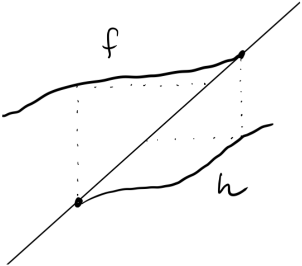
Entropy for one dimensional group actions

Cristóbal Rivas
(Work in progress with A. Navas, M. Triestino, ...)



f and h are in overlapping position

(metric) Entropy

Let $G = \langle S \rangle$, where $S \subset G$ is a finite symetric generating set.

- For $g \in G$ we denote $|g| = |g|_S$ the word length of g with respecto to S.
- Also, we denote $B_G(n) = B_{(G,S)}(n) := \{g \in G \mid |g|_S \leq n\}$ the ball of radius n in G.

Assume G is acting by homeomorphisms in a compact metric space (X, d).

- A subset $A \subseteq X$ is (n, ϵ)**-separated** if for all $x \neq y$ in A, there is $g \in B_G(n)$ such that $d(g(x), g(y)) \geq \epsilon$.
- Let $s(n, \epsilon)$ be the maximal cardinality of an (n, ϵ)-separated set.
- Let $h_\epsilon = \text{lím sup}_{n\to\infty} \frac{\log(s(n,\epsilon))}{n}$

The entropy for the G-action on X is

$$\mathfrak{h}(G \curvearrowright X) = \lim_{\epsilon \to 0} h_\epsilon.$$

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Some facts

- $\mathfrak{h}$ may depend on the generating set, but being zero, positive or infinite does not.
- $\mathfrak{h}(G \curvearrowright X)$ is a conjugation invariant.
- If every $s \in S$ is K-bi-Lipchitz, then $\mathfrak{h} \preceq \log K$.

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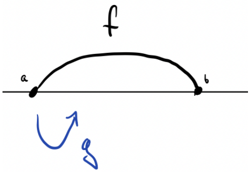
Yet for more general group actions this is not the case

Entropy for 1-d actions: crossings

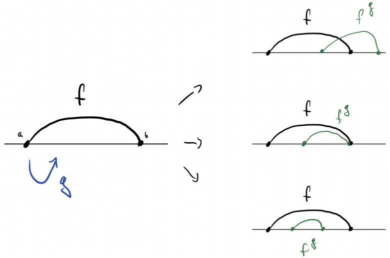
Definition

Two homeomorphisms of f and g are crossed if there is a component of support of f, say I, such that

$$g(\partial I) \cap I \neq \emptyset.$$



Having crossed elements is the same as having elements in overlapping position



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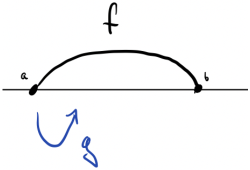

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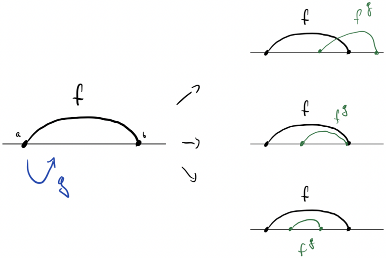
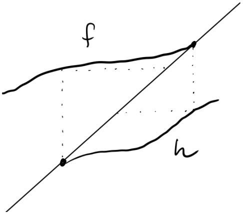




local version of a crossing

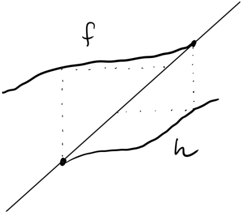
f and h are in **overlapping** position

Entropy for 1-d actions: crossings

If f and h are in overlapping position then $\mathfrak{h} > 0$.

Forward images of the interval I under f and h produces an exponentially-growing family of disjoint intervals:

The semi-group $\langle f, h\rangle^+$ is free and has a wandering interval.



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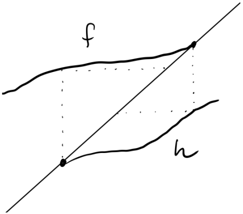

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Theorem (Ghys, Langevin, Walczak '88)

Suppose $G \leq \mathrm{Diff}^2(S^1)$ is a finitely generated group. Then $\mathfrak{h}(G)$ is positive if and only if G has overlapping elements.

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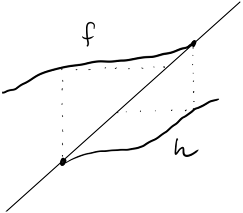

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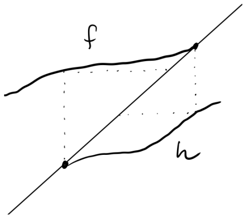

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Full disclosure: S. Hurder claim to have prove GLW result in $C^{1+\alpha}$ and even C^1 regularity.

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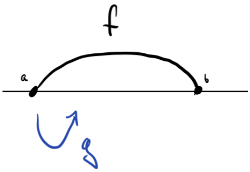

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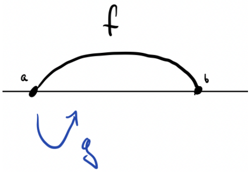

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This **entails** that actions without crossings have a **level** structure. For instance we know:

Theorem (Holder, Conrad, Solodov)

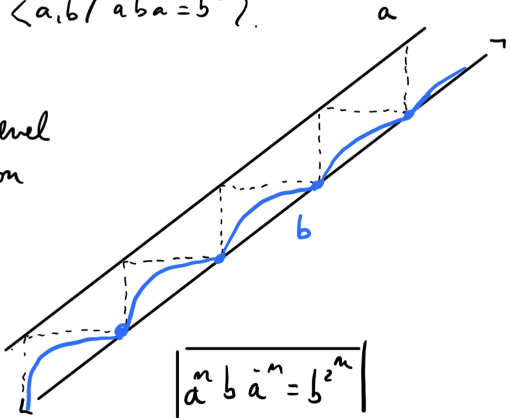
An action of a finitely generated subgroup of $Homeo_+(\mathbb{R})$ without crossings is semi-conjugated to an action by translations.

Entropy in absence of crossings:

Baumslag-Solitar actions

$G = \langle a, b \mid a b a^{-1} = b^2 \rangle$.

2 level action



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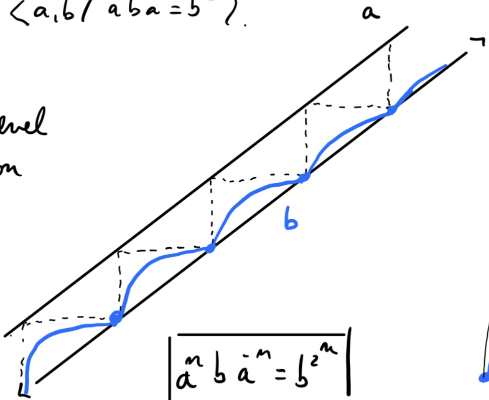

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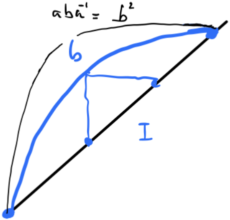

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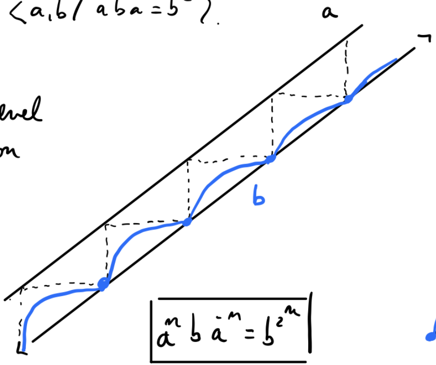

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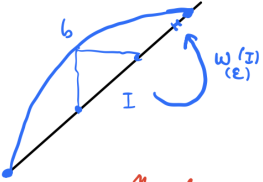

2^n elements in $B(3n)$.

$(\varepsilon_i) \in \{0,1\}^n$, $w_{(\varepsilon)} = a^n b^{\varepsilon_n} \cdots b^{\varepsilon_2} a^{-1} b^{\varepsilon_1} a^{-1} b^{\varepsilon_0}$

$= (a^n b^{\varepsilon_n} a^{-n})(a^{n-1} b^{\varepsilon_{n-1}} a^{-(n-1)}) \cdots (a b^{\varepsilon_1} a^{-1}) b^{\varepsilon_0} = b^{\sum_{i=0}^{n} 2^i \varepsilon_i}$

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But this action is not C^1 in the compact interval $[0, 1]$.

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Indeed, we can assume that I is very close to 0 and hence derivatives of a and b are δ-close to 1.

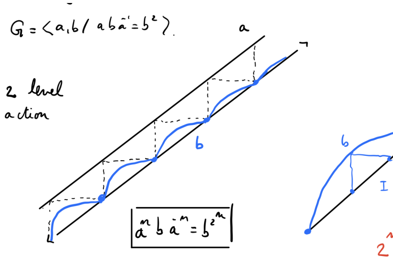
So, by MVT,

$$|w_{(\epsilon)}(I)| \geq (1-\delta)^{3n}|I|$$

Hence, adding up all of them, we get $\forall n$

$$1 \geq \sum_{(\epsilon)} |w_{(\epsilon)}(I)| \geq 2^n(1-\delta)^{3n}|I|$$

which is a contradictions for small δ.



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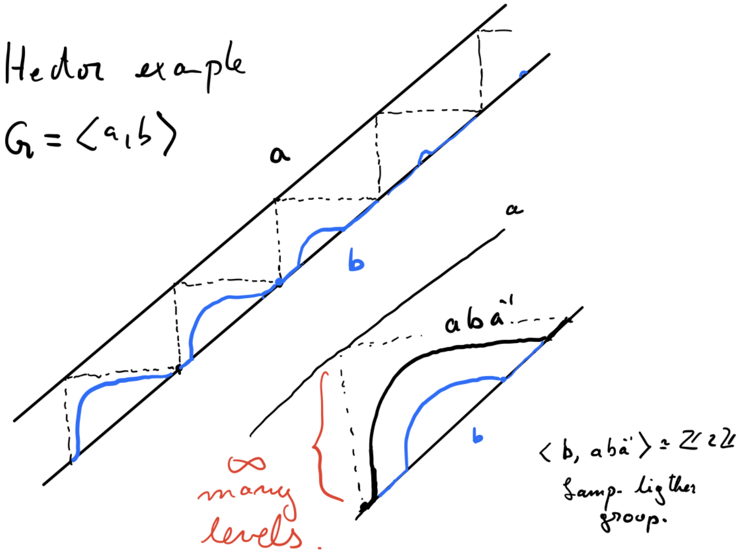
Entropy in absence of crossings:

Wreath-product actions: has entropy but is not C^1 in $[0, 1]$



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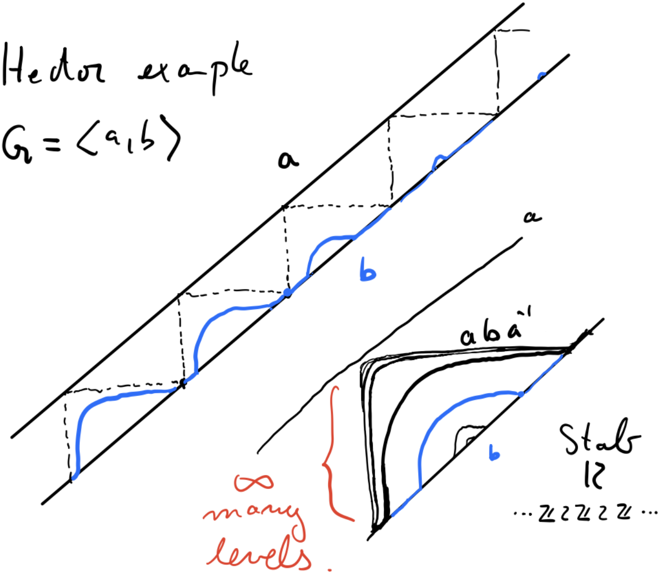
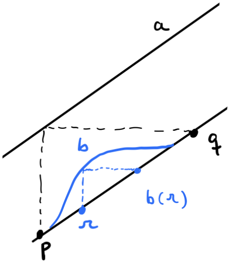


A key ingredient in C^2 regularity:

Level-compactness of bumps



Theorem (Ghys, Langevin, Walczak)

Supposse $G \leq \mathrm{Diff}^2([0,1])$ is a finitely generated group. Then $\mathfrak{h}(G)$ is positive if and only if G has overlapping elements.

Level-compactness principle: For G as above, $a, b \in G$ as in the picture, $\exists\ C \geq 0$ (depending on b) such that

$$\frac{|b(r) - r|}{|p-q|} \leq C|p-q|.$$

Proof: By MVT $\frac{b(r)-b(p)}{r-p} = Db(\xi)$, $1 = Db(\xi')$.

$$\left|\frac{b(r)-p}{r-p} - \frac{r-p}{r-p}\right| \leq |Db(\xi) - Db(\xi')| \leq C|\xi - \xi'| \leq C|p-q|$$

$\square$

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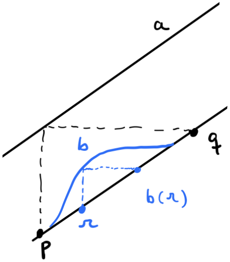

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But, for a of class C^2 and I a wandering interval inside $[p, q]$, the proportion of

$$\frac{|a^n(I)|}{|a^n([p,q])|}$$

stay away from 0. (control of distortion)

Thus, the element b can only move finitely many of these iterates of I.

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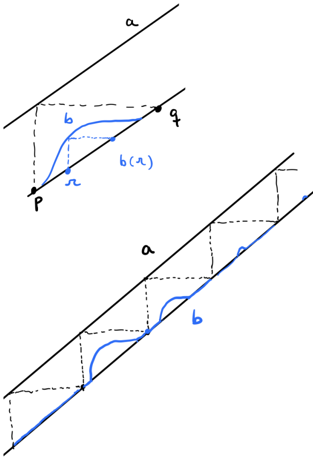
Hence, by control of distortion, b can only move finitely many intervals in the a-orbit of a wandering interval I.



Conclusion: at macroscopic level, C^2 the action without crossings behaves like an action with finitely many levels (which are *easier*).

Hope in C^1 or $C^{1+\alpha}$ regularity ?

Try to conjugate the action C^1-close, or bi-Lipchitz close to the identity.

In $C^{1+\alpha}$: Still a challenge. Some level-compactness of bumps.

In C^1: (even) more challenging.

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Merci, Obrigado, Gracias