is decidable.

Consequences of factor maps

Corollary (C., J. Leroy (2024))

Let ζ be an aperiodic primitive constant-shape substitution. Then $\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S\rangle$ *is finite.*

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B. Host, F. Parreau (1989): (1-D) Certain constant-length substitutions.
V. Cyr, B. Kra (2015): (1-D) Transitive subshifts with sublinear complexity.
E. M. Coven, A. Quas, R. Yassawi (2016): (1-D) Infinite minimal subshifts with sublinear complexity.
S. Donoso, F. Durand, A. Maass, and S. Petite (2016): (1-D) Subshifts with non-superlinear complexity. Using asymptotic pairs.
C. (2023): Certain constant-shape substitutions.

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We say that a constant-shape substitution is **injective on letters** if $a \neq b \in \mathcal{A}$, then $\zeta(a) \neq \zeta(b)$.

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Proposition (F. Blanchard, F. Durand, and A. Maass (2004))

Let ζ be an aperiodic primitive constant-shape substitution. There exists an aperiodic **injective** *primitive constant-shape substitution $\tilde{\zeta}$ such that $(X_\zeta, S, \mathbb{Z}^d)$ and $(X_{\tilde{\zeta}}, S, \mathbb{Z}^d)$ are topologically conjugate.*

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Remark

Under the assumption of injectivity, we can improve the bound $\bar{r} + R_{\zeta_2}$ to $3\bar{r} + 1$.

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Aperiodic symbolic factors of substitutive subshifts are substitutive (with the same expansion matrix and support of the substitution), up to conjugation.

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F. Durand (1999): Characterization of the topological Cantor factors of 1-D aperiodic primitive substitutions.

C. Mülner, R. Yassawi (2021): Refinement on the constant-length case.

Consequences of factor maps

Lemma (C., J. Leroy(2024))

Substitutive subshifts $(X_\zeta, S, \mathbb{Z}^d)$ *have finitely many aperiodic symbolic factors, up to conjugacy.*

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Substitutive subshifts $(X_\zeta, S, \mathbb{Z}^d)$ have finitely many aperiodic symbolic factors, up to conjugacy.

Remark

We can provide a list containing the aperiodic primitive injective substitution that are topological factors of an aperiodic primitive constant-shape substitution.

Theorem (C., J. Leroy (2024))

Let ζ_1, ζ_2 be two aperiodic primitive constant-shape substitution with the same expansion matrix L. It is decidable to know whether there exists a factor map between $(X_{\zeta_1}, S, \mathbb{Z}^d)$ and $(X_{\zeta_2}, S, \mathbb{Z}^d)$.

$$\zeta_1 : \quad 0 \mapsto \begin{array}{cc} 0 & 2 \\ 1 & 3 \\ 1 & 3 \\ 0 & 2 \end{array} \qquad 1 \mapsto \begin{array}{cc} 1 & 3 \\ 0 & 2 \\ 0 & 2 \\ 1 & 3 \end{array} \qquad 2 \mapsto \begin{array}{cc} 2 & 0 \\ 3 & 1 \\ 3 & 1 \\ 2 & 0 \end{array} \qquad 3 \mapsto \begin{array}{cc} 3 & 1 \\ 2 & 0 \\ 2 & 0 \\ 3 & 1 \end{array}$$

$$\zeta_2 : \quad 0 \mapsto \begin{array}{cccc} 1 & 3 & 3 & 1 \\ 0 & 2 & 2 & 0 \end{array} \qquad 1 \mapsto \begin{array}{cccc} 0 & 2 & 2 & 0 \\ 1 & 3 & 3 & 1 \end{array}$$

$$2 \mapsto \begin{array}{cccc} 3 & 1 & 1 & 3 \\ 2 & 0 & 0 & 2 \end{array} \qquad 3 \mapsto \begin{array}{cccc} 2 & 0 & 0 & 2 \\ 3 & 1 & 1 & 3 \end{array}$$

Conjugate substitutions such that $L_1^n \neq L_2^m$ for all $n, m \geq 1$.


Recaudación solidaria
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nuestra amiga y compañera
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su tratamiento
contra el cáncer
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THANKS